Where Mathematics Comes From

How the Embodied Mind Brings Mathematics into Being

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BASIC BOOKS
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0465037704fm.qxd 8/23/00 9:49 AM Page iv

Published by Basic Books,
A Member of the Perseus Books Group

Designed by Rachel Hegarty

Library of Congress cataloging-in-publication data
Lakoff, George.
Where mathematics comes from : how the embodied mind brings mathematics into being / George Lakoff and Rafael E. Núñez.
p. cm.
Includes bibliographical references and index.
ISBN 0-465-03770-4
1. Number concept. 2. Mathematics—Psychological aspects. 3. Mathematics—Philosophy. I. Núñez, Rafael E., 1960– . II. Title.
QA141.15.L37 2000
510—dc21 00-034216
CIP

FIRST EDITION

00 01 02 03 / 10 9 8 7 6 5 4 3 2 1

Preface

We are cognitive scientists—a linguist and a psychologist—each with a long-standing passion for the beautiful ideas of mathematics. As specialists within a field that studies the nature and structure of ideas, we realized that despite the remarkable advances in cognitive science and a long tradition in philosophy and history, there was still no discipline of *mathematical idea analysis* from a cognitive perspective—no cognitive science of mathematics.

With this book, we hope to launch such a discipline.

A discipline of this sort is needed for a simple reason: Mathematics is deep, fundamental, and essential to the human experience. As such, it is crying out to be understood.

It has not been.

Mathematics is seen as the epitome of precision, manifested in the use of symbols in calculation and in formal proofs. Symbols are, of course, just symbols, not ideas. The intellectual content of mathematics lies in its ideas, not in the symbols themselves. In short, the intellectual content of mathematics does not lie where the mathematical rigor can be most easily seen—namely, in the symbols. Rather, it lies in human ideas.

But mathematics by itself does not and cannot *empirically* study human ideas; human cognition is simply not its subject matter. It is up to cognitive science and the neurosciences to do what mathematics itself cannot do—namely, apply the science of mind to human mathematical ideas. That is the purpose of this book.

One might think that the nature of mathematical ideas is a simple and obvious matter, that such ideas are just what mathematicians have consciously taken them to be. From that perspective, the commonplace formal symbols do as good a job as any at characterizing the nature and structure of those ideas. If that were true, nothing more would need to be said.

But those of us who study the nature of concepts within cognitive science know, from research in that field, that the study of human ideas is not so simple. Human ideas are, to a large extent, grounded in sensory-motor experience. Abstract human ideas make use of precisely formulatable cognitive mechanisms such as conceptual metaphors that import modes of reasoning from sensory-motor experience. It is *always* an empirical question just what human ideas are like, mathematical or not.

The central question we ask is this: How can cognitive science bring systematic *scientific rigor* to the realm of human mathematical ideas, which lies outside the rigor of mathematics itself? Our job is to help make precise what mathematics itself cannot—the nature of mathematical ideas.

Rafael Núñez brings to this effort a background in mathematics education, the development of mathematical ideas in children, the study of mathematics in indigenous cultures around the world, and the investigation of the foundations of embodied cognition. George Lakoff is a major researcher in human conceptual systems, known for his research in natural-language semantics, his work on the embodiment of mind, and his discovery of the basic mechanisms of everyday metaphorical thought.

The general enterprise began in the early 1990s with the detailed analysis by one of Lakoff's students, Ming Ming Chiu (now a professor at the Chinese University in Hong Kong), of the basic system of metaphors used by children to comprehend and reason about arithmetic. In Switzerland, at about the same time, Núñez had begun an intellectual quest to answer these questions: How can human beings understand the idea of actual infinity—infinity conceptualized as a thing, not merely as an unending process? What is the concept of actual infinity in its mathematical manifestations—points at infinity, infinite sets, infinite decimals, infinite intersections, transfinite numbers, infinitesimals? He reasoned that since we do not encounter actual infinity directly in the world, since our conceptual systems are finite, and since we have no cognitive mechanisms to perceive infinity, there is a good possibility that metaphorical thought may be necessary for human beings to conceptualize infinity. If so, new results about the structure of metaphorical concepts might make it possible to precisely characterize the metaphors used in mathematical concepts of infinity. With a grant from the Swiss NSF, he came to Berkeley in 1993 to take up this idea with Lakoff.

