Foundations of Neuroimaging

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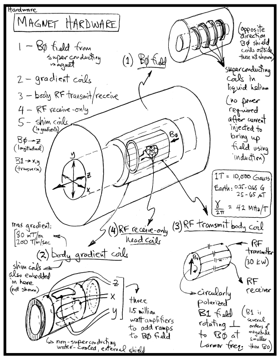
(124 pages)



MAGNET HARDWARE

1 – B0 field from superconducting magnet

2 – gradient coils

3 – body RF transmit/receive

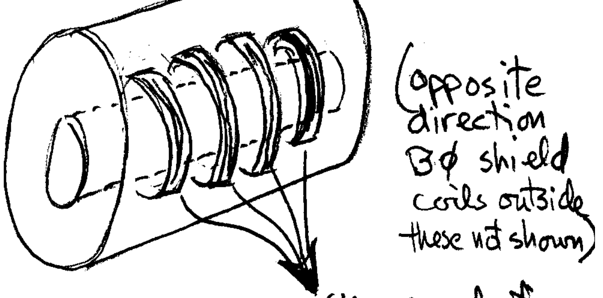
4 – RF receive-only

5 – shim coils (in gradients)

B0 → z (longitudinal)

B1 → x,y (transverse)

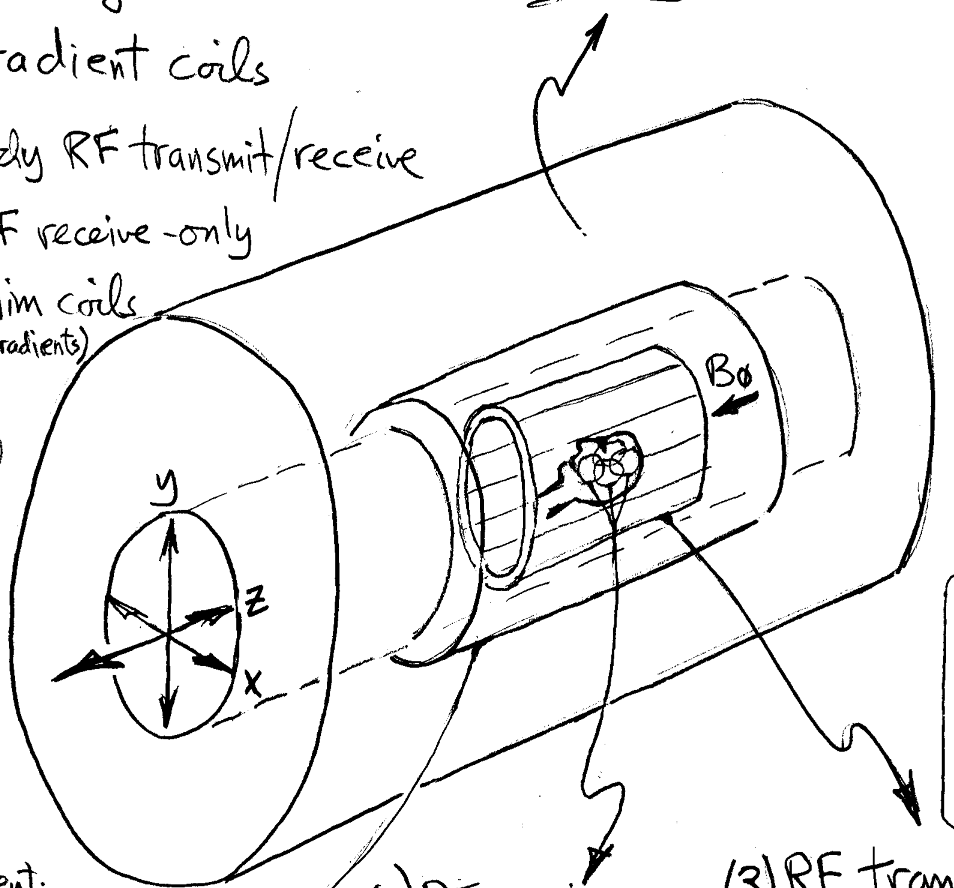
(1) B0 field



(opposite direction B0 shield coils outside these not shown)

superconducting coils in liquid helium

(no power required after current injected to bring up field using induction)



1T = 10,000 Gauss

Earth: 0.25 - 0.65 G
25 - 65 μT

$\frac{\gamma}{2\pi}$ = 42 MHz/T

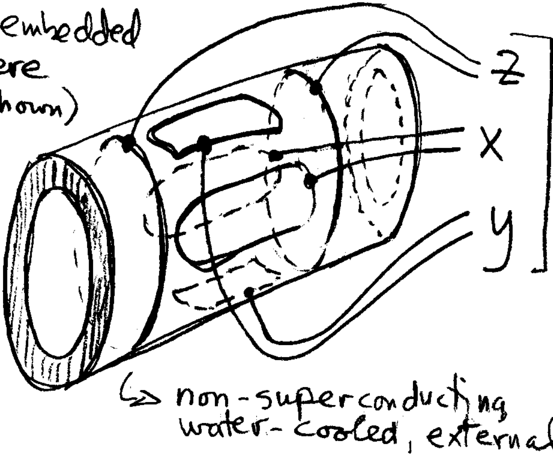
(3) RF transmit body coil

(4) RF receive-only head coils

max gradient:
[80 mT/m
200 T/m/sec

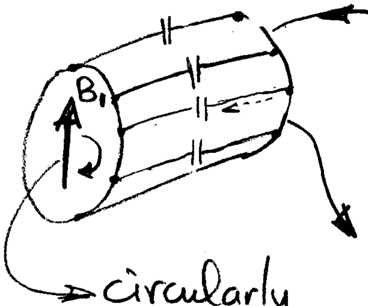
(2) body gradient coils

shim coils also embedded in here (not shown)



three 1.5 million watt amplifiers to add ramps to B0 field

non-superconducting water-cooled, external shield



RF transmitter (30 kW)

RF receiver

circularly polarized B1 field rotating ⊥ to B0 at Larmor freq.

(B1 is several orders of magnitude smaller than B0)

SPIN & PRECESSION

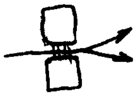
N.B.: compared to top & gravity magnetic relaxation is:

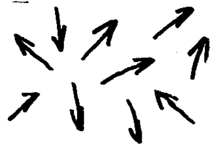
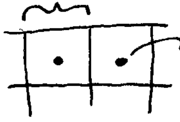
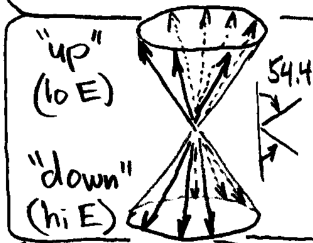
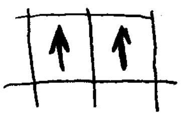
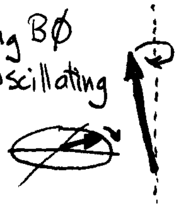
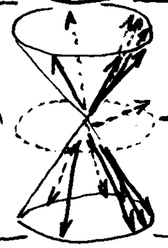
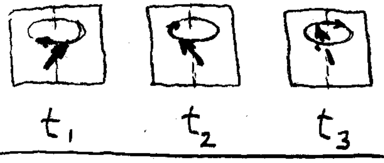
- frictionless spin, doesn't slow
- signed gravity
  - can change precession dir
  - can stick under floor
- neighbor bumping causes decay (T2)

- nuclei act like a spinning sphere of matter with an embedded equatorial charge (nuclei w/ odd atomic weight or odd proton numbers)
- moving charge creates magnetic field

classical picture

- current loop from spinning charge (right-hand rule)
- N.B.: classically this would cause EM radiation, spindown

- Stern-Gerlach experiment  pass silver atoms thru strong mag. field → split into just 2 beams

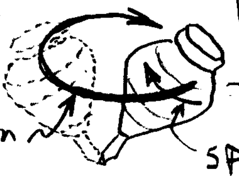
	Microscopic picture	Macroscopic picture
no strong magnetic field B0 = 0	 all vectors same length, random directions ⇨	e.g. 1mm  bulk magnetization = 0
strong magnetic field, B0 = 	 "up" (lo E) "down" (hi E) 54.4° Slight excess of "up" (3 ppm) *already precessing x,y components still random ⇨	 bulk magnetization = M_z^0
strong B0 plus oscillating B1 	(still up/down bias)  precessing vectors are "bunched" at any one moment around circle ⇨	bulk magnetization precesses  t_1 t_2 t_3

$$\omega_{top} = \frac{\overbrace{r \cdot m}^{\text{"}\gamma\text{"}} \cdot \overset{\text{"}B0\text{"}}{g}}{\underbrace{\underset{\text{moment inertia}}{I} \cdot \omega_{spin}}_{L = \text{ang. mom.}}}$$

Precession

- distinguish precession (slow) from spin (fast)
- treat classically, like spinning top

$$\underbrace{2\pi f_0}_{\text{Larmor freq (e.g. 63 MHz)}} = \underbrace{\omega_0}_{\text{Same in radians/sec}} = \underbrace{\gamma}_{\text{gyromagnetic ratio}} \underbrace{B_0}_{\text{static field (e.g. 1.5T)}}$$



Left-hand rule:
thumb = B0
fingers = precess.

like top: precession faster w/ more gravity

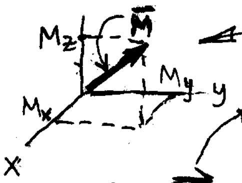
- bulk equilibrium magnetization (parallel to B_0)

$$M_z^0 = |\vec{M}| = \frac{\gamma^2 h^2 \overbrace{\boxed{B_0}\boxed{N_s}}^{\text{two non-constants}}}{4 K T_s}$$

where $I = \pm 1/2$ $\frac{I(I+1)}{3}$ → 4

γ = gyromagnetic ratio
h = Planck's const.
B_0 → i.e., M_z^0 proportional to B0 strength
N_s → i.e., M_z^0 proportional to number spins
K = Boltzmann const.
T_s = abs. temperature sample

BLOCH EQUATION



vector sum of microscopic magnetization vectors

rewritten by components

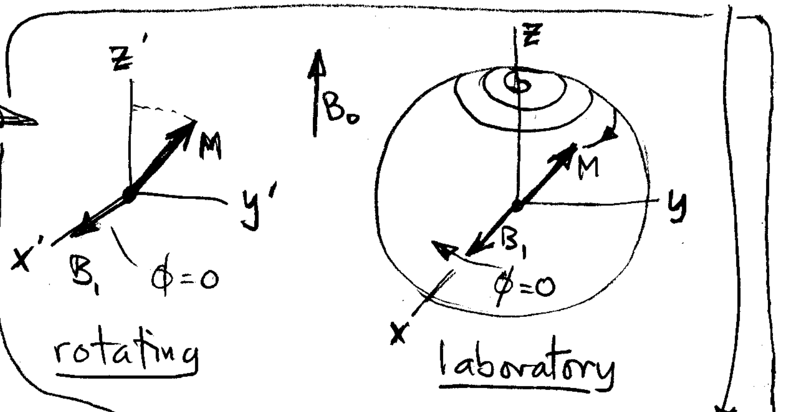
$$\vec{M} = M_x\vec{i} + M_y\vec{j} + M_z\vec{k}$$

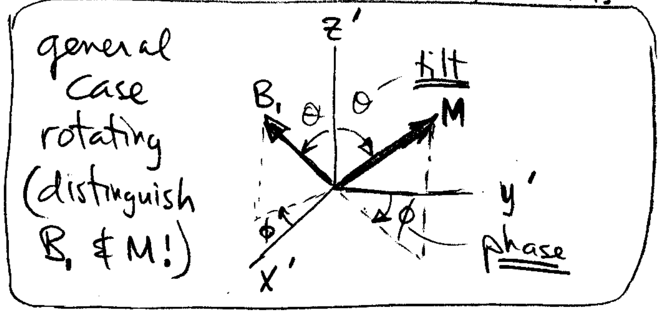
- time-dependent behavior of **M** in the presence of an applied magnetic field (excitation & relaxation)

$$\underset{\text{(laboratory frame)}}{}\ \underbrace{\frac{d\vec{M}}{dt}}_{\text{change in mag vector}} = \overbrace{\underbrace{\vec{M}}_{\text{existing mag vector sum}} \times \underbrace{\gamma\vec{B}}_{\text{applied field (typically rotating)}}}^{\boxed{\text{precession: } B=B_0 \quad \text{excitation: } B=B_0+B_1}} \underbrace{- \overset{\text{"transverse" "spin-spin"}}{\frac{M_x\vec{i} + M_y\vec{j}}{T_2}} - \overset{\text{"longitudinal" "spin-lattice"}}{\frac{(M_z - \underset{\text{equilibrium } \vec{M} \text{ value in only the } B_0 \text{ field}}{M_z^0})\underset{\text{unit vector z-dir.}}{\vec{k}}}{T_1}}}$$

these can be ignored during short excitation, so magnetic vector stays same length as it spirals down (vs. relaxation, where it <u>shrinks</u>, then <u>grows</u>)

- in the Larmor-rotating coordinate system, a tilt w/o a phase shift for a standard B_1 excitation is rotation around **x**-axis





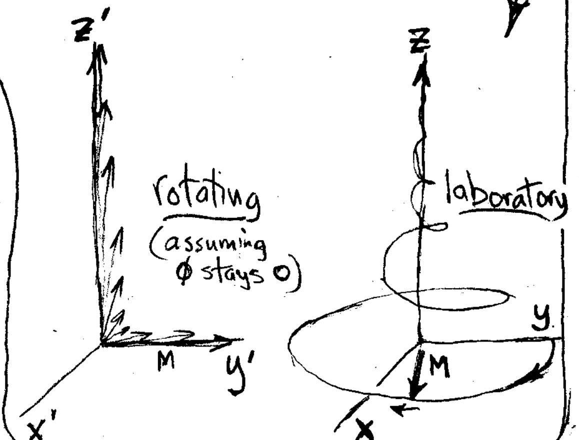
- longitudinal and transverse relaxations

drop i, j, k

$$\underset{\text{(rotating frame)}}{\boxed{M_L}} \quad \frac{dM_{z'}(t)}{dt} = -\frac{M_{z'}(t) - M_z^0}{T_1}$$

$$\boxed{M_T} \quad \frac{dM_{x'y'}(t)}{dt} = -\frac{M_{x'y'}(t)}{T_2}$$

from Bloch equation after dropping applied field term



- solution to equations above (given init M_L, M_T): <u>time-dependent free precession eq's</u>

lab frame: same!

