Spin & Precession

- Nuclei act like a spinning sphere of matter with an embedded equatorial charge. (Nuclei with odd atomic weight or odd proton numbers)
- Moving charge creates magnetic field.
- Current loop from spinning charge (right-hand rule).
- N.B.: Classically, this would cause EM radiation, spin-down.
- Stern-Gerlach experiment: Pass silver atoms through strong magnetic field → split into just 2 beams.

Microscopic picture:
- All vectors of same length, random directions.
- Slight excess of "up" (3 ppm).
- Precessing vectors are "bunched" at any one moment around circle.

Macroscopic picture:
- Bulk magnetization: \( M^0_z \) in y component:
- Bulk magnetization precesses.

Precession:
- Distinguish precession (slow) from spin (fast).
- Treat classically, like spinning top.
- \( \omega_p = \frac{\gamma B_0}{2\pi} \) (Larmor frequency).
- \( \gamma \) = Gyromagnetic ratio. (eg. 63 MHz/T).

Bulk equilibrium magnetization:
- \( M^0_z = \frac{\gamma^2 h^2 B_0 N_I}{4 KT_s} \)
- Two non-constants:
- \( Y \) = Gyromagnetic ratio.
- \( h \) = Planck's constant.

N.B.: Compared to top & gravity:
- Frictionless spin, doesn't slow.
- Signed gravity usually changes precession dir.
- Can stick under floor.
- Neighboring bumping causes decay (~T2).

B1 = Strong oscillating field.
The Bloch equation describes the time-dependent behavior of the magnetization vector \( \vec{M} \) in the presence of an applied magnetic field and an applied field. The equation is given by:

\[
\frac{d\vec{M}}{dt} = \vec{M} \times \vec{Y}_B - \frac{M_x i + M_y j}{T_2} - \frac{(M_z - M_z^0) k}{T_1}
\]

where \( M_x, M_y, M_z \) are the components of the magnetization in the laboratory frame, \( T_2 \) is the transverse relaxation time, and \( T_1 \) is the longitudinal relaxation time.

In the Larmor-rotating coordinate system, the magnetization vector experiences precession with angular frequency \( \omega = \gamma B_0 \), where \( \gamma \) is the gyromagnetic ratio and \( B_0 \) is the static magnetic field. A phase shift \( \phi \) is introduced due to the precession.

- **Longitudinal and Transverse Relaxations**

  - **Longitudinal Relaxation**
    
    \[
    \frac{dM_z(t)}{dt} = -\frac{M_z(t) - M_z^0}{T_1}
    \]

  - **Transverse Relaxation**
    
    \[
    \frac{dM_x y(t)}{dt} = -\frac{M_x y(t)}{T_2}
    \]

- **Solution to Equations**

  - In the laboratory frame, the solution for \( M_z(t) \) and \( M_x y(t) \) re-grows from 0, re-growing from 0 after pulse-decaying.

  - \( M_z(t) = M_z^0 e^{-t/T_1} \)
  - \( M_x y(t) = M_x y(0) e^{-t/T_2} \)

- **Initial Conditions**

  - At time \( t = 0 \), the magnetization is given by \( M_x(0), M_y(0), M_z(0) \).

- **Decay and Growth**

  - After a pulse, the magnetization decays.

  - \( M_z(t) = 63\% M_z^0 \)
  - \( M_y(t) = 37\% M_y(0) \)

- **Equilibrium Value**

  - In the presence of only the applied field \( B_0 \), the equilibrium value can be ignored during short excitation, as the magnetic vector stays the same length as it spirals down (vs. relaxation, where it shrinks then grows).
Vector addition and multiplication:

- Adding vectors is easy:
  \[ \vec{c} = \vec{a} + \vec{b} = [a_x + b_x, a_y + b_y] \]
  - Just add components (Vector)
  - Applies to complex numbers
  - Generalizes to any D
  \[ \|\vec{c}\| = \sqrt{(a_x + b_x)^2 + (a_y + b_y)^2} \]

- Multiple ways to multiply vectors: here are 3

**Dot Product**

\[ \vec{c} = \vec{a} \cdot \vec{b} = [b_x, b_y, b_z] \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = a_x b_x + a_y b_y + a_z b_z \]
- Scalar
- Generalizes to any D
- Length of C
- C = \|\vec{a}\| \|\vec{b}\| \cos \theta
- Zero if \( \vec{a}, \vec{b} \) orthogonal

**Cross Product**

\[ \vec{c} = \vec{a} \times \vec{b} = \begin{bmatrix} 0 & -b_z & b_y \\ -b_x & 0 & b_z \\ b_y & -b_z & 0 \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = [a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_x b_y - a_y b_x] \]
- Vector
- Geometric Algebra: bivector plane area
- Right hand rule: curl fingers from \( \vec{a} \) to \( \vec{b} \): thumb is \( \vec{c} \)
- Unique orthogonal specific to 3D
- \( \|\vec{c}\| = \|\vec{a}\| \|\vec{b}\| \sin \theta \)
- Max if orthogonal

**Complex Multiply**

\[ \vec{c} = \vec{a} \cdot \overline{\vec{b}} = \begin{bmatrix} b_x & -b_y \\ b_y & b_x \end{bmatrix} \begin{bmatrix} a_x \\ a_y \end{bmatrix} = [a_x b_x - a_y b_y, a_x b_y + a_y b_x] \]
- Angle add
- Magnitudes multiply
- Specific to 2D
- \( \|\vec{c}\| = \|\vec{a}\| \|\overline{\vec{b}}\| \)
- Like real num

Cf. not affected by angle between
- Contrast (PD, T1, T2, T2*) depends on magnetization not getting back to equilibrium, and then differences in how far away each tissue type is at measurement time.