We soon realized that such a question could not be answered in isolation. We would need to develop enough of the foundations of mathematical idea analysis so that the question could be asked and answered in a precise way. We would need to understand the cognitive structure not only of basic arithmetic but also of symbolic logic, the Boolean logic of classes, set theory, parts of algebra, and a fair amount of classical mathematics: analytic geometry, trigonometry, calculus, and complex numbers. That would be a task of many lifetimes. Because of other commitments, we had only a few years to work on the project—and only part-time.

So we adopted an alternative strategy. We asked, What would be the minimum background needed

- to answer Núñez's questions about infinity,
- to provide a serious beginning for a discipline of mathematical idea analysis, and
- to write a book that would engage the imaginations of the large number of people who share our passion for mathematics and want to understand what mathematical ideas are?

As a consequence, our discussion of arithmetic, set theory, logic, and algebra are just enough to set the stage for our subsequent discussions of infinity and classical mathematics. Just enough for that job, but not trivial. We seek, *from a cognitive perspective,* to provide answers to such questions as, Where do the laws of arithmetic come from? Why is there a unique empty class and why is it a subclass of all classes? Indeed, why is the empty class a class at all, if it cannot be a class *of* anything? And why, in formal logic, does every proposition follow from a contradiction? Why should anything at all follow from a contradiction?

From a cognitive perspective, these questions cannot be answered merely by giving definitions, axioms, and formal proofs. That just pushes the question one step further back: How are those definitions and axioms understood? To answer questions at this level requires an account of ideas and cognitive mechanisms. Formal definitions and axioms are *not* basic cognitive mechanisms; indeed, they themselves require an account in cognitive terms.

One might think that the best way to understand mathematical ideas would be simply to ask mathematicians what they are thinking. Indeed, many famous mathematicians, such as Descartes, Boole, Dedekind, Poincaré, Cantor, and Weyl, applied this method to themselves, introspecting about their own thoughts. Contemporary research on the mind shows that as valuable a method as this can be, it can at best tell a partial and not fully accurate story. Most of our thought and our systems of concepts are part of the cognitive unconscious (see Chapter 2). We human beings have no direct access to our deepest forms of understanding. The analytic techniques of cognitive science are necessary if we are to understand how we understand.

One of the great findings of cognitive science is that our ideas are shaped by our bodily experiences—not in any simpleminded one-to-one way but indirectly, through the grounding of our entire conceptual system in everyday life. The cognitive perspective forces us to ask, Is the system of mathematical ideas also grounded indirectly in bodily experiences? And if so, exactly how?

The answer to questions as deep as these requires an understanding of the cognitive superstructure of a whole nexus of mathematical ideas. This book is concerned with how such cognitive superstructures are built up, starting for the most part with the commonest of physical experiences.

To make our discussion of classical mathematics tractable while still showing its depth and richness, we have limited ourselves to one profound and central question: What does Euler's classic equation, $e^{\pi i} + 1 = 0$, mean? This equation links all the major branches of classical mathematics. It is proved in introductory calculus courses. The equation itself mentions only numbers and mathematical operations on them. What is lacking, from a cognitive perspective, is an analysis of the *ideas* implicit in the equation, the *ideas* that characterize those branches of classical mathematics, the way those *ideas* are linked in the equation, and why the truth of the equation follows from those *ideas*. To demonstrate the utility of mathematical idea analysis for classical mathematics, we set out to provide an initial idea analysis for that equation that would answer all these questions. This is done in the case-study chapters at the end of the book.

To show that mathematical idea analysis has some importance for the philosophy of mathematics, we decided to apply our techniques of analysis to a pivotal moment in the history of mathematics—the arithmetization of real numbers and calculus by Dedekind and Weierstrass in 1872. These dramatic developments set the stage for the age of mathematical rigor and the Foundations of Mathematics movement. We wanted to understand exactly what ideas were involved in those developments. We found the answer to be far from obvious: The modern notion of mathematical rigor and the Foundations of Mathematics movement both rest on a sizable collection of crucial conceptual metaphors.