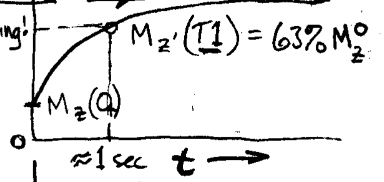
$$\underset{\text{(rotating frame)}}{\boxed{M_L}} \quad M_{z'}(t) = \overbrace{M_z^0\left(1 - e^{-t/T1}\right)}^{\text{re-growing from 0}} + \overbrace{M_{z'}(0_+)e^{-t/T1}}^{\text{leftover after pulse — decaying!}}$$

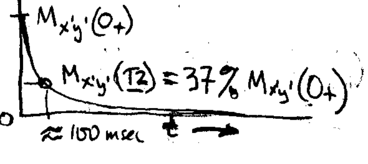
$$[\text{equiv}] = M_z^0 + \left[M_{z'}(0_+) - M_z^0\right]e^{-t/T1}$$

$$\boxed{M_T} \quad M_{x'y'}(t) = \underbrace{M_{x'y'}(0_+)}_{\text{initial condition}}e^{-t/T2}$$

lab frame: times $e^{-i\omega_0 t}$

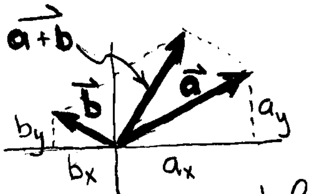
M_L, M_T at time immed. after pulse





Bloch-1c

VECTOR ADD, MULTIPLY



- adding vectors is easy

$\vec{c} = \vec{a} + \vec{b} = [a_x+b_x,\ a_y+b_y]$ (vector)

[- just add components
- applies to complex numbers

- generalizes to any D

get length

$\|\vec{c}\| = \sqrt{(a_x+b_x)^2+(a_y+b_y)^2}$

- multiple ways to multiply vectors: here are 3

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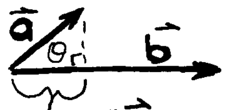
dot product

(= inner product)
(= "scaled projection onto")

$c = \vec{a}\cdot\vec{b} = [b_x\ b_y\ b_z]\begin{bmatrix}a_x\\a_y\\a_z\end{bmatrix} = a_xb_x + a_yb_y + a_zb_z$ (scalar)



$p = \|\vec{a}\|\cos\theta$

$c = p\|\vec{b}\|$ → $c = p$ if $\|\vec{b}\| = 1$

- generalizes to any D

N.B. equals: $\sqrt{\vec{a}\cdot\vec{a}}$

$\sqrt{a_x^2+a_y^2+a_z^2}$

length of $\vec{a}$

$c = \|\vec{a}\|\,\|\vec{b}\|\cos\theta$

↳ zero if $\vec{a},\vec{b}$ orthogonal

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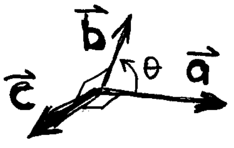
cross product

(= outer product)
(can be generalized: see "geometric algebra")

$\vec{c} = \vec{a}\times\vec{b} = \begin{bmatrix}0 & b_z & -b_y\\ -b_z & 0 & b_x\\ b_y & -b_x & 0\end{bmatrix}\begin{bmatrix}a_x\\a_y\\a_z\end{bmatrix} = [a_yb_z - a_zb_y,\ a_zb_x - a_xb_z,\ a_xb_y - a_yb_x]$ (vector)

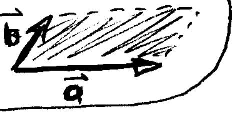
↳ skew-symmetric: $A^T = -A$



right hand rule: curl fingers from $\vec{a}$ to $\vec{b}$: thumb is $\vec{c}$

- unique orthogonal specific to 3D

geometric algebra: [bivector, plane, area



$\|\vec{c}\| = \|\vec{a}\|\,\|\vec{b}\|\sin\theta$

↳ max if orthogonal

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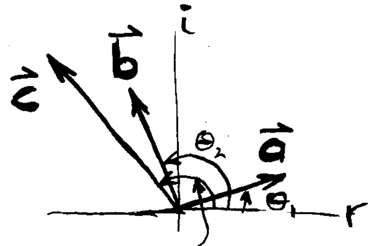
complex multiply

(see also quaternions, geometric algebra generalization)

$\vec{c} = \vec{a}\cdot\vec{b} = \begin{bmatrix}b_x & -b_y\\ b_y & b_x\end{bmatrix}\begin{bmatrix}a_x\\a_y\end{bmatrix} = [a_xb_x - a_yb_y,\ a_xb_y + a_yb_x]$ (vector)



sum of angles: $\theta_1 + \theta_2$

- angles add
- magnitudes multiply

- specific to 2D

$\|\vec{c}\| = \|\vec{a}\|\,\|\vec{b}\|$

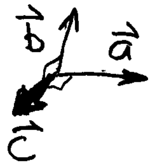
↳ like real nums

cf. not affected by angle between

EFFECTS OF $\vec{M}$, $\vec{B}$, and Θ ON PRECESSION FREQ.

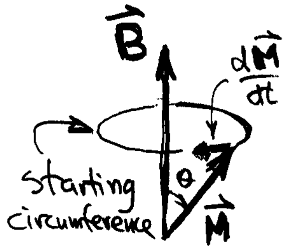
Bloch 1st term

$$\frac{dM}{dt} = \vec{M} \times \gamma\vec{B}$$



cross prod. properties review

$$\left\| \frac{d\vec{M}}{dt} \right\| = \|\vec{M}\| \cdot \|\gamma\vec{B}\| \cdot \sin\Theta$$

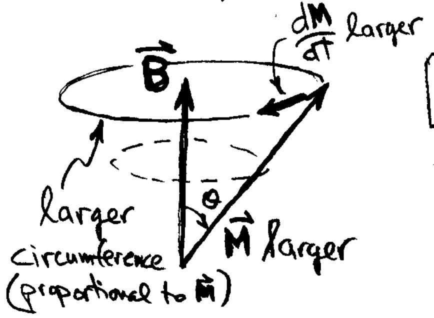


Starting condition

↳ now see effects of changing $\|\vec{M}\|, \|\vec{B}\|, \theta$

N.B. length

vectors define this

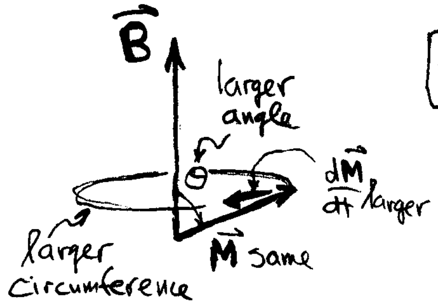


Change $\vec{M}$ length

↳ $\frac{d\vec{M}}{dt}$ proportionally larger, so cancels effect of larger $\vec{M}$

↳ same precession freq. as starting cond.

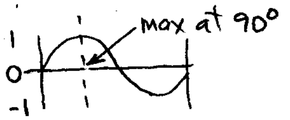
b/c $\vec{M}$ on both sides eq.



change Θ between $\vec{M}$ and $\vec{B}$

↳ $\frac{d\vec{M}}{dt}$ goes up (then down) as $\sin\Theta$

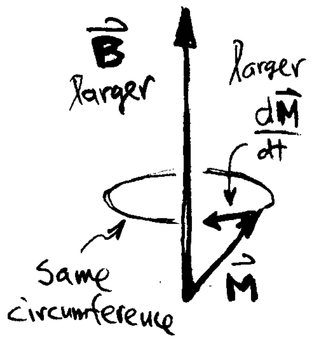
↳ cross prod.



↳ but circumference also goes up as $\sin\Theta$, cancelling again

opposite = $\sin\Theta$ = radius so circumference proportional to $\sin\Theta$

↳ same precession freq.



Change $\vec{B}$ length

↳ $\frac{d\vec{M}}{dt}$ goes up, proportional to $\vec{B}$

↳ but circumference is same at starting cond.

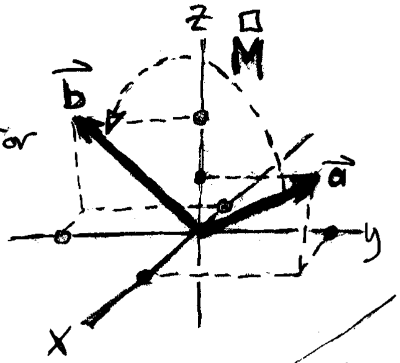
↳ increased precession freq. ($\omega = \gamma B_0$)

SIMPLE MATRIX OPERATIONS

Basic idea

- a matrix [rotates / scales] a vector

$$\vec{b} = \overset{\square}{M}\vec{a}$$



3D example

to get a pure rotation (no scaling) these 3 col. vectors must be orthogonal

$$\begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

$$b_x = M_{11}a_x + M_{12}a_y + M_{13}a_z$$
$$b_y = M_{21}a_x + M_{22}a_y + M_{23}a_z$$
$$b_z = M_{31}a_x + M_{32}a_y + M_{33}a_z$$

Add translate (after rotate/scale)

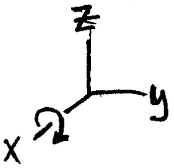
- commonly used "hack" for aligning vols
- a 4D matrix [rotates/scales then translates] a 3D vector (4th D = 1)
- N.B.: have to keep track of order!!
  - → rotate/scale then trans ≠ trans, then rot/scale
  - → change rot component: untranslate, rot, retranslate

translate?

$$\begin{bmatrix} b_x \\ b_y \\ b_z \\ 1 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} & \mathbf{3\times3} & \\ & \text{rot/scale} & \end{bmatrix} & \begin{matrix} dx \\ dy \\ dz \end{matrix} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \\ 1 \end{bmatrix}$$

3 special cases (3D): rotate around each major axis without changing length (scale = 1.0)

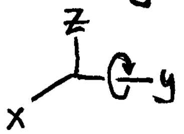
- rotate around x-axis:



$$\overset{\square}{\mathbf{R}}_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}$$

e.g., 90° flip

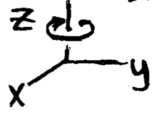
- rotate around y-axis:



$$\overset{\square}{\mathbf{R}}_y(\alpha) = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

e.g., 180° flip to avoid add 180° phase after 90° flip on x'

- rotate around z-axis:

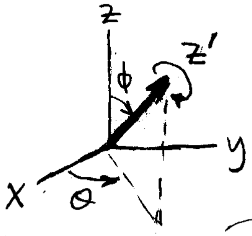


$$\overset{\square}{\mathbf{R}}_z(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

e.g., precession with B0 along z'

General case

- rotate around general z'-axis:



$$\mathbf{R}_{z'}(\alpha) = \mathbf{R}_z(-\theta)\,\mathbf{R}_y(-\phi)\,\mathbf{R}_z(\alpha)\,\mathbf{R}_y(\phi)\,\mathbf{R}_z(\theta)$$

[unrot. axis to z
z-rot.
re-rot.]

(quaternions are more efficent)

SOLUTIONS TO SIMPLE DIFFERENTIAL EQ.

diff. eq: $dM_{x'y'}(t) = -\frac{M_{x'y'}(t)}{T2}$

solution: $M_{x'y'}(t) = M_{x'y'}(0_+)\cdot e^{-t/T2}$

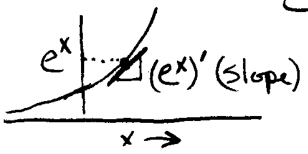
?? How this works, and where $M_{x'y'}(0_+)$ comes from...

Goal: 1) find eq. whose derivative satifies diff. eq.