- Simple saturation/recovery w/ no echo

- Initial conditions:
  \[ M_z \text{ before first pulse} = M_z^0 \]
  \[ M_z = 0 \text{ immed. after first pulse (i.e., 90° pulse)} \]

- From Bloch eq, \( M_z \) just before second pulse:

  \[ M_z^n(0^-) = M_z^0(1 - e^{-TR/T1}) + M_z^n(0^+)e^{-TR/T1} \]

- Given:
  
  - (1) 90° pulse
  - (2) no \( M_{xy} \) left

  \[ \text{pure tip: } M_{xy} = M_z \]

- Tip existing mag

  \[ M_z'(0^-) = M_x'y'(0^+) = M_z^0(1 - e^{-TR/T1}) \]

- That is, the not-completely-regrown longitudinal magnetization, which depends on T1, but which we cannot record, is completely converted to recordable transverse magnetization.

\[ I(r) = C g(r) (1 - e^{-TR/T1(r)}) \]

Spectral density \( g(r) \): p.d. density, underlying equilibrium, \( M_z^0 \).
\[ \Phi(t) = \int_{\text{obj}} \vec{B}(\vec{r}) \cdot \hat{m}(\vec{r}, t) \, d\vec{r} \]

magnetic flux through coil → scalar
(integral magnetic field perpendicular to area)

\[ V(t) = -\frac{\partial \Phi(t)}{\partial t} = -\frac{\partial}{\partial t} \int_{\text{obj}} \vec{B}(\vec{r}) \cdot \hat{m}(\vec{r}, t) \, d\vec{r} \]

Fermat's law of Induction

- ignore change in z-comp. \( \hat{m} \) because so slow → i.e., we only see \( M_{xy} \), not \( M_z \)
- substitute \( \hat{m}(t) \) with lab frame \( \hat{m}_{xy}(t) = \hat{M}_{xy}(t) e^{i\omega t} e^{-i\theta} \)
- simplify:
  1) ignore decay (assume this \( t=0 \))
  2) assume phase-sensitive detection

\[ \vec{S}(t) = \int_{\text{obj}} B_{xy}(\vec{r}) M_{xy}(\vec{r}, 0) e^{-i \frac{\omega(t)}{2}} \, d\vec{r} \]

Laboratory frame Bloch solutions:
\( M_L = \text{same} \)
\( M_T = M_{xy}(0) e^{i\theta} e^{-i\phi} \)

\[ \vec{S}(t) = \int_{\text{obj}} M_{xy}(\vec{r}, 0) e^{-i \frac{\omega(t)}{2}} \, d\vec{r} \]

Spatially-dependent resonant freq in rotating frame → i.e., after subtraction of \( \omega_0 = \gamma B_0 \)

i.e., at a single time point, RF signal is vector sum across object of local transverse magnetization vectors

Phase angle in rotating frame
\[ \omega_t = \text{radians} \times \text{sec} = \text{radians} \left( \frac{\text{radians}}{\text{sec}} \right) \]

\[ \gamma = \text{sec} \]

Getting difference converts lab → rotating frame
**PHASE-SENSITIVE DETECTION**

- Method for moving very high-frequency Larmor oscillations down to tractable frequency range

\[ V(t) = 123 \text{ MHz} \]

**Low-Pass Filter**

\[ S(t) \]

\[ \sim 50 \text{ kHz} \]

digitization

Demodulated signal \( \propto \) RF coil signal \( \cdot \) Reference (transmitter)

\[ \propto \sin[(\omega_0 + \delta \omega)t] \cdot \sin[\omega_0 t] \]

\[ \propto \frac{1}{2} \left( \cos \delta \omega t - \cos(2\omega_0 + \delta \omega)t \right) \]

This signal is digitized

**Chirp - Time Domain**

- Chirp input
- Center
- Demodulated
- Signal
- Filter
- \( \Rightarrow \) phase!