In addition, we wanted to see if mathematical idea analysis made any difference at all in how mathematics is understood. We discovered that it did: What is called the *real-number line* is not a line as most people understand it. What is called the *continuum* is not continuous in the ordinary sense of the term. And what are called *space-filling curves* do not fill space as we normally conceive of it. These are not mathematical discoveries but discoveries about how mathematics is conceptualized—that is, discoveries in the cognitive science of mathematics.

Though we are not primarily concerned here with mathematics education, it *is* a secondary concern. Mathematical idea analysis, as we seek to develop it, asks what theorems *mean* and *why* they are true *on the basis of what they mean*. We believe it is important to reorient mathematics teaching more toward understanding mathematical ideas and understanding *why* theorems are true.

In addition, we see our job as helping to make mathematical ideas precise in an area that has previously been left to "intuition." Intuitions are not necessarily vague. A cognitive science of mathematics should study the precise nature of clear mathematical intuitions.

The Romance of Mathematics

In the course of our research, we ran up against a mythology that stood in the way of developing an adequate cognitive science of mathematics. It is a kind of "romance" of mathematics, a mythology that goes something like this.

- Mathematics is abstract and disembodied—yet it is real.
- Mathematics has an objective existence, providing structure to this universe and any possible universe, independent of and transcending the existence of human beings or any beings at all.
- Human mathematics is just a part of abstract, transcendent mathematics.
- Hence, mathematical proof allows us to discover transcendent truths of the universe.
- Mathematics is part of the physical universe and provides rational structure to it. There are Fibonacci series in flowers, logarithmic spirals in snails, fractals in mountain ranges, parabolas in home runs, and π in the spherical shape of stars and planets and bubbles.
- Mathematics even characterizes logic, and hence structures reason itself—any form of reason by any possible being.
- To learn mathematics is therefore to learn the language of nature, a mode of thought that would have to be shared by any highly intelligent beings anywhere in the universe.
- Because mathematics is disembodied and reason is a form of mathematical logic, reason itself is disembodied. Hence, machines can, in principle, think.

It is a beautiful romance—the stuff of movies like *2001*, *Contact*, and *Sphere*. It initially attracted us to mathematics.

But the more we have applied what we know about cognitive science to understand the cognitive structure of mathematics, the more it has become clear that this romance cannot be true. Human mathematics, the only kind of mathematics that human beings know, cannot be a subspecies of an abstract, transcendent mathematics. Instead, it appears that mathematics as we know it arises from the nature of our brains and our embodied experience. As a consequence, *every* part of the romance appears to be false, for reasons that we will be discussing.

Perhaps most surprising of all, we have discovered that a great many of the most fundamental mathematical ideas are inherently metaphorical in nature:

- The *number line*, where numbers are conceptualized metaphorically as points on a line.
- Boole's *algebra of classes*, where the formation of classes of objects is conceptualized metaphorically in terms of algebraic operations and elements: plus, times, zero, one, and so on.
- *Symbolic logic*, where reasoning is conceptualized metaphorically as mathematical calculation using symbols.
- *Trigonometric functions*, where angles are conceptualized metaphorically as numbers.
- The *complex plane*, where multiplication is conceptualized metaphorically in terms of rotation.

And as we shall see, Núñez was right about the centrality of conceptual metaphor to a full understanding of infinity in mathematics. There are two infinity concepts in mathematics—one literal and one metaphorical. The literal concept ("in-finity"—lack of an end) is called "potential infinity." It is simply a process that goes on without end, like counting without stopping, extending a line segment indefinitely, or creating polygons with more and more sides. No metaphorical ideas are needed in this case. Potential infinity is a useful notion in mathematics, but the main event is elsewhere. The idea of "actual infinity," where infinity becomes a *thing*—an infinite set, a point at infinity, a transfinite number, the sum of an infinite series—is what is really important. Actual infinity is fundamentally a metaphorical idea, just as Núñez had suspected. The surprise for us was that *all* forms of actual infinity—points at infinity, infinite intersections, transfinite numbers, and so on—appear to be special cases of just one Basic Metaphor of Infinity. This is anything but obvious and will be discussed at length in the course of the book.