2) also find sol'n (one of many) that passes thru init condition

→ since our diff.eq. is: derivative of funct. = const. same funct

→ try exponential, since derivative $(e^x) = e^x$ ☺



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deriv.

diff eq. $M'(t) = \frac{-1}{T2}\cdot \boxed{M(t)}$

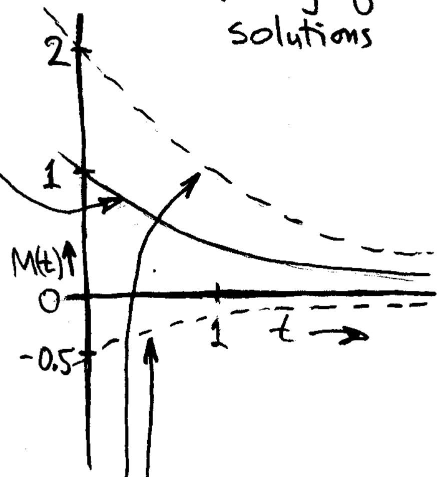
N.B. this function is the "unknown" like the x in x+1=3

one sol'n $M(t) = e^{-t/T2} = e^{\frac{-1}{T2}\cdot t}$

chain rule deriv of exponent

take deriv. to check $M'(t) = \frac{-1}{T2}\cdot \underbrace{e^{-t/T2}}_{M(t)}$

family of solutions



diff eq. $M'(t) = \frac{-1}{T2}\cdot M(t)$

another sol'n $M(t) = const\cdot e^{-t/T2}$

So: any const is OK!

chain rule

take deriv. to check $M'(t) = \frac{-1}{T2}\cdot const\cdot e^{-t/T2}$

for ex.:
const = 2
const = -0.5

---

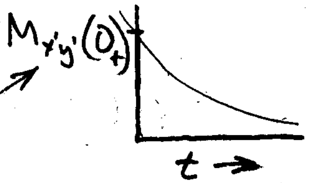
const is "initial condition"

information added to sol'n (not from diff eq!)

const = $M_{x'y'}(0_+)$

magnetization immed. after RF (B1) ends

$M't) = M_{x'y'}(0_+)\cdot e^{-t/T2}$



Bloch-1c2

VERIFY SOLUTION TO T1 REGROWTH

- slightly more complex T1 soln compared to T2 soln

T2 soln verify (from prev.)		T1 solution verify
$\frac{dM}{dt} = \frac{M_{xy}}{T2}$	original diff eq.	$\frac{dM}{dt} = \frac{-(M_z - M_z^0)}{T1}$
$M'(t) = \frac{-1}{T2} \cdot M(t)$	make unknown funct M(t) more visible	$M'(t) = \frac{-1}{T1}\left(M(t) - M_z^0\right)$
$M(t) = \boxed{M_{xy}(0_+)e^{-t/T2}}$	proposed solution	$M(t) = M_z^0\left(1 - e^{-t/T1}\right) + \overset{\text{init cond.}}{M_z(0_+)}e^{-t/T1}$ $= \boxed{\underset{\text{const}}{M_z^0} - \underset{\text{chain rule as before}}{M_z^0 e^{-t/T1}} + \underset{\text{chain rule}}{M_z(0_+)e^{-t/T1}}}$
$M'(t) = \boxed{\frac{-1}{T2} M_{xy}(0_+)e^{-t/T2}}$	test by take deriv.	$M'(t) = 0 + \frac{1}{T1}M_z^0 e^{-t/T1} - \frac{1}{T1}M_z(0_+)e^{-t/T1}$ $= \boxed{\frac{-1}{T1}\left(-M_z^0 e^{-t/T1} + M_z(0_+)e^{-t/T1}\right)}$

$\Uparrow$

- derivative in original T1 eq. says M(t) minus M_z^0

$$M'(t) = \frac{-1}{T1}\left(M(t) - M_z^0 \right)$$

solution $\rightarrow \boxed{M_z^0 - M_z^0 e^{-t/T1} + M_z(0_+)e^{-t/T1}}$

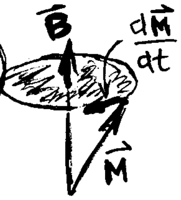
- which equals our re-calculated derivative:

$$M'(t) = \frac{-1}{T1}\left(-M_z^0 e^{-t/T1} + M_z(0_+)e^{-t/T1} \right)$$

Bloch-1d

BLOCH EQ. – MATRIX VERSION

Just Precession



$$\frac{d\vec{M}}{dt} = \vec{M} \times \gamma \vec{B}_0$$

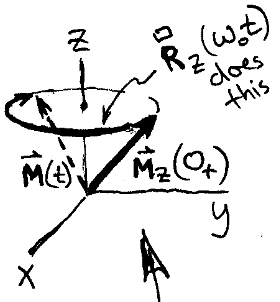
$$\vec{a}\times\vec{b} = \begin{bmatrix} 0 & b_z & -b_y \\ -b_z & 0 & b_x \\ b_y & -b_x & 0 \end{bmatrix}\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

assume only z component of B_0 present

Differential Eq.:

$= \omega_0$

$$\frac{d\vec{M}}{dt} = \begin{bmatrix} \frac{dM_x}{dt} \\ \frac{dM_y}{dt} \\ \frac{dM_z}{dt} \end{bmatrix} = \begin{bmatrix} 0 & \gamma B_0 & 0 \\ -\gamma B_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix}$$



$\vec{R}_z(\omega_0 t)$ does this

initial condition

Solution:

$$\vec{M}(t) = \begin{bmatrix} M_x(t) \\ M_y(t) \\ M_z(t) \end{bmatrix} = \begin{bmatrix} \cos\omega_0 t & \sin\omega_0 t & 0 \\ -\sin\omega_0 t & \cos\omega_0 t & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} M_x(0_+) \\ M_y(0_+) \\ M_z(0_+) \end{bmatrix} = \vec{R}_z(\omega_0 t)\,\vec{M}(0_+)$$

Include Relaxation

$$\frac{d\vec{M}}{dt} = \vec{M} \times \gamma\vec{B}_0 - \frac{M_x\vec{i} + M_y\vec{j}}{T2} - \frac{(M_z - M_z^0)\vec{k}}{T1}$$

Differential Eq.:

$$\frac{d\vec{M}}{dt} = \begin{bmatrix} -1/T2 & \gamma B_0 & 0 \\ -\gamma B_0 & -1/T2 & 0 \\ 0 & 0 & -1/T1 \end{bmatrix}\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ M_z^0/T1 \end{bmatrix}$$

initial condition

(as above)

Solution:

$$\vec{M}(t) = \begin{bmatrix} e^{-t/T2} & 0 & 0 \\ 0 & e^{-t/T2} & 0 \\ 0 & 0 & e^{-t/T1} \end{bmatrix}\left[\begin{bmatrix} \cos\omega_0 t & \sin\omega_0 t & 0 \\ -\sin\omega_0 t & \cos\omega_0 t & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} M_x(0_+) \\ M_y(0_+) \\ M_z(0_+) \end{bmatrix}\right] + \begin{bmatrix} 0 \\ 0 \\ M_z(0_+)(1 - e^{-t/T1}) \end{bmatrix}$$

Bloch-1f

EXCITATION IN THE ROTATING FRAME

Remember:

$$\vec{a} \times \vec{b} = \begin{bmatrix} 0 & b_z & -b_y \\ -b_z & 0 & b_x \\ b_y & -b_x & 0 \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

- original Bloch eq. in laboratory frame

$$\frac{d\vec{M}_{lab}}{dt} = \vec{M} \times \gamma \vec{B} \quad \begin{cases} B\phi \\ B1 \\ \text{gradients} \end{cases}$$

- add on-resonance B1 to Bϕ

$$\vec{B} = B1(t)\left(\cos \omega_o t\, \vec{i} - \sin \omega_o t\, \vec{j}\right) + B\phi\, \vec{k}$$

lab frame < on-resonance, no gradient

*basic excite

↳ matrix version

$$\frac{d\vec{M}}{dt} = \begin{bmatrix} \frac{dM_x'}{dt} \\ \frac{dM_y'}{dt} \\ \frac{dM_z'}{dt} \end{bmatrix} = \begin{bmatrix} 0 & \overset{\gamma B\phi}{\omega_o} & \overset{\gamma B1(t)}{\boxed{\omega_1(t)}} \sin \omega_o t \\ -\omega_o & 0 & \omega_1(t) \cos \omega_o t \\ \underbrace{-\omega_1(t) \sin \omega_o t}_{\text{B1 y comp.}} & \underbrace{-\omega_1(t) \cos \omega_o t}_{\text{B1 x comp.}} & 0 \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix}$$

b/c CW.

- substitution to convert to the rotating frame

$$\begin{aligned} \vec{M} &= \overset{\square}{R}_z(\omega_o t) \overset{\text{matrix multiply}}{\cdot} \vec{M}_{rot} \\ \vec{B} &= \overset{\square}{R}_z(\omega_o t) \cdot \vec{B}_{rot} \end{aligned}$$

"subtract off" rotating from both $\vec{M}$ and $\vec{B}$

- after substitution any off-resonance appears as residual Bϕ (B_z) (see off-res notes page)

$$\frac{d\vec{M}_{rot}}{dt} = \vec{M}_{rot} \times \gamma \vec{B}_{eff} \quad \begin{cases} \text{off-res (appears as } B\phi) \\ B1 \\ \text{gradients} \end{cases}$$

rotating frame < on-resonance, no gradient

*basic excite, B1x-only

↳ removes ω_o, cos/sin

$$\frac{d\vec{M}_{rot}}{dt} = \begin{bmatrix} \overset{\text{ignore T2, T1}}{0} & 0 & 0 \\ 0 & 0 & \omega_1(t) \\ \underset{\text{B1y = 0}}{0} & -\omega_1(t) & 0 \end{bmatrix} \begin{bmatrix} M_x' \\ M_y' \\ M_z' \end{bmatrix}$$

rotating frame < off-resonance, incl gradients

*general, B1x-only

$$\frac{d\vec{M}_{rot}}{dt} = \begin{bmatrix} 0 & \overset{\text{off-res.}}{\omega_o - \omega} + \overset{\text{gradient}}{\omega(z)} & 0 \\ -(\omega_o - \omega + \omega(z)) & 0 & \omega_1(t) \\ 0 & -\omega_1(t) & 0 \end{bmatrix} \begin{bmatrix} M_x' \\ M_y' \\ M_z' \end{bmatrix}$$

gradient: $\omega(z) = \gamma G_z z$

off-res: appears as residual B_z, lifting $\vec{B1}$ vect. out of x-y plane

↳ this means $\vec{M}$ vect. update will contain component that rotates $\vec{M}$ around z-axis (in rotating coords ≡ phase)

rotating frame < on-resonance, incl gradient, small tip < $M_z \cong M_z^0$, $\frac{dM_z}{dt} \cong 0$ → zeros this line in matrix

*small tip approx.

$$\frac{d\vec{M}_{rot}}{dt} = \begin{bmatrix} 0 & \omega(z) & 0 \\ -\omega(z) & 0 & \omega_1(t) \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} M_x' \\ M_y' \\ \underset{\text{small tip}}{M_z^0} \end{bmatrix}$$

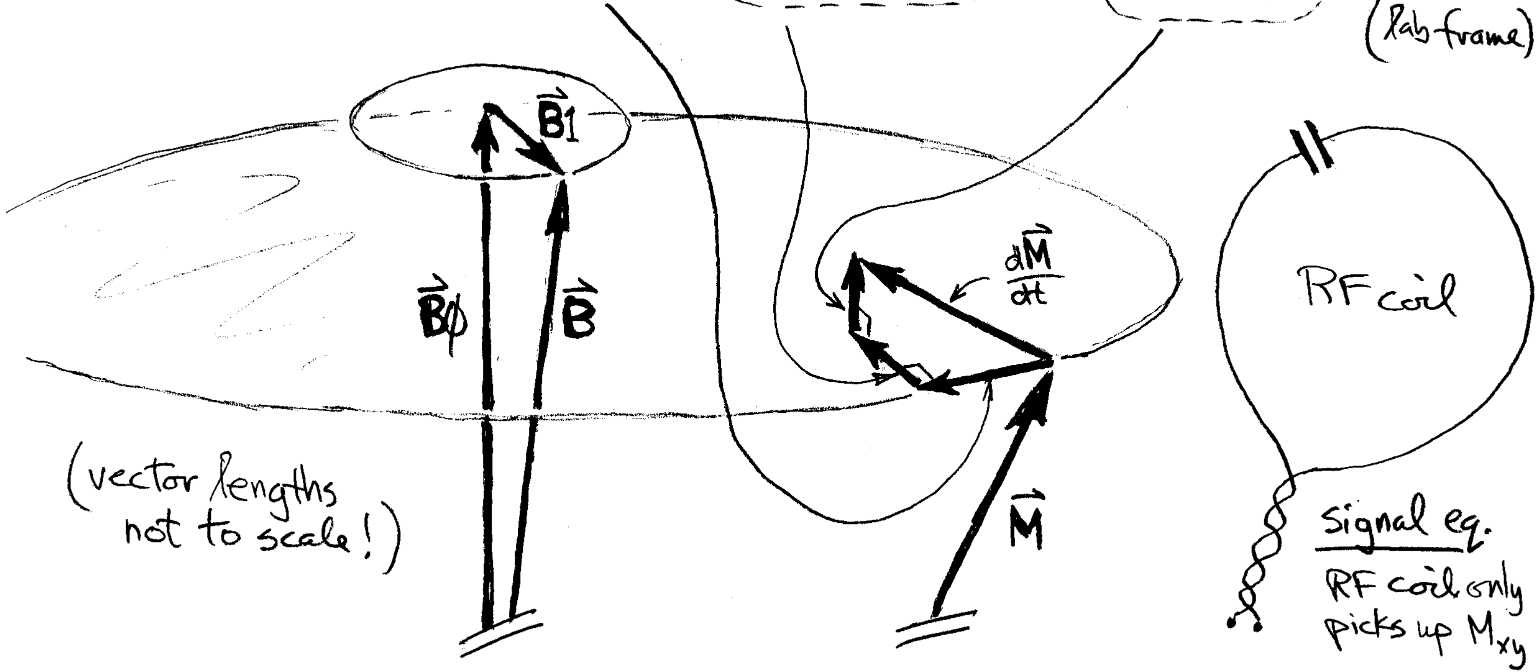
↳ small tip ⇒ easier to solve!