- Two signals are made from a single receiving RF coil
- A quadrature coil can be treated the same way (OK to combine after adding \( \frac{1}{2} \) phase, then PSD)
- Quadrature coil has better S/N since noise in each part is uncorrelated (\( \frac{1}{2} \) better)

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ECHOES - spin echo

90° - τ - 180°, T2/T2* & echo

Rotating coords

Just after 90° x' pulse f_{e0} + f_{hi} have same phase

Relaxation + phase dispersion of f_{e0} + f_{hi}
(both from B>B_0)

Just after 180° y' pulse
(y' pulse like x' pulse but RF has +90° phase)

Echo caused by re-phasing of f_{e0} + f_{hi}
(w/ decay due to T2)

- Remember brief RF just tips vectors while retaining length
  relaxation includes tips and shrinks (M_t) and grows (M, echo)

- 180° x' pulse works, too, but echo will have +π phase (left side in figs above)

- Echo generated even if second pulse not 180° (see next)

FID decay (and echo growth/decay)
described by T2*
from inhomogeneity

Reduction in height of echo
compared to initial
described by T2,
echo fixes the 'star'
**Gradient Echoes** - $T_2^*$, GE chains

- Initial negative gradient dephases spins
- After $t = T$ of positive gradient, spins rephase
- Does not correct for $T_2^*$ inhomogeneities so echo amplitude is 
  $$A_E = e^{-t/T_2^*}$$
- The initial "FID" is not "free" since it is being actively dephased by gradient, so FID decay

- Key difference between spin-echo (SE) and gradient echo (GE) is that $B_0$ inhomogeneities not cancelled
  $\Rightarrow$ hence, echoes are $T_2^*$-weighted, not $T_2$-weighted $\Rightarrow$ more susceptible to inhomogeneities

- Echo trains possible w/ gradient echo (CPMG-like)

- The faster the gradients are switched, the more echoes you get

- EPI hardware $\Rightarrow$ 64 echoes
**Complex Algebra**

- **Real/Imaginary**
  - Add: \((r_1, i_1) + (r_2, i_2) = (r_1 + r_2, i_1 + i_2)\)
  - Multiply: \((r_1, i_1) \times (r_2, i_2) = (r_1 r_2 - i_1 i_2, r_1 i_2 + i_1 r_2)\)

- **Angle/Phase**
  - Add: \((A_1, \phi_1) + (A_2, \phi_2) = (A_1 \cos \phi_1 + A_2 \cos \phi_2, A_1 \sin \phi_1 + A_2 \sin \phi_2)\)
  - Multiply: \((A_1, \phi_1) \times (A_2, \phi_2) = (A_1 A_2, \phi_1 + \phi_2)\)
  - Divide: \((A_1, \phi_1) \div (A_2, \phi_2) = (A_1 / A_2, \phi_1 - \phi_2)\)

- Complex to Real power: \((A, \phi)^n = (A^n, n \phi)\)

- \(e^{i\phi} = \cos \phi + i \sin \phi\)
  - Expand as series
  - Recognize \(\cos, \sin\) series
  - The real "e" to "purely imaginary" power

- \(e^{i\phi^n} = (\cos \phi + i \sin \phi)^n = \cos n \phi + i \sin n \phi\)

**Fourier Transform**

- **Fourier Transform**
  - \(H(f) = \int h(t) e^{-2\pi i ft} \, dt\)
  - \(\mathcal{F}\{h(t)\} = H(f)\)

- **Convolution Theorem**
  - \(F\{g(x) \ast h(x)\} = G(k) \ast H(k)\)

- **Fourier Transform of convolution**
  - \(f(x) = g(x) \ast h(x) = \int g(z) \cdot h(x-z) \, dz\)
  - \(\mathcal{F}\{g(x) \ast h(x)\} = \mathcal{F}\{g(x)\} \cdot \mathcal{F}\{h(x)\}\)

- **Convolutions**
  - \(f(x) \ast g(x) = \int f(x-z) g(z) \, dz\)

- **Convolution Theorem**
  - \(\mathcal{F}\{g(x) \ast h(x)\} = \mathcal{F}\{g(x)\} \cdot \mathcal{F}\{h(x)\}\)

- **Phase**
  - For arbitrary amplitude, multiply \(A e^{i\phi}\)

- **Phase**
  - Phase is integral of freq. variable \(\phi = \int \omega \, dt\)

- **Fourier transform of two functions multiplied by each other equals the convolution of the Fourier transform of each function**
**Fourier Transform (1)**

For one $f$:

$$H(f) = \int_{-\infty}^{\infty} h(t) \cdot e^{-j \frac{2\pi ft}{t_0}} dt$$

**Sum across $t$ (complex add)**

- **How to calculate $H(f)$ for one $f$ ($f=3$):**
  (real signal: only need 2 correlations)

1. **Real signal**
   - $t \rightarrow$ real
   - $t \rightarrow \sin(3t)$
   - $t \rightarrow \cos(3t)$

2. **Imaginary signal (zero)**
   - $t \rightarrow \sin(3t)$
   - $t \rightarrow \cos(3t)$

**Complex multiply**

- integrate/sum these multiplies across all $t$

**Cartesian (r,t)**
- $(\text{amplitude frequency domain})$

**Polar coords (A,|$\phi$|)**
- $(\text{phase frequency domain})$

**Like correlating with sin and cos (at each freq) so we get phase (at each freq.)**
**Sampling**

- Must consider effects of sampling in the time domain:
  - limited number of samples in k-space
  - limited range of frequencies sampled ($k_{min}$ to $k_{max}$)
  - limited rate of sampling ($\Delta k$)

- N.B. aliasing less familiar when result of limited frequency domain sampling than limited space or time domain sampling.