As we have learned more and more about the nature of human mathematical cognition, the Romance of Mathematics has dissolved before our eyes. What has emerged in its place is an even more beautiful picture—a picture of what mathematics really is. One of our main tasks in this book is to sketch that picture for you.

None of what we have discovered is obvious. Moreover, it requires a prior understanding of a fair amount of basic cognitive semantics and of the overall cognitive structure of mathematics. That is why we have taken the trouble to write a book of this breadth and depth. We hope you enjoy reading it as much as we have enjoyed writing it.

Introduction: Why Cognitive Science Matters to Mathematics

MATHEMATICS AS WE KNOW IT HAS BEEN CREATED and used by human beings: mathematicians, physicists, computer scientists, and economists—all members of the species *Homo sapiens*. This may be an obvious fact, but it has an important consequence. Mathematics as we know it is limited and structured by the human brain and human mental capacities. The only mathematics we know or can know is a brain-and-mind-based mathematics.

As cognitive science and neuroscience have learned more about the human brain and mind, it has become clear that the brain is not a general-purpose device. The brain and body co-evolved so that the brain could make the body function optimally. Most of the brain is devoted to vision, motion, spatial understanding, interpersonal interaction, coordination, emotions, language, and everyday reasoning. Human concepts and human language are not random or arbitrary; they are highly structured and limited, because of the limits and structure of the brain, the body, and the world.

This observation immediately raises two questions:

1. Exactly what mechanisms of the human brain and mind allow human beings to formulate mathematical ideas and reason mathematically?
2. Is brain-and-mind-based mathematics all that mathematics *is*? Or is there, as Platonists have suggested, a disembodied mathematics transcending all bodies and minds and structuring the universe—this universe and every possible universe?

Question 1 asks where mathematical ideas come from and how mathematical ideas are to be analyzed from a cognitive perspective. Question 1 is a scientific question, a question to be answered by cognitive science, the interdisciplinary science of the mind. As an empirical question about the human mind and brain, it cannot be studied purely within mathematics. And as a question for empirical science, it cannot be answered by an a priori philosophy or by mathematics itself. It requires an understanding of human cognitive processes and the human brain. Cognitive science matters to mathematics because only cognitive science can answer this question.

Question 1 is what this book is mostly about. We will be asking how normal human cognitive mechanisms are employed in the creation and understanding of mathematical ideas. Accordingly, we will be developing techniques of mathematical idea analysis.

But it is Question 2 that is at the heart of the philosophy of mathematics. It is the question that most people want answered. Our answer is straightforward:

- Theorems that human beings prove are within a human mathematical conceptual system.
- All the mathematical knowledge that we have or can have is knowledge within human mathematics.
- There is no way to know whether theorems proved by human mathematicians have any objective truth, external to human beings or any other beings.

The basic form of the argument is this:

1. The question of the existence of a Platonic mathematics cannot be addressed *scientifically*. At best, it can only be a matter of faith, much like faith in a God. That is, Platonic mathematics, like God, cannot in itself be perceived or comprehended via the human body, brain, and mind. Science alone can neither prove nor disprove the existence of a Platonic mathematics, just as it cannot prove or disprove the existence of a God.
2. As with the conceptualization of God, all that is possible for human beings is an understanding of mathematics in terms of what the human brain and mind afford. The only conceptualization that we can have of mathematics is a human conceptualization. Therefore, mathematics as we know it and teach it can only be humanly created and humanly conceptualized mathematics.

3. What human mathematics is, is an empirical scientific question, not a mathematical or a priori philosophical question.
4. Therefore, it is only through cognitive science—the interdisciplinary study of mind, brain, and their relation—that we can answer the question: What is the nature of the only mathematics that human beings know or can know?
5. Therefore, if you view the nature of mathematics as a scientific question, then mathematics *is* mathematics as conceptualized by human beings using the brain's cognitive mechanisms.
6. However, you may view the nature of mathematics itself not as a scientific question but as a philosophical or religious one. The burden of scientific proof is on those who claim that an external Platonic mathematics does exist, and that theorems proved in human mathematics are objectively true, external to the existence of any beings or any conceptual systems, human or otherwise. At present there is no known way to carry out such a scientific proof in principle.