Bloch-1g

BLOCH EQ. SUMMARY

$$\frac{d\vec{M}}{dt} = \vec{M}\times\gamma\vec{B} - \frac{M_x\vec{i}+M_y\vec{j}}{T2} - \frac{(M_z - M_z^{0})\vec{k}}{T1}$$

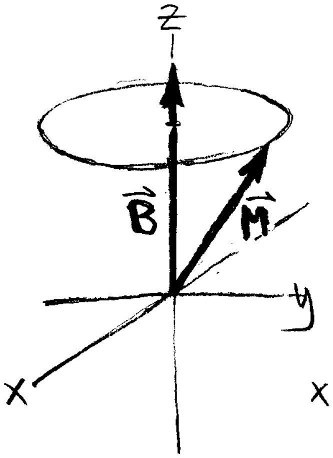
(lab frame)



- full lab frame picture is complex:
  - 3 component of $\frac{d\vec{M}}{dt}$ update vector
  - Larmor freq. component 7-9 orders magnitude larger than T2, T1 decay
  - $\vec{B1}$ is also rapidly wiggling
- conceptual simplification in 4 stages:

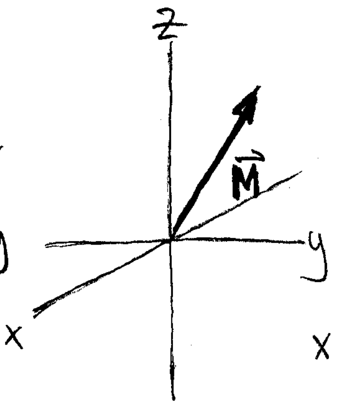
1) lab frame

- just precession



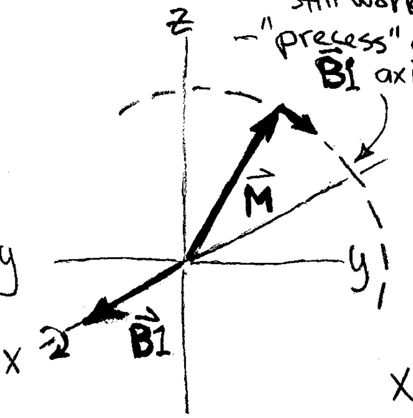
2) rotating frame

- $\vec{M}$ stopped
- that is, $B\phi = 0$

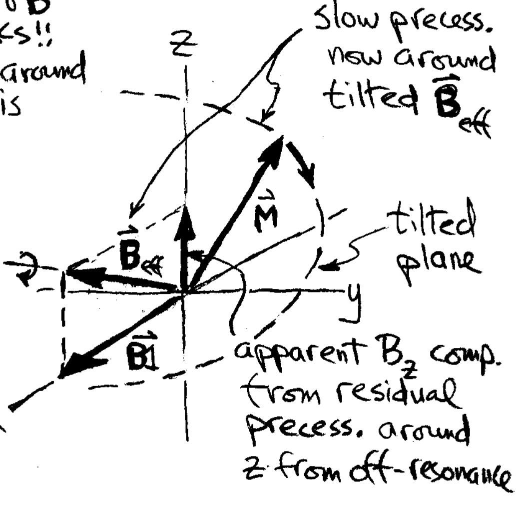


3) add $\vec{B1}$

- $\vec{B1}$ also stopped!
- but $\vec{M}\times\gamma\vec{B}$ still works!!
- "precess" around $\vec{B1}$ axis



4) off-resonance

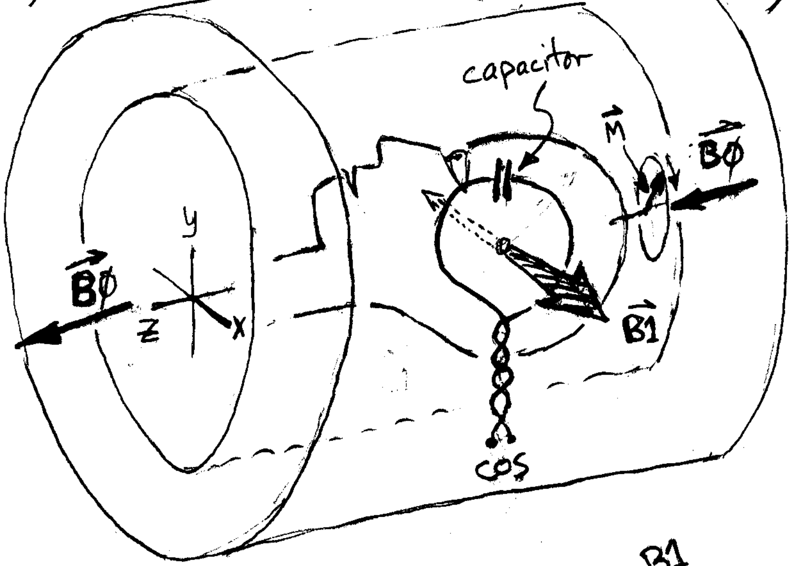


Bloch-2

RF FIELD POLARIZATION

B1 generated/recorded by RF coil

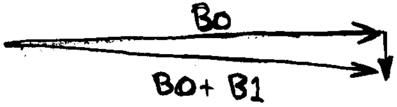
- polarization (change of direction) of magnetic field (vs. electric field)



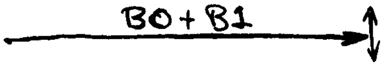
- linearly polarized field

$$\overrightarrow{B1}^{lin}(t) = \underbrace{B1}_{\text{magn. strength}} \cdot \underbrace{\cos \overset{rad/sec}{\omega} \overset{\to sec}{t}}_{\{-1,1\}} \cdot \underbrace{\vec{x}}_{1}$$

- N.B.: $\vec{B1}$ adds to much larger $\vec{B0}$



one instant in time

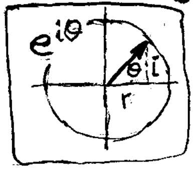


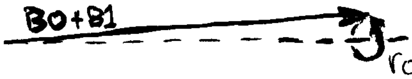
wiggles L/R across time

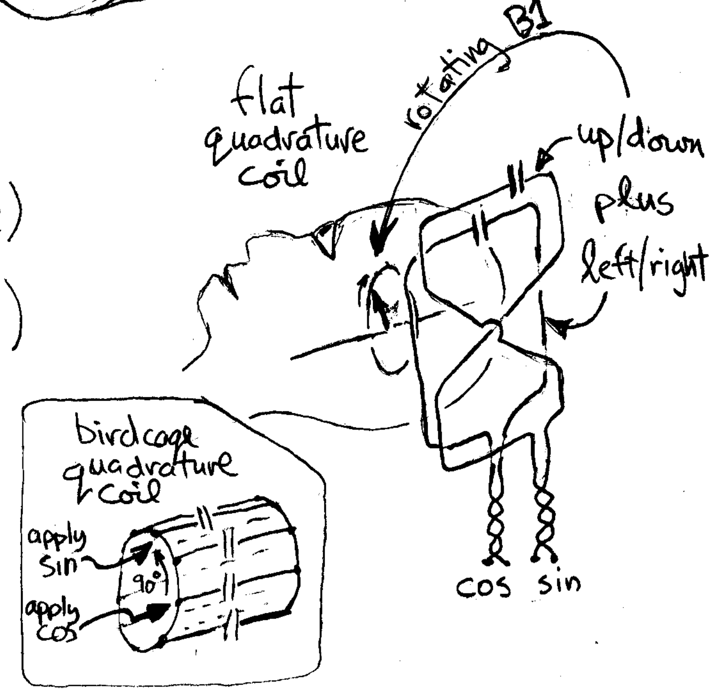
- circularly polarized field (quadrature)

$$\overrightarrow{B1}^{circ}(t) = B1 \left(\cos \omega t \, \vec{x} - \sin \omega t \, \vec{y}\right)$$

$$= B1 \cdot e^{-i\overset{angle}{\omega t}}$$

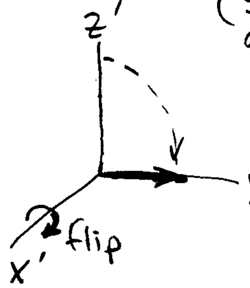






- in the rotating coordinate syst, flipping around x-axis vs. y-axis is just difference in phase of RF stim

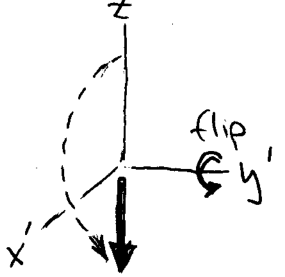
typical 90° flip (around x-axis)



rotating frame

(same coords as above, z at top)

vs.



rotating frame

typical 180° flip (around opposite y-axis)

180° flip req's ~6X power of 90°

Bloch-3

SIGNAL EQUATION

one "voxel"

for a particular instant in time vs. Bloch M×B which is change of **M** w/t

i.e., projection of mag. vector at each point onto coil magnetic field direction at each point, summed across object

$$\underbrace{\Phi(t)}_{\substack{\text{magnetic "flux"}\\ \text{thru coil} \rightarrow \underline{\text{scalar}}\\ \text{(integral mag. field}\\ \text{perpendicular to area)}}} = \int_{obj} \underbrace{\vec{\mathbf{B}}(\vec{\mathbf{r}})}_{\substack{\text{magn. field}\\ \text{detected (=generated)}\\ \text{by coil geometry}\\ \text{at each point}\\ \text{in object}}} \cdot \underbrace{\vec{\mathbf{M}}(\vec{\mathbf{r}},t)}_{\substack{\text{local}\\ \text{magnetization}\\ \text{of object}\\ \text{(time-dependent)}}} \underbrace{d\vec{\mathbf{r}}}$$

position: $\vec{\mathbf{r}} \rightarrow x, y, z$

$$\underbrace{V(t) = -\frac{\partial \Phi(t)}{\partial t}}_{\text{Faraday law of Induction}} = \underbrace{-\frac{\partial}{\partial t}} \int_{obj} \overset{\text{(const retained in deriv)}}{\vec{\mathbf{B}}(\vec{\mathbf{r}})} \cdot \vec{\mathbf{M}}(\vec{\mathbf{r}},t)\, d\vec{\mathbf{r}}$$

- time deriv. inside
- use Bloch precess. soln
- [deriv. cos → −sin
  deriv. sin → cos

[or using $e^{i\theta}$]

$\frac{d}{dt}(e^{i\omega t}) = i\omega e^{i\omega t}$

mult by i → add 90°

- evaluate using free precession eqs. (solution to Bloch) ignoring relaxation
- rewrite w/ complex notation w/ <u>time-dependence</u> from lab frame Bloch

- ignore change in z-comp. $\vec{\mathbf{M}}$ because so slow → i.e., we only see M_{xy}, not M_z
- substitute $\vec{\mathbf{M}}(t)$ with lab frame $\underbrace{\vec{M}_{xy}(t)}_{\text{complex}} = \underbrace{M_{xy}(0_+)}_{\text{scalar}}\, \underbrace{e^{-t/T2}}\, e^{-i\omega_0 t}$
- simplify:
  1) ignore decay (assume this t=0) → so equals 1 (scalar)
  2) assume phase-sensitive detection [$\delta\omega$ (difference from ω_0) → rotating frame! makes data complex → even from one coil!

omit receive and init excite flip phase offsets

laboratory frame Bloch solutions:
M_L → same
$M_T = M_{xy}(0_+)\, e^{-t/T2}\, e^{-i\omega_0 t}$

demodulate

$$\underbrace{\vec{\mathbf{S}}(t)}_{\text{complex: 2 V(t)'s}} = \int_{obj} \underbrace{B_{xy}(\vec{\mathbf{r}})}_{\substack{\text{assume}\\ \text{homogeneous}\\ \text{(ignore)}}} \underbrace{M_{xy}(\vec{\mathbf{r}},0)}_{\substack{\text{lab frame}\\ \text{transverse}\\ \text{magnetization}\\ \text{at time t=0}}} \underbrace{e^{-i\,\overline{\underbrace{\delta\omega}_{\text{freq diff from } \gamma B_0}(\vec{\mathbf{r}})}\,t}}_{\substack{\omega_0 \text{ subtracted}\\ \text{off by PSD}}} \underbrace{d\vec{\mathbf{r}}}_{\substack{\text{sum}\\ \text{across}\\ \underline{\text{object}}}}$$

spatially-dependent resonant freq in <u>rotating</u> frame — i.e. after subtraction of $\omega_0 = \gamma B_0$