This book aspires to tell you what human mathematics, conceptualized via human brains and minds, is like. Given the present and foreseeable state of our scientific knowledge, human mathematics *is* mathematics. What human mathematical concepts are is what mathematical concepts are.

We hope that this will be of interest to you whatever your philosophical or religious beliefs about the existence of a transcendent mathematics.

There is an important part of this argument that needs further elucidation. What accounts for what the physicist Eugene Wigner has referred to as "the unreasonable effectiveness of mathematics in the natural sciences" (Wigner, 1960)? How can we make sense of the fact that scientists have been able to find or fashion forms of mathematics that accurately characterize many aspects of the physical world and even make correct predictions? It is sometimes assumed that the effectiveness of mathematics as a scientific tool shows that mathematics itself exists *in the structure of the physical universe*. This, of course, is not a scientific argument with any empirical scientific basis.

We will take this issue up in detail in Part V of the book. Our argument, in brief, will be that whatever "fit" there is between mathematics and the world occurs in the minds of scientists who have observed the world closely, learned the appropriate mathematics well (or invented it), and fit them together (often effectively) using their all-too-human minds and brains.

Finally, there is the issue of whether human mathematics is an instance of, or an approximation to, a transcendent Platonic mathematics. This position presupposes a nonscientific faith in the existence of Platonic mathematics. We will argue that even this position cannot be true. The argument rests on analyses we will give throughout this book to the effect that human mathematics makes fundamental use of conceptual metaphor in characterizing mathematical concepts. Conceptual metaphor is limited to the minds of living beings. Therefore, human mathematics (which is constituted in significant part by conceptual metaphor) cannot be a part of Platonic mathematics, which—if it existed—would be purely literal.

Our conclusions will be:

1. Human beings can have no access to a transcendent Platonic mathematics, if it exists. A belief in Platonic mathematics is therefore a matter of faith, much like religious faith. There can be no scientific evidence for or against the existence of a Platonic mathematics.
2. The only mathematics that human beings know or can know is, therefore, a *mind-based mathematics,* limited and structured by human brains and minds. The only scientific account of the nature of mathematics is therefore an account, via cognitive science, of human mind-based mathematics. Mathematical idea analysis provides such an account.
3. Mathematical idea analysis shows that human mind-based mathematics uses conceptual metaphors as part of the mathematics itself.
4. Therefore human mathematics cannot be a part of a transcendent Platonic mathematics, if such exists.

These arguments will have more weight when we have discussed in detail what human mathematical concepts are. That, as we shall see, depends upon what the human body, brain, and mind are like. A crucial point is the argument in (3)—that conceptual metaphor structures mathematics as human beings conceptualize it. Bear that in mind as you read our discussions of conceptual metaphors in mathematics.

Recent Discoveries about the Nature of Mind

In recent years, there have been revolutionary advances in cognitive science—advances that have an important bearing on our understanding of mathematics. Perhaps the most profound of these new insights are the following:

1. *The embodiment of mind*. The detailed nature of our bodies, our brains, and our everyday functioning in the world structures human concepts and human reason. This includes mathematical concepts and mathematical reason.
2. *The cognitive unconscious*. Most thought is unconscious—not repressed in the Freudian sense but simply inaccessible to direct conscious introspection. We cannot look directly at our conceptual systems and at our low-level thought processes. This includes most mathematical thought.
3. *Metaphorical thought*. For the most part, human beings conceptualize abstract concepts in concrete terms, using ideas and modes of reasoning grounded in the sensory-motor system. The mechanism by which the abstract is comprehended in terms of the concrete is called *conceptual metaphor*. Mathematical thought also makes use of conceptual metaphor, as when we conceptualize numbers as points on a line.

This book attempts to apply these insights to the realm of mathematical ideas. That is, we will be taking mathematics as a subject matter for cognitive science and asking how mathematics is created and conceptualized, especially how it is conceptualized metaphorically.

As will become clear, it is only with these recent advances in cognitive science that a deep and grounded mathematical idea analysis becomes possible. Insights of the sort we will be giving throughout this book were not even imaginable in the days of the old cognitive science of the disembodied mind, developed in the 1960s and early 1970s. In those days, thought was taken to be the manipulation of purely abstract symbols and all concepts were seen as literal—free of all biological constraints and of discoveries about the brain. Thought, then, was taken by many to be a form of symbolic logic. As we shall see in Chapter 6, symbolic logic is itself a mathematical enterprise that requires a cognitive analysis. For a discussion of the differences between the old cognitive science and the new, see *Philosophy in the Flesh* (Lakoff & Johnson, 1999) and *Reclaiming Cognition* (Núñez & Freeman, eds., 1999).

Mathematics is one of the most profound and beautiful endeavors of the imagination that human beings have ever engaged in. Yet many of its beauties and profundities have been inaccessible to nonmathematicians, because most of the cognitive structure of mathematics has gone undescribed. Up to now, even the basic ideas of college mathematics have appeared impenetrable, mysterious, and paradoxical to many well-educated people who have approached them. We believe that cognitive science can, in many cases, dispel the paradoxes and clear away the shrouds of mystery to reveal in full clarity the magnificence of those ideas. To do so, it must reveal how mathematics is grounded in embodied experience and how conceptual metaphors structure mathematical ideas.

Many of the confusions, enigmas, and seeming paradoxes of mathematics arise because conceptual metaphors that are part of mathematics are not recognized as metaphors but are taken as literal. When the full metaphorical character of mathematical concepts is revealed, such confusions and apparent paradoxes disappear.

But the conceptual metaphors themselves do not disappear. They cannot be analyzed away. Metaphors are an essential part of mathematical thought, not just auxiliary mechanisms used for visualization or ease of understanding. Consider the metaphor that Numbers Are Points on a Line. Numbers don't have to be conceptualized as points on a line; there are conceptions of number that are not geometric. But the number line is one of the most central concepts in all of mathematics. Analytic geometry would not exist without it, nor would trigonometry.

Or take the metaphor that Numbers Are Sets, which was central to the Foundations movement of early-twentieth-century mathematics. We don't have to conceptualize numbers as sets. Arithmetic existed for over two millennia without this metaphor—that is, without zero conceptualized as being the empty set, 1 as the set containing the empty set, 2 as the set containing 0 and 1, and so on. But if we do use this metaphor, then forms of reasoning about sets can also apply to numbers. It is only by virtue of this metaphor that the classical Foundations of Mathematics program can exist.

Conceptual metaphor is a cognitive mechanism for allowing us to reason about one kind of thing as if it were another. This means that metaphor is not simply a linguistic phenomenon, a mere figure of speech. Rather, it is a cognitive mechanism that belongs to the realm of thought. As we will see later in the book, "conceptual metaphor" has a technical meaning: It is a *grounded, inference-preserving cross-domain mapping*—a neural mechanism that allows us to use the inferential structure of one conceptual domain (say, geometry) to reason about another (say, arithmetic). Such conceptual metaphors allow us to apply what we know about one branch of mathematics in order to reason about another branch.

Conceptual metaphor makes mathematics enormously rich. But it also brings confusion and apparent paradox if the metaphors are not made clear or are taken to be literal truth. Is zero a point on a line? Or is it the empty set? Or both? Or is it just a number and neither a point nor a set? There is no one answer. Each answer constitutes a choice of metaphor, and each choice of metaphor provides different inferences and determines a different subject matter.

Mathematics, as we shall see, layers metaphor upon metaphor. When a single mathematical idea incorporates a dozen or so metaphors, it is the job of the cognitive scientist to tease them apart so as to reveal their underlying cognitive structure.

This is a task of inherent scientific interest. But it also can have an important application in the teaching of mathematics. We believe that revealing the cognitive structure of mathematics makes mathematics much more accessible and comprehensible. Because the metaphors are based on common experiences, the mathematical ideas that use them can be understood for the most part in everyday terms.

The cognitive science of mathematics asks questions that mathematics does not, and cannot, ask about itself. How do we understand such basic concepts as infinity, zero, lines, points, and sets using our everyday conceptual apparatus? How are we to make sense of mathematical ideas that, to the novice, are paradoxical—ideas like space-filling curves, infinitesimal numbers, the point at infinity, and non-well-founded sets (i.e., sets that "contain themselves" as members)?