M_{xy} is now scalar, direction is here

ω_0 = freq = radians/sec
$\omega = \omega_0 + \delta\omega$ ← gradients B1, ΔB0
$\omega_0 t$ = angle
$(\omega_0 + \delta\omega)t$ = angle
$\omega_0 t + \phi$ = angle

$$\vec{\mathbf{S}}(t) = \int_{obj} M_{xy}(\vec{\mathbf{r}},0)\, e^{-i\,\underbrace{\delta\omega(\vec{\mathbf{r}})\,t}}\, d\vec{\mathbf{r}}$$

standard signal expression

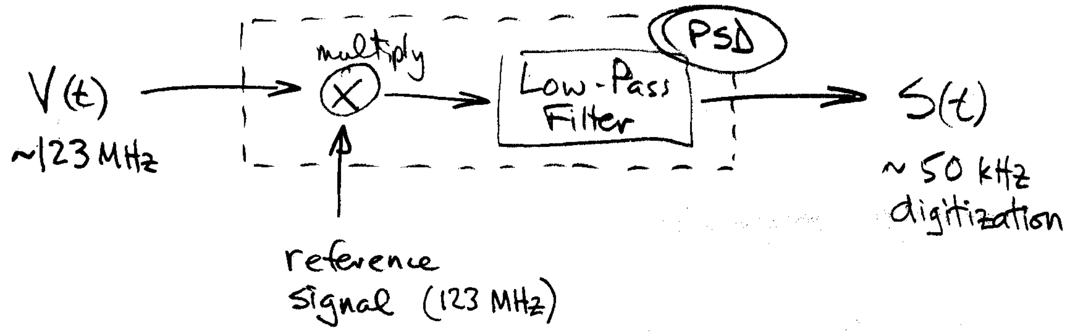
i.e., at a <u>single</u> time point, RF signal is **vector sum** across object of local transverse magnetization vectors

phase <u>angle</u> in <u>rotating</u> frame

[$\underline{\underline{\omega t}} = \frac{\text{radians}}{\text{sec}} \times \text{sec} = \text{radians}$ $\quad (\phi = \int \omega\, dt)$
getting difference converts lab → rotating

PHASE-SENSITIVE DETECTION

how we get rotating frame



- method for moving very high frequency Larmor oscillations down to tractable frequency range

demodulated signal $\propto$ RF coil signal $\cdot$ reference (transmitter)

$$\propto \sin[\overbrace{(\omega_0+\delta\omega)t}^{a}] \cdot \sin[\overbrace{\omega_0 t}^{b}]$$

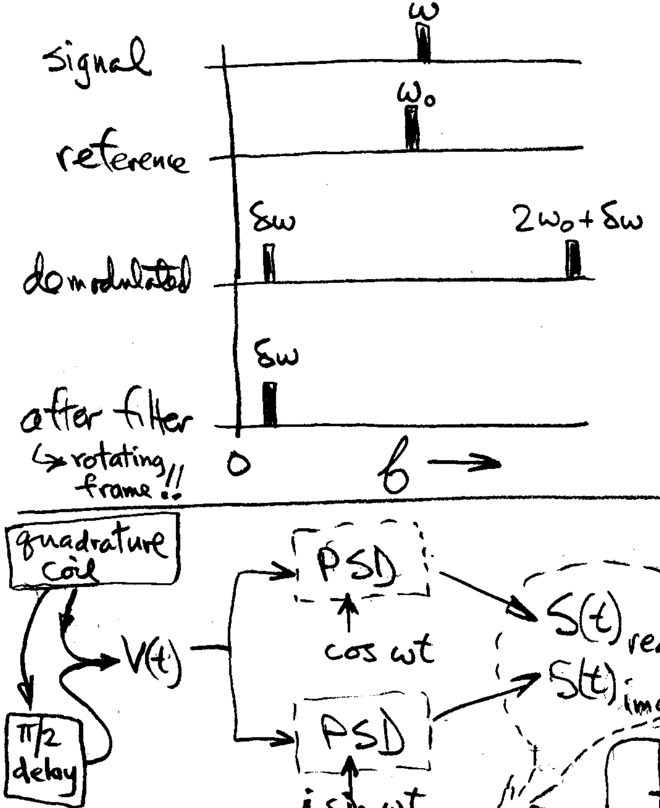
$$\Downarrow \xrightarrow{\text{trig identity}} \boxed{\begin{array}{l} \sin a \sin b = \tfrac{1}{2}[\cos(a-b) - \cos(a+b)] \\ \sin a \cos b = \tfrac{1}{2}[\sin(a-b) + \sin(a+b)] \end{array}}$$

$$\propto \tfrac{1}{2}[\underbrace{\cos \delta\omega t}_{\text{freq diff between RF input \& ref}} - \underbrace{\cos(2\omega_0+\delta\omega)t}_{\text{filter this one out w/ low pass filter}}]$$

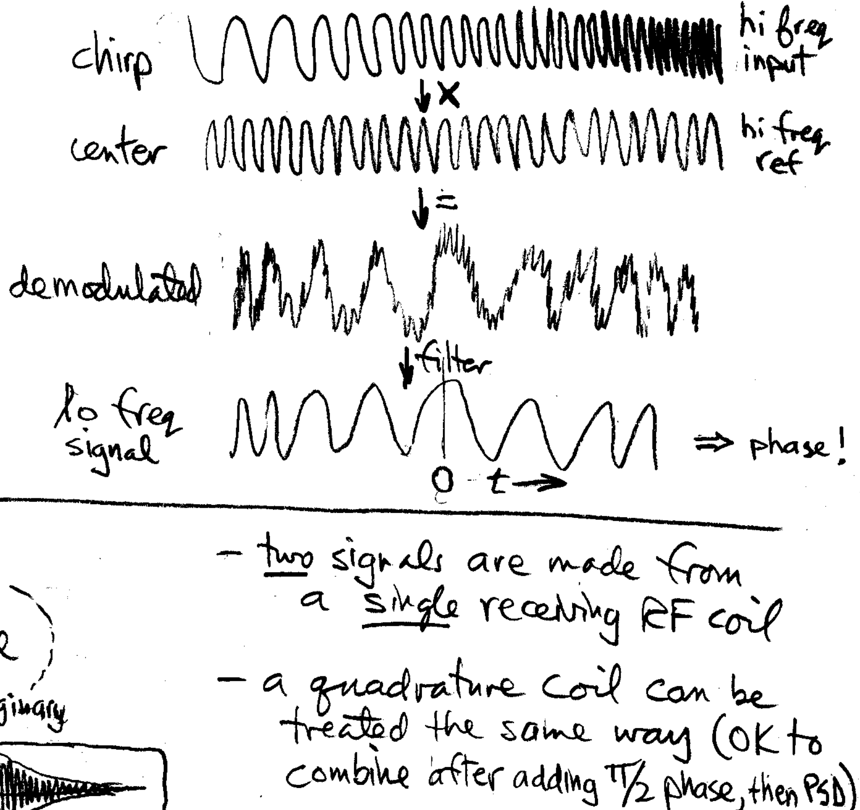
this signal is digitized ←

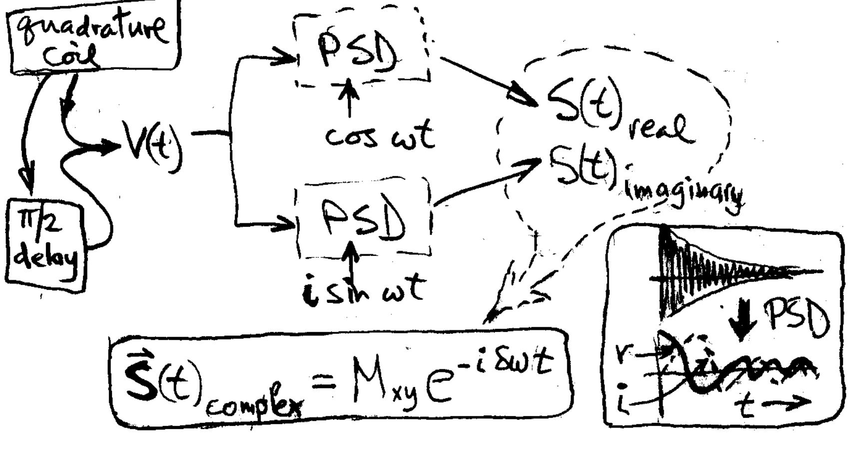
---

one freq — freq domain



chirp — time domain





$$\boxed{\vec{S}(t)_{complex} = M_{xy} e^{-i\delta\omega t}}$$

- two signals are made from a single receiving RF coil
- a quadrature coil can be treated the same way (OK to combine after adding π/2 phase, then PSD)
- quadrature coil has better S/N since noise in each part is uncorrelated ($\sqrt{2}$ better)

echoes-1

FID – FREE INDUCTION DECAY, T2*

T2 – unrecoverable (= rapid)
T2* – add recoverable (= rapid + static)

– Signal (FID) resulting from RF pulse w/angle α

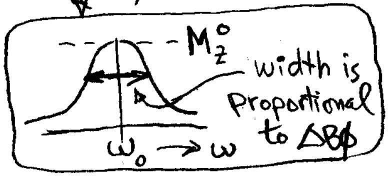
$$\underbrace{\vec{S}(t)}_{\substack{\text{recorded}\\\text{complex}\\\text{signal}}} = \underbrace{\sin\alpha}_{\text{amount of tip}} \int_{\omega=-\infty}^{\omega=\infty} \underbrace{\rho(\omega)}_{\substack{\text{spectral}\\\text{density}\\\text{funct.}}} \cdot \underbrace{e^{-t/T2(\omega)}}_{\substack{\text{time-dep}\\\text{decay}}} \cdot \underbrace{e^{-i\omega t}}_{\substack{\text{rapid}\\\text{oscillations}}} \cdot \underbrace{d\omega}_{\substack{\text{incr}\\\text{freq.}}}$$

ignore space in obj

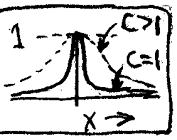
– An example spectral density ("Lorentzian inhomogeneity") → all atoms assumed to interact in same way in freq. range under curve

$$\rho(\omega) = M_z^0 \frac{(\gamma \Delta B0)^2}{(\gamma \Delta B0)^2 + (\omega - \omega_0)^2}$$

ω = γB0 (Bloch)
Δω = γΔB0



ρ(ω) behavior: $y = \frac{c}{c+x^2}$
↳ fixed height



[subst ρ(ω), rearrange to extract ω_0, take integral]

$$\underbrace{\vec{S}(t)}_{\substack{\text{complex}\\\text{signal}}} = \underbrace{\pi}_{\substack{\text{from integral of:}\\ \frac{c^2}{c^2+\omega^2}}} \cdot M_z^0 \cdot \underbrace{\gamma\Delta B0}_{\substack{\Delta\omega\\\text{(big }\Delta\rightarrow\text{big sig.)}\\\text{(b/c fixed height)}}} \cdot \underbrace{\sin\alpha}_{\text{flip}} \cdot \underbrace{e^{-t\gamma\Delta B0}}_{\substack{\text{extra decay}\\\text{from static}\\\text{(big }\Delta\rightarrow\text{big decay)}}} \cdot \underbrace{e^{-t/T2}}_{\substack{\text{regular}\\\text{T2 decay}}} \cdot \underbrace{e^{-i\omega_0 t}}_{\substack{\text{oscillations}\\\text{(complex)}\\\text{(one freq!)}}}$$

[combine T2 + static terms]

$$\boxed{\vec{S}(t) = \pi \cdot M_z^0 \cdot \gamma\Delta B0 \cdot \sin\alpha \cdot e^{-t/T2^*} \cdot e^{-i\omega_0 t}}$$

N.B. center freq., not original integration variable

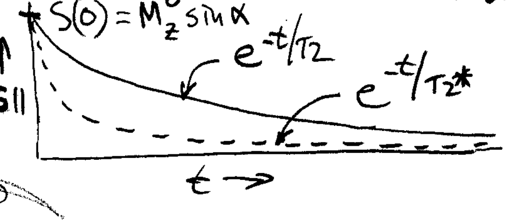
$$\underbrace{\frac{1}{T2^*}}_{\substack{\text{overall decay}\\\text{rate including}\\\text{inhomogeneous B0}}} = \underbrace{\frac{1}{T2}}_{\substack{\text{unrecoverable}\\\text{"intrinsic"}}} + \underbrace{\frac{1}{T2'}}_{\substack{\text{recoverable}\\\text{"static"}}}$$

$\frac{1}{T2'} \rightarrow \gamma\Delta B0$

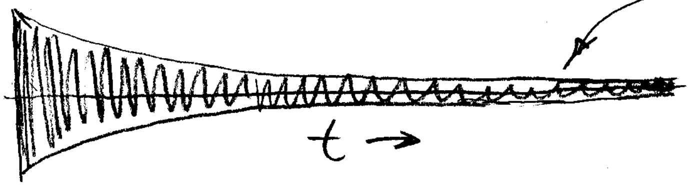
T2' ↳ e.g. extra decay from BOLD

"intrinsic" ↳ spin-spin

"static" → echo can cancel/fix!



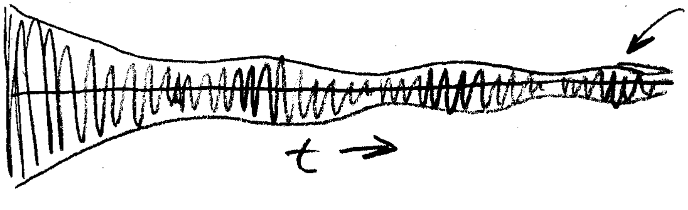
ρ(ω) =



suggestive, since 100 million cycles per second
↳ N.B. complex
↳ (r,i)

ρ(ω) =

(N.B. not Lorentzian!)

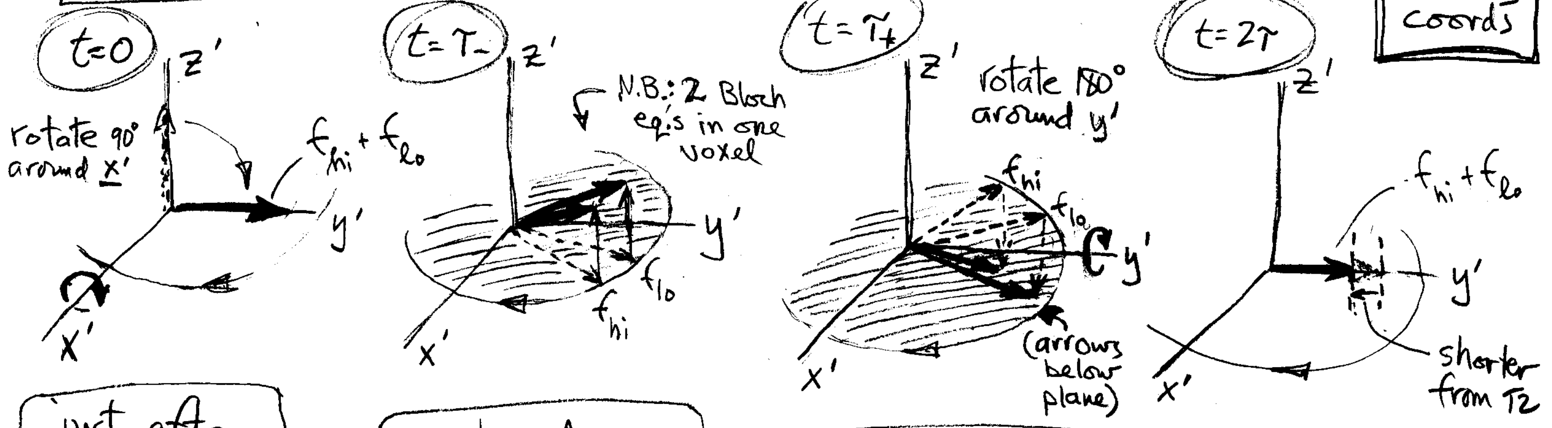


only approximated by $e^{-t/T2^*}$

ECHOES — spin echo

$90° - \tau - 180°$, $T2/T2^*$ & echo

rotating coords



just after $90° x'$ pulse $f_{lo} + f_{hi}$ have *same phase*

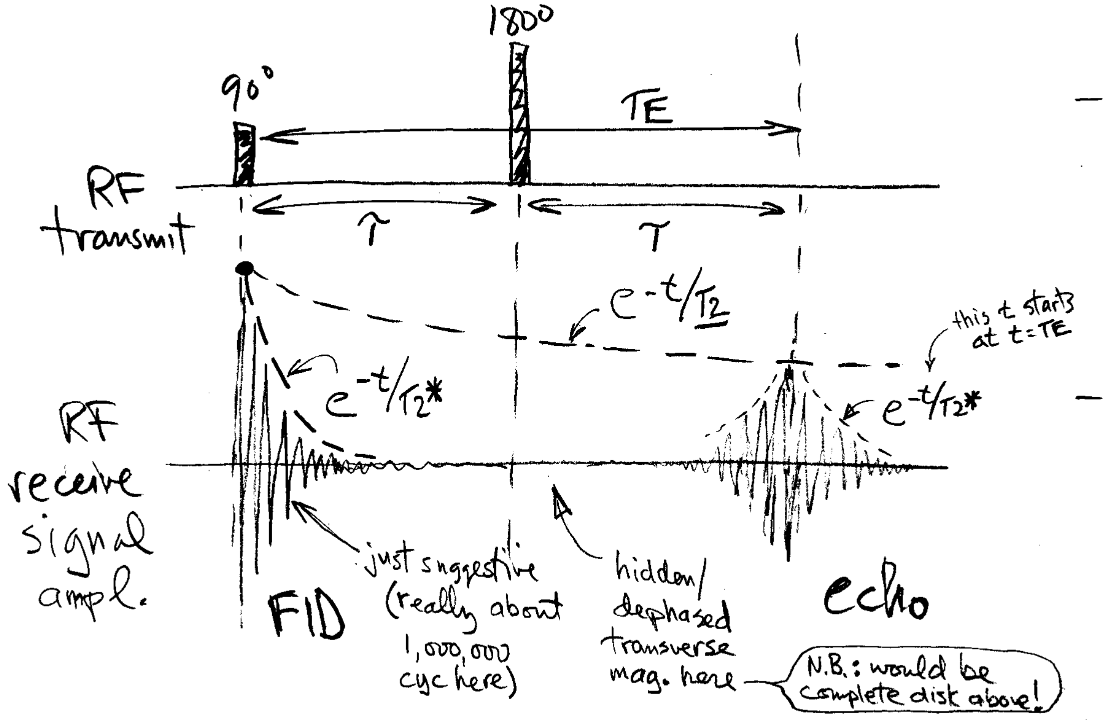
relaxation + phase dispersion of $f_{lo} + f_{hi}$ (both from $B > B_0$)

just after $180° y'$ pulse

(y' pulse like x' pulse but RF has +90° phase)

echo caused by re-phasing of $f_{lo} + f_{hi}$ (w/ decay due to T2)

- remember [brief RF just tips vectors while retaining length
  relaxation includes tips *and* shrinks (M_T) *and* grows (M_L, echo)
- 180° *x'* pulse works, too, but echo will have $+\pi$ phase (*left* side in figs above)
- echo generated even if second pulse *not* 180° (see next)



- FID decay (and echo growth/decay) described by $T2^*$, from inhomogeneity
- reduction in *height* of echo compared to initial described by T2, echo fixes the "star"

ECHOES — spin echo

$\alpha_1 - \tau - \alpha_2 - \tau$ (both pulses along y' for simplicity)

general transforms (operators)

effect of α_y pulse → see $\overset{\square}{\mathbf{R}}_y(\alpha)$

$$M_{x'} \rightarrow M_{x'} \cos\alpha - M_{z'} \sin\alpha$$
$$M_{y'} \rightarrow M_{y'}$$
$$M_{z'} \rightarrow M_{x'} \sin\alpha + M_{z'} \cos\alpha$$

↳ (etc for α_x, α_z)

effect of τ delay ↗ see $\overset{\square}{\mathbf{R}}_z(\alpha)$

indicates rotating coord system

slow precession in rotating coords

decay

$$M_{x'} \rightarrow \left(M_{x'} \cos\omega\tau + M_{y'} \sin\omega\tau\right) e^{-\tau/T_2}$$
$$M_{y'} \rightarrow \left(-M_{x'} \sin\omega\tau + M_{y'} \cos\omega\tau\right) e^{-\tau/T_2}$$
$$M_{z'} \rightarrow M_z^0\left(1 - e^{-\tau/T_1}\right) + M_{z'} e^{-\tau/T_1}$$

---

immediately after α_1 pulse

$$M_{x'}(\omega, 0_+) = -M_z^0(\omega) \sin\alpha_1$$
$$M_{y'}(\omega, 0_+) = 0$$
$$M_{z'}(\omega, 0_+) = M_z^0(\omega) \cos\alpha_1$$

for one isochromat of freq. ω

after τ delay

$$M_{x'}(\omega, \tau) = -M_z^0(\omega) \sin\alpha_1 \cos\omega\tau\, e^{-\tau/T_2}$$
$$M_{y'}(\omega, \tau) = M_z^0(\omega) \sin\alpha_1 \sin\omega\tau\, e^{-\tau/T_2}$$
$$M_{z'}(\omega, \tau) = M_z^0(\omega)\left[1 - (1-\cos\alpha_1) e^{-\tau/T_1}\right]$$

immediately after α_2 pulse (no effect on $M_{y'}$; rewrite y; combine x and y eq.s)

$$M_{x'y'}(\omega, \tau_+) = M_z^0(\omega) \sin\alpha_1 \left(\sin^2\frac{\alpha_2}{2} e^{-i\omega\tau} - \cos^2\frac{\alpha_2}{2} e^{i\omega\tau}\right) e^{-\tau/T_2}$$
$$- M_z(\omega)\left[1 - (1-\cos\alpha_1) e^{-\tau/T_1}\right] \sin\alpha_2$$

*** time dependent free precession around z' (rewrite $M_{x'y'}(\omega, \tau_+)$)**

$$M_{x'y'}(\omega, t) = M_{x'y'}(\omega, \tau_+) e^{-(t-\tau)/T_2} e^{-i\omega(t-\tau)}$$

terms

$$= M_z^0(\omega) \sin\alpha_1 \sin^2\frac{\alpha_2}{2} e^{-t/T_2} e^{-i\omega(t-2\tau)} \quad ①$$
$$- M_z^0(\omega) \sin\alpha_1 \cos^2\frac{\alpha_2}{2} e^{-t/T_2} e^{-i\omega t} \quad ②$$
$$- M_z^0(\omega)\left[1-(1-\cos\alpha_1) e^{-\tau/T_1}\right] \sin\alpha_2 e^{-(t-\tau)/T_2} e^{-i\omega(t-\tau)} \quad ③$$

– for a large num of freq's:

[terms ② & ③ are dephasing → FID of echo
term ① rephasing ⟶ rephase at $t = 2\tau$]

echo signal from ①

$$S(t) = \sin\alpha_1 \sin^2\frac{\alpha_2}{2} \int_{-\infty}^{\infty} \rho(\omega) e^{-t/T_2(\omega)} e^{-i\omega(t-TE)}\, d\omega$$

⇓ $t = TE$

peak ampl

$$A_E = \sin\alpha_1 \sin^2\frac{\alpha_2}{2} \int_{-\infty}^{\infty} \rho(\omega) e^{-TE/T_2(\omega)}\, d\omega = \boxed{M_z^0 \sin\alpha_1 \sin^2\frac{\alpha_2}{2} e^{-TE/T_2}}$$

***

(echo amplitude, ignoring freq dependence of T2

↑ e.g., special case $\alpha_2 = 180°$, $\alpha_1 = 90°$ → $\overset{2\tau}{\Downarrow}$ e^{-TE/T_2}

* $90°_{y'} - \tau - 90°_{y'}$ $\quad S_1(t) = \frac{1}{2} \int_{-\infty}^{\infty} \rho(\omega) e^{-t/T_2} e^{-i\omega(t-TE)}\, d\omega$

$90°_{y'} - \tau - 180°_{y'}$ $\quad S_2(t) =$ no ½ factor

$90°_{x'} - \tau - 180°_{y'}$ $\quad S_3(t) =$ multiply by i → add $\pi/2$ phase → etc for A_E ... like ↑

ECHO TRAINS – spin-echo trains

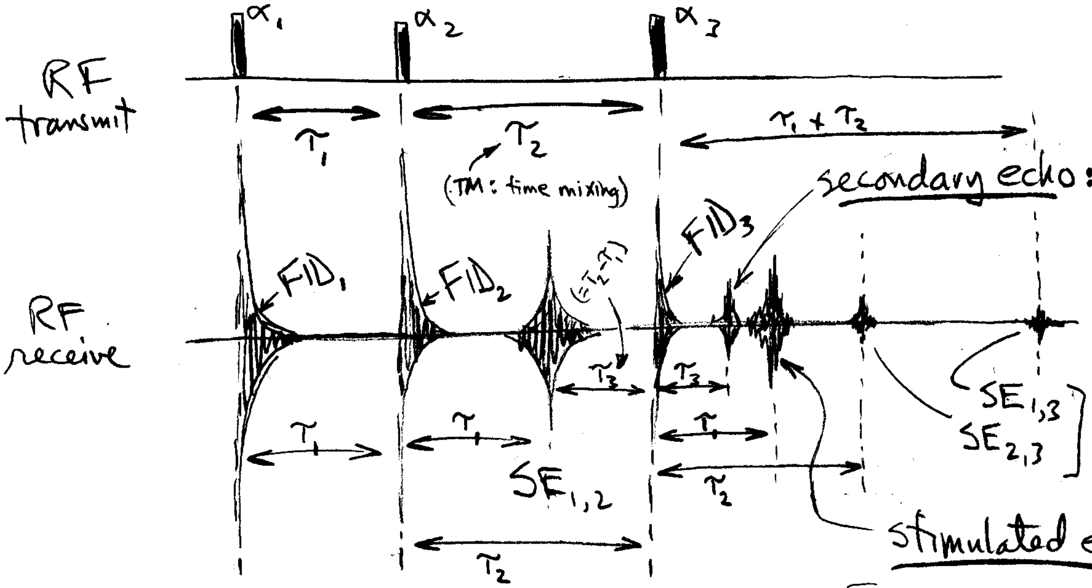
$E_n = \frac{3^{(n-1)} - 1}{2}$

echos after end of nth pulse

3 RF's → 4 echos (here)

6 RF's → 121 echos (!!)

- it's (too) easy to make echos...