Consider, for example, one of the deepest equations in all of mathematics, the Euler equation, $e^{\pi i} + 1 = 0$, e being the infinite decimal 2.718281828459045. . . , a far-from-obvious number that is the base for natural logarithms. This equation is regularly taught in elementary college courses. But what exactly does it mean? We are usually told that an exponential of the form q^n is just the number q multiplied by itself n times; that is, $q \cdot q \cdot \ldots \cdot q$. This makes perfect sense for 2^5, which would be $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$, which multiplies out to 32. But this definition of an exponential makes no sense for $e^{\pi i}$. There are at least three mysteries here.

1. What does it mean to multiply an infinite decimal like e by itself? If you think of multiplication as an algorithmic operation, where do you start? Usually you start the process of multiplication with the last digit on the right, but there is no last digit in an infinite decimal.
2. What does it mean to multiply any number by itself π times? π is another infinite nonrepeating decimal. What could "π times" for performing an operation mean?
3. And even worse, what does it mean to multiply a number by itself an imaginary ($\sqrt{-1}$) number of times?

And yet we are told that the answer is –1. The typical proof is of no help here. It proves that $e^{\pi i} + 1 = 0$ is true, but it does not tell you what $e^{\pi i}$ means! In the course of this book, we will.

In this book, unlike most other books about mathematics, we will be concerned not just with *what* is true but with what mathematical ideas *mean*, how they can be understood, and *why* they are true. We will also be concerned with the nature of mathematical truth from the perspective of a mind-based mathematics.

One of our main concerns will be the concept of infinity in its various manifestations: infinite sets, transfinite numbers, infinite series, the point at infinity, infinitesimals, and objects created by taking values of sequences "at infinity," such as space-filling curves. We will show that there is a single Basic Metaphor of Infinity that all of these are special cases of. This metaphor originates outside mathematics, but it appears to be the basis of our understanding of infinity in virtually all mathematical domains. When we understand the Basic Metaphor of Infinity, many classic mysteries disappear and the apparently incomprehensible becomes relatively easy to understand.

The results of our inquiry are, for the most part, not mathematical results but results in the cognitive science of mathematics. They are results about the human conceptual system that makes mathematical ideas possible and in which mathematics makes sense. But to a large extent they are not results reflecting the conscious thoughts of mathematicians; rather, they describe the *unconscious* conceptual system used by people who do mathematics. The results of our inquiry should not change mathematics in any way, but they may radically change the way mathematics is understood and what mathematical results are taken to mean.

Some of our findings may be startling to many readers. Here are some examples:

- Symbolic logic is not the basis of all rationality, and it is not absolutely true. It is a beautiful metaphorical system, which has some rather bizarre metaphors. It is useful for certain purposes but quite inadequate for characterizing anything like the full range of the mechanisms of human reason.
- The real numbers do not "fill" the number line. There is a mathematical subject matter, the hyperreal numbers, in which the real numbers are rather sparse on the line.
- The modern definition of *continuity* for functions, as well as the so-called *continuum*, do not use the idea of continuity as it is normally understood.

- So-called *space-filling curves* do not fill space.
- There is no absolute yes-or-no answer to whether 0.99999. . . . = 1. It will depend on the conceptual system one chooses. There is a mathematical subject matter in which 0.99999. . . . = 1, and another in which 0.99999. . . . ≠ 1.

These are not new mathematical findings but new ways of understanding well-known results. They are findings in the cognitive science of mathematics—results about the conceptual structure of mathematics and about the role of the mind in creating mathematical subject matters.

Though our research does not affect mathematical results in themselves, it does have a bearing on the understanding of mathematical results and on the claims made by many mathematicians. Our research also matters for the philosophy of mathematics. *Mind-based mathematics*, as we describe it in this book, is not consistent with any of the existing philosophies of mathematics: Platonism, intuitionism, and formalism. Nor is it consistent with recent postmodernist accounts of mathematics as a purely social construction. Based on our findings, we will be suggesting a very different approach to the philosophy of mathematics. We believe that the philosophy of mathematics should be consistent with scientific findings about the only mathematics that human beings know or can know. We will argue in Part V that the *theory of embodied mathematics*—the body of results we present in this book—determines an empirically based philosophy of mathematics, one that is coherent with the "embodied realism" discussed in Lakoff and Johnson (1999) and with "ecological naturalism" as a foundation for embodiment (Núñez, 1995, 1997).