secondary echo: SE$_{1,2}$ acts like RF pulse α_3 makes an echo from it

SE$_{1,3}$, SE$_{2,3}$ – two more conventional two pulse spin echoes

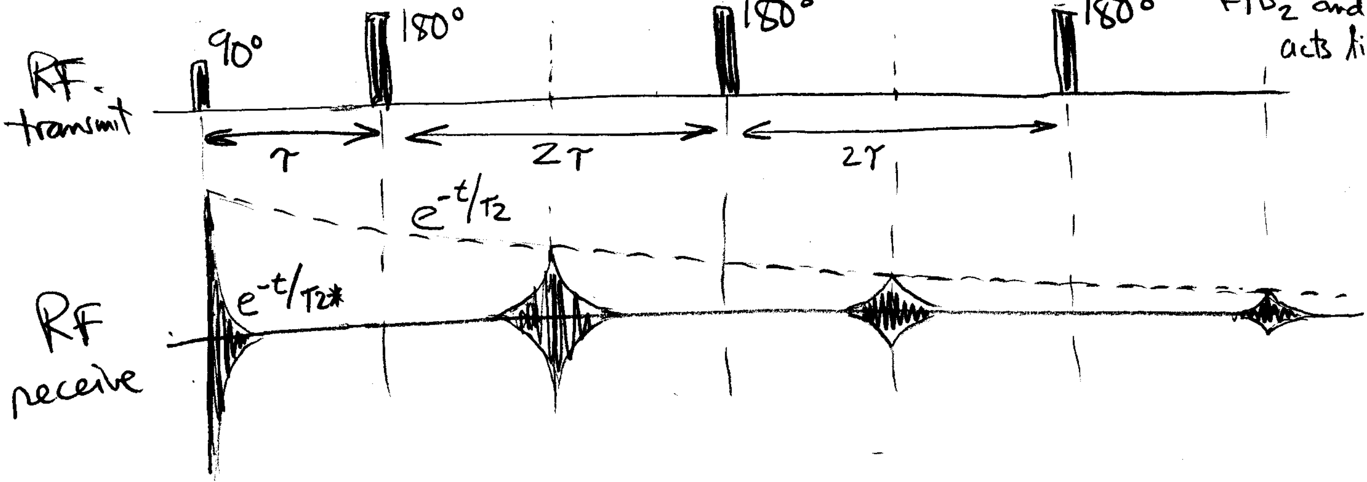
stimulated echo: combined effect of 3

α_1 : $M_L \rightarrow M_T$

α_2 : leftover $\underline{M_T}$ flipped to $\underline{M_L}$ ("saved")

α_3 : flip saved $M_L \rightarrow M_T$ which can then begin to cancel delays (after being held "in limbo" between FID$_2$ and FID$_3$); acts like 2-pulse echo

- a useful multi-echo sequence (CPMG) is a 90° followed by 180° at 2τ spacing



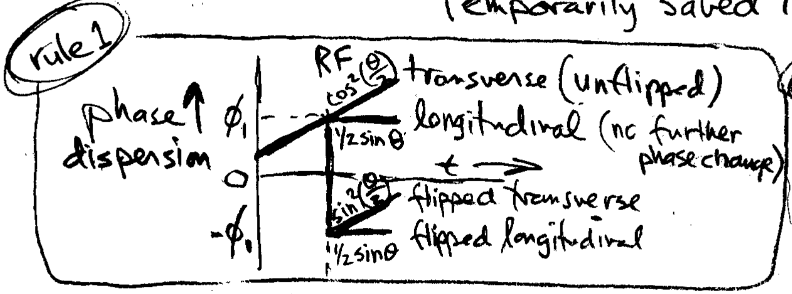
- typically, 90° and 180° applied in different axes (x', then $y', y', y', \ldots$) which reduces phase errors due to imperfect 180° pulses (since slightly-off rotation around y' affects phase less)

EXTENDED PHASE GRAPHS

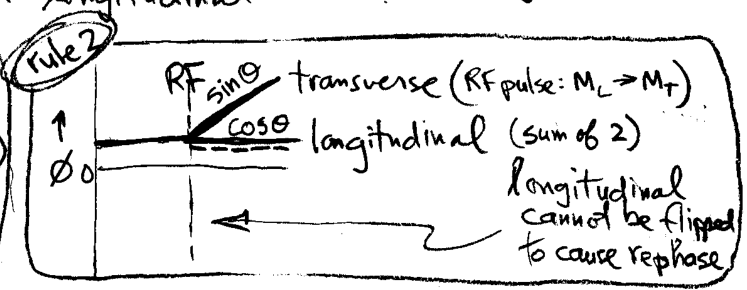
- using full Bloch eq. solutions is tedious ☺ (need 1000 copies)
- pictorial method for visualizing effects of a series of α pulses
- starting point: initial RF creates new transverse ↳ vs. easier to visualize 90/180°

effects on existing → α pulse rotates a portion of existing transverse mag. into a position that results in rephasing and another portion into M_L ↳ QM view helps

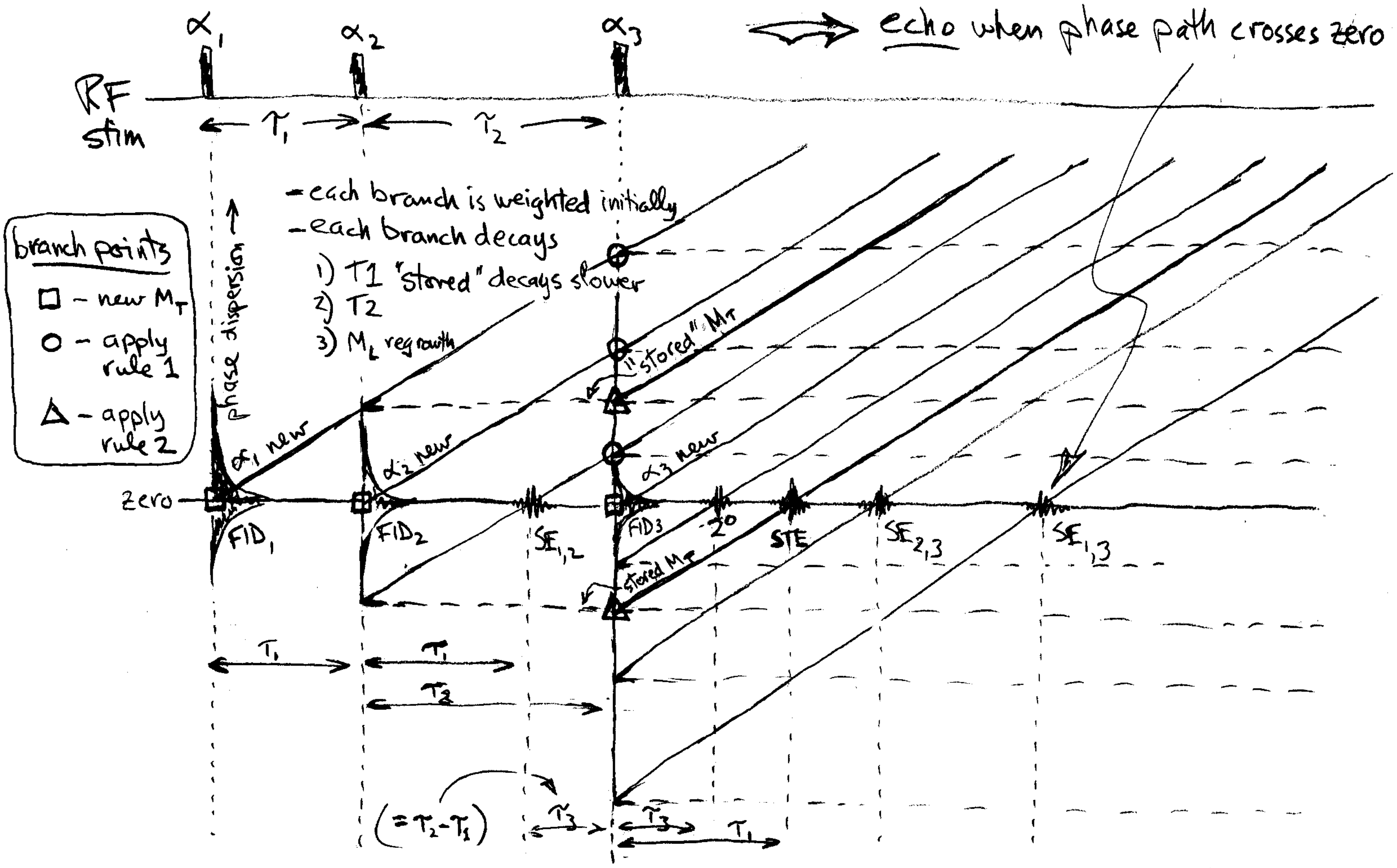
effects on existing → third pulse can uncover and rephase transverse mag. temporarily saved in longitudinal



↳ branching rule for effect of α RF pulse on transverse mag



↳ branching rule for effect of α RF pulse on longitudinal mag

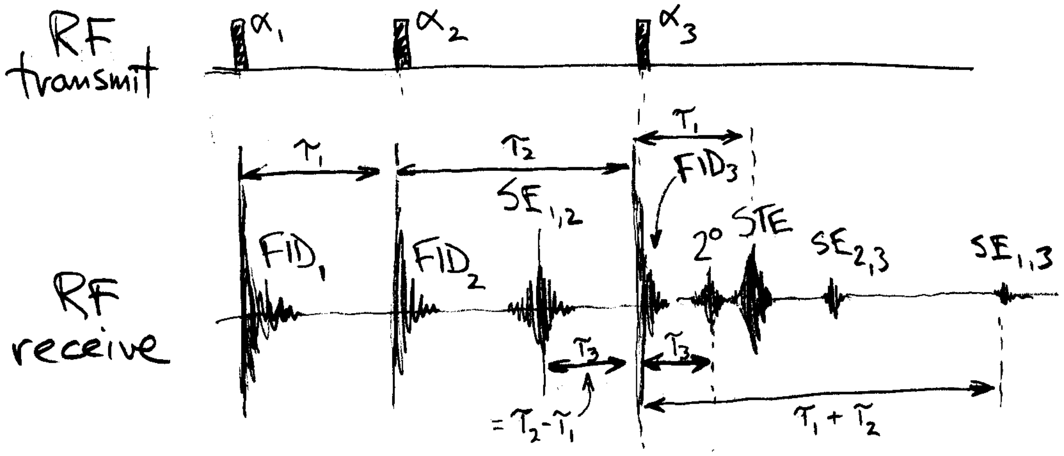


- each branch is weighted initially
- each branch decays
  1) T1 "stored" decays slower
  2) T2
  3) M_L regrowth

echoes-4b

3-PULSE ECHO AMPLITUDES

- assume $M_z^0 = 1$

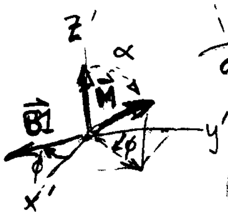


echo	time	amplitude
$SE_{1,2}$	$(t = 2\tau_1)$	$\sin\alpha_1 \sin^2\frac{\alpha_2}{2} e^{-2\tau_1/T2}$ ↳ special cases: $\boxed{\alpha_1 = 90°,\ \alpha_2 = 180°}$ $\boxed{e^{-2\tau_1/T2}}$ $\boxed{2\alpha_1 = \alpha_2}$ $\boxed{\sin^3\alpha_1 \cdot e^{-2\tau_1/T2}}$
2° ("secondary")	$(t = 2\tau_2)$ $(t = 2\tau_1 + 2\tau_3)$	$-\sin\alpha_1 \sin^2\frac{\alpha_2}{2} \sin^2\frac{\alpha_3}{2} e^{-2\tau_2/T2}$
STE ("stimulated")	$(t = 2\tau_1 + \tau_2)$	$\frac{1}{2}\sin\alpha_1 \sin\alpha_2 \sin\alpha_3 e^{-\tau_2/\overset{\text{N.B.: T1}}{T1}} e^{2\tau_1/T2}$
$SE_{2,3}$	$(t = \tau_1 + 2\tau_2)$	$[1 - (1-\cos\alpha_1)e^{-\tau_1/\overset{\text{N.B.: T1}}{T1}}] \sin\alpha_2 \sin^2\frac{\alpha_3}{2} e^{-(\tau_1 + 2\tau_2)/T2}$
$SE_{1,3}$	$(t = 2(\tau_1 + \tau_2))$	$\sin\alpha_1 \cos^2\frac{\alpha_2}{2} \sin^2\frac{\alpha_3}{2} e^{-2(\tau_1+\tau_2)/T2}$

- T1-dependence in STE (but also $SE_{2,3}$) from temporary "storage" of M_T in M_L, then recovery by third pulse

HYPER ECHOES

(N.B. right-handed coord syst vs. left-handed in Bloch eq. notes)

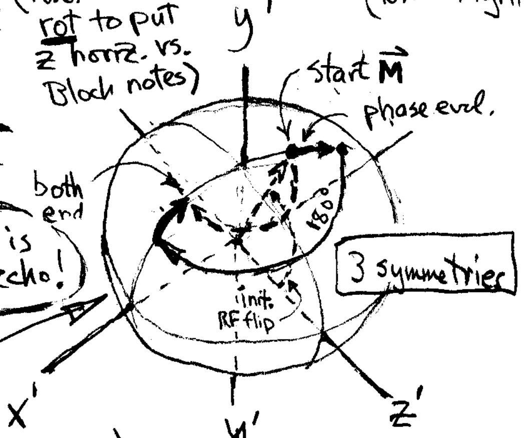


ignore amplitude

(N.B.: coord rot to put z horiz. vs. Bloch notes)

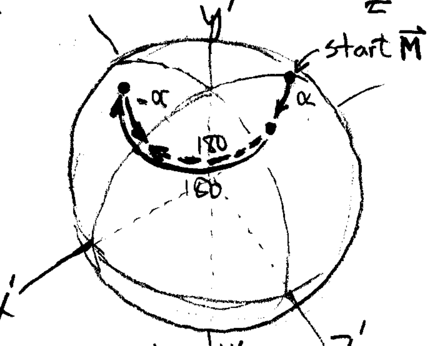
1. $\underbrace{\varphi_z - 180_y - \varphi_z}_{\text{3 solid lines}} \equiv \underbrace{180^\circ_y}_{\text{1 dashed line}}$

this is spin echo!

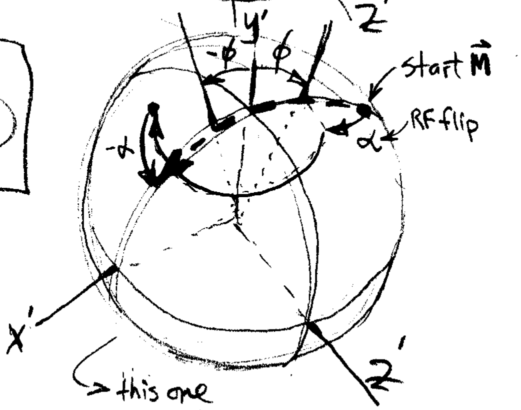


- Hennig & Scheffler (2001)
- normalize $\vec{M}$ amplitude → 1
- sphere surface defines 2D space for $\vec{M}$ moved by:
  1) vect. rotation of $\vec{M}$ around tilted axis in transverse x-y plane by RF with flip, α, and phase, φ : P(α, φ)
  2) rotation around z by phase evolution due to freq offset, ω (B0 offset) and time, t : φ(ω, t)
- three symmetries:
  - solid lines: phase evol or RF flip → 180° → phase evol or RF
  - dashed lines: just 180° equiv.

2. $\alpha_y - 180_y - -\alpha_y \equiv 180^\circ_y$



3. $\alpha_\phi - 180_y - -\alpha_\phi \equiv 180^\circ_y$

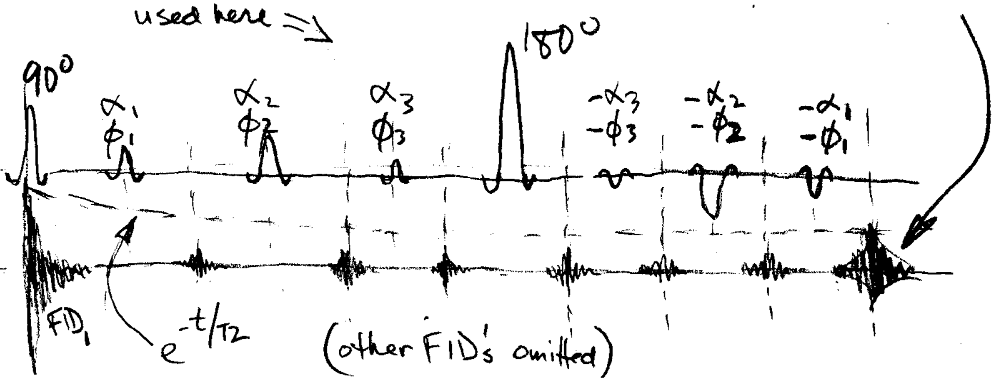


- by combining long sequences observing these symmetries, can generate as strong echo even w/ many inserted α-pulses in between

this one used here ⇒

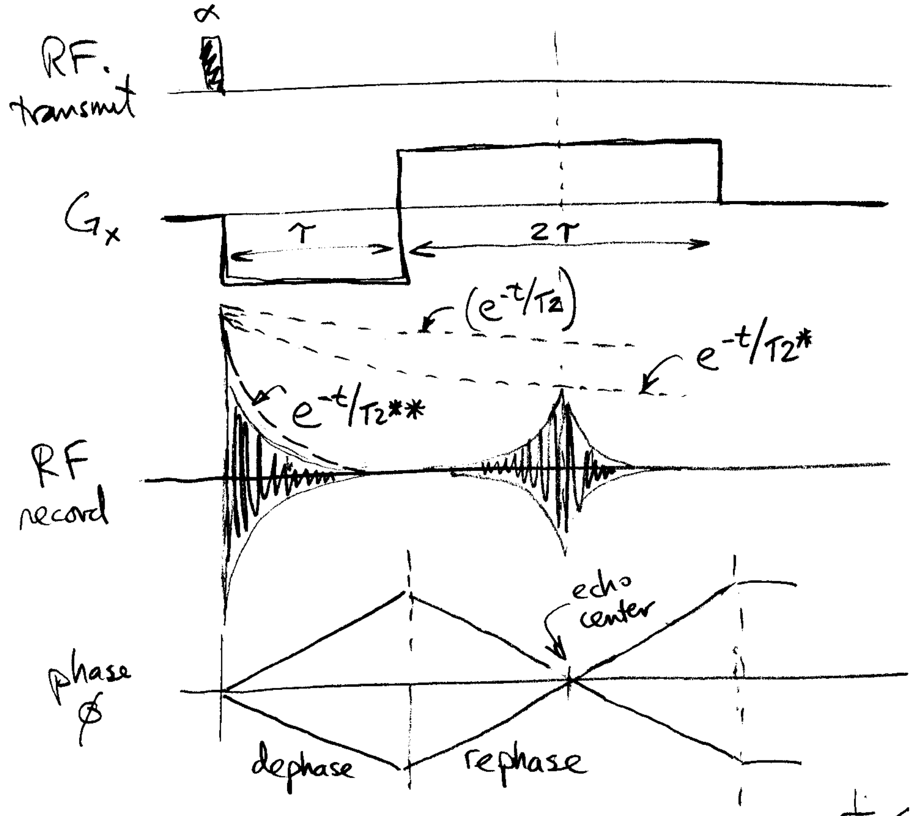
Practical use

- multi-echo example
- can also use to prepare, then separate read-out



- practical prob: 180° pulses deposit a lot of RF (6X 90°)
  ↳ prob at high fields
- by arranging to get big echo in middle of k-space can get by with *much less* RF power

GRADIENT ECHOES — T2*, GE chains

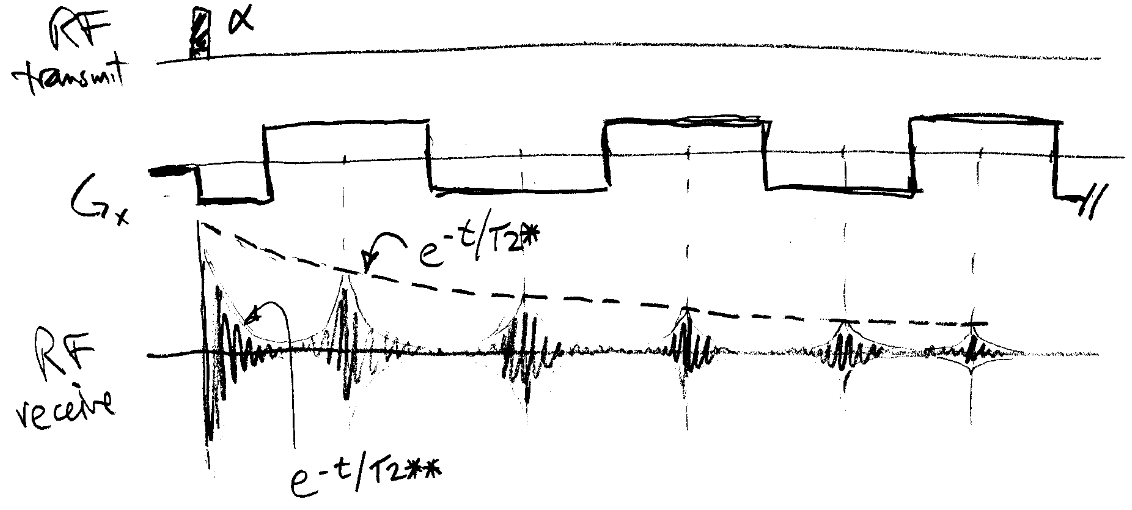


- initial negative gradient ~~de~~ dephases spins
- after $t = \tau$ of positive gradient, spins rephase
- does not correct for T2* inhomogeneities so echo amplitude is $\boxed{A_E = e^{-t/T2^*}}$
- the initial "FID" is not "free" since it is being actively de-phased by gradient, so FID decay

$$\frac{1}{T2} < \frac{1}{T2^*} < \frac{1}{T2^{**}} \leftarrow \boxed{A_E = e^{-t/T2^{**}}}$$

- key difference between spin-echo (SE) and gradient echo (GE) is that B_0 inhomogeneities not corrected
  - → hence, echoes are T2*-weighted, not T2-weighted ⟹ more susceptible to inhomogeneities

- echo trains possible w/ gradient echo (CPMG-like)

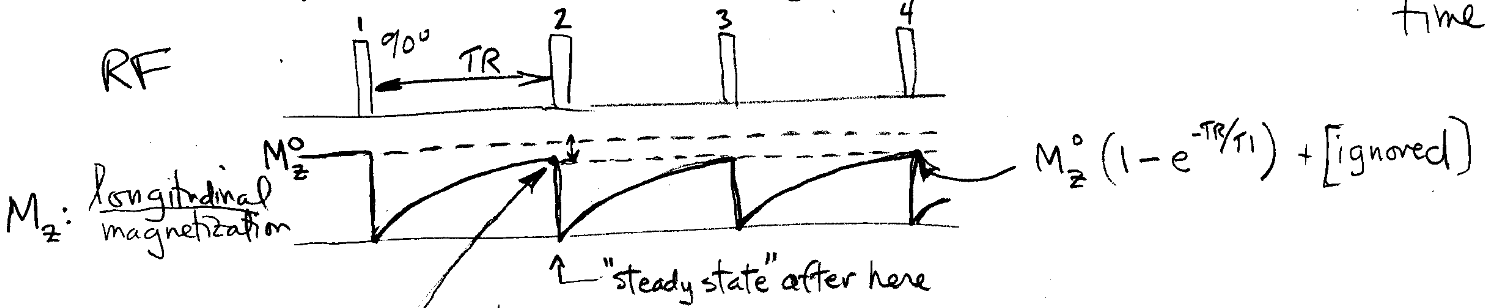


- the faster the gradients are switched, the more echoes you get
- EPI hardware ⟹ 64 echoes

IMAGE CONTRAST

T1 saturation-recovery (no echo, just FID)

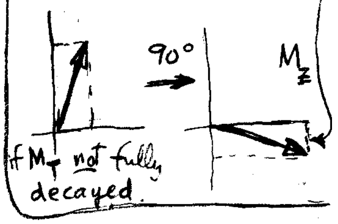
- contrast (PD, T1, T2, T2*) depends on magnetization <u>not</u> getting back to equilibrium, and then <u>differences</u> in how far away each tissue type is at measurement time



- simple saturation/recovery w/ <u>no</u> echo
- initial conditions: [M_z before first pulse = M_z^0; $M_z = 0$ immed. after first pulse (i.e., 90° pulse)]
- from Bloch eq, M_z just before second pulse:

$$\underbrace{M_{z'}^{\overset{2}{(n)}}(0_-)}_{M_L \text{ before current pulse}} = \underbrace{M_z^{(0)}\left(1-e^{-\overset{t}{TR}/T1}\right)}_{M_L \text{ "regrowth-from-zero" term}} + \underbrace{M_z^{\overset{1}{(n-1)}}(0_+)\,e^{-\overset{t}{TR}/T1}}_{M_L \text{ "left-immed.-after-pulse" term (N.B. decaying)}}$$

assume this is 0 because we assume that M_{xy} (<u>transverse</u>) completely decayed so that a 90° pulse doesn't generate any <u>initial</u> <u>longitudinal</u>



- given [(1) 90° pulse (2) no M_{xy} left] → pure tip: $M_{xy} = M_z$

- tip existing mag

$$* \boxed{\underbrace{M_{z'}^{(n)}(0_-)}_{\text{longitudinal mag just before pulse}} = \underbrace{M_{x'y'}(0_+)}_{\text{transverse we can record after pulse}} = \underbrace{M_z^0\left(1-e^{-TR/T1}\right)}_{\text{transverse mag depends on } \underline{\underline{T1}}!}}$$

- that is, the <u>not-completely-regrown</u> longitudinal magnetization, which depends on T1, but which we <u>cannot</u> record, is completely converted to recordable transverse magnetization

assume immediate recording of signal

$$I(\mathbf{r}) = \underbrace{C}_{\text{recon const.}}\ \underbrace{\rho(\mathbf{r})}_{\text{spectral dens}}\left(1-e^{-TR/T1(\mathbf{r})}\right)$$

spectral dens $\rho(\mathbf{r})$ ≈ p. density: underlies equilib. M_z^0

Contrast-1b

IMAGE CONTRAST