Mathematics as we know it is human mathematics, a product of the human mind. Where does mathematics come from? It comes from us! We create it, but it is not arbitrary—not a mere historically contingent social construction. What makes mathematics nonarbitrary is that it uses the basic conceptual mechanisms of the embodied human mind as it has evolved in the real world. Mathematics is a product of the neural capacities of our brains, the nature of our bodies, our evolution, our environment, and our long social and cultural history.

By the time you finish this book, our reasons for saying this should be clear.

The Structure of the Book

Part I is introductory. We begin in Chapter 1 with the brain's innate arithmetic—the ability to subitize (i.e., to instantly determine how many objects are in a very small collection) and do very basic addition and subtraction. We move on in Chapter 2 to some of the basic results in cognitive science on which the remainder of the book rests. We then take up basic metaphors grounding our understanding of arithmetic (Chapter 3) and the question of where the laws of arithmetic come from (Chapter 4).

In Part II, we turn to the grounding and conceptualization of sets, logic, and forms of abstract algebra such as groups (Chapters 5, 6, and 7).

Part III deals with the concept of infinity—as fundamental a concept as there is in sophisticated mathematics. The question we ask is how finite human cognitive capacities and everyday conceptual mechanisms can give rise to the full range of mathematical notions of infinity: points at infinity, infinite sets, mathematical induction, infinite decimals, limits, transfinite numbers, infinitesimals, and so on. We argue that the concept of actual infinity is metaphorical in nature and that there is a single conceptual metaphor—the Basic Metaphor of Infinity (Chapter 8)—underlying most if not all infinite notions in mathematics (Chapters 8 through 11). We will then, in Part IV, point out the implications of this type of analysis for an understanding of the continuum (Chapter 12) and for continuity and the real numbers (Chapters 13 and 14).

At this point in the book, we take a break from our line of argumentation to address a commonly noticed apparent contradiction, which we name the Length Paradox. We call this interlude *le trou normand,* after the course in a rich French meal where a sorbet with calvados is served to refresh the palate.

We now have enough results for Part V, a discussion of an overall *theory of embodied mathematics* (Chapter 15) and a new philosophy of mathematics (Chapter 16).

To demonstrate the real power of the approach, we end the book with Part VI, a detailed case study of the equation that brings together the ideas at the heart of classical mathematics: $e^{\pi i} + 1 = 0$. To show exactly what this equation means, we have to look at the cognitive structure—especially the conceptual metaphors—underlying analytic geometry and trigonometry (Case Study 1), exponentials and logarithms (Case Study 2), imaginary numbers (Case Study 3), and the cognitive mechanisms combining them (Case Study 4).

We chose to place this case study at the end for three reasons. First, it is a detailed illustration of how the cognitive mechanisms described in the book can shed light on the structure of classical mathematics. We have placed it after our discussion of the philosophy of mathematics to provide an example to the reader of how a change in the nature of what mathematics is can lead to a new understanding of familiar mathematical results.

Second, it is in the case study that mathematical idea analysis comes to the fore. Though we will be analyzing mathematical ideas from a cognitive perspective throughout the book, the study of Euler's equation demonstrates the power of the analysis of ideas in mathematics, by showing how a single equation can bring an enormously rich range of ideas together—even though the equation itself contains nothing but numbers: e, π, $\sqrt{-1}$, 1, and 0. We will be asking throughout how mere *numbers* can express *ideas*. It is in the case study that the power of the answer to this question becomes clear.

Finally, there is an educational motive. We believe that classical mathematics can best be taught with a cognitive perspective. We believe that it is important to teach mathematical ideas and to explain why mathematical truths follow from those ideas. This case study is intended to illustrate to teachers of mathematics how this can be done.

We see our book as an early step in the development of a cognitive science of mathematics—a discipline that studies the cognitive mechanisms used in the human creation and conceptualization of mathematics. We hope you will find this discipline stimulating, challenging, and worthwhile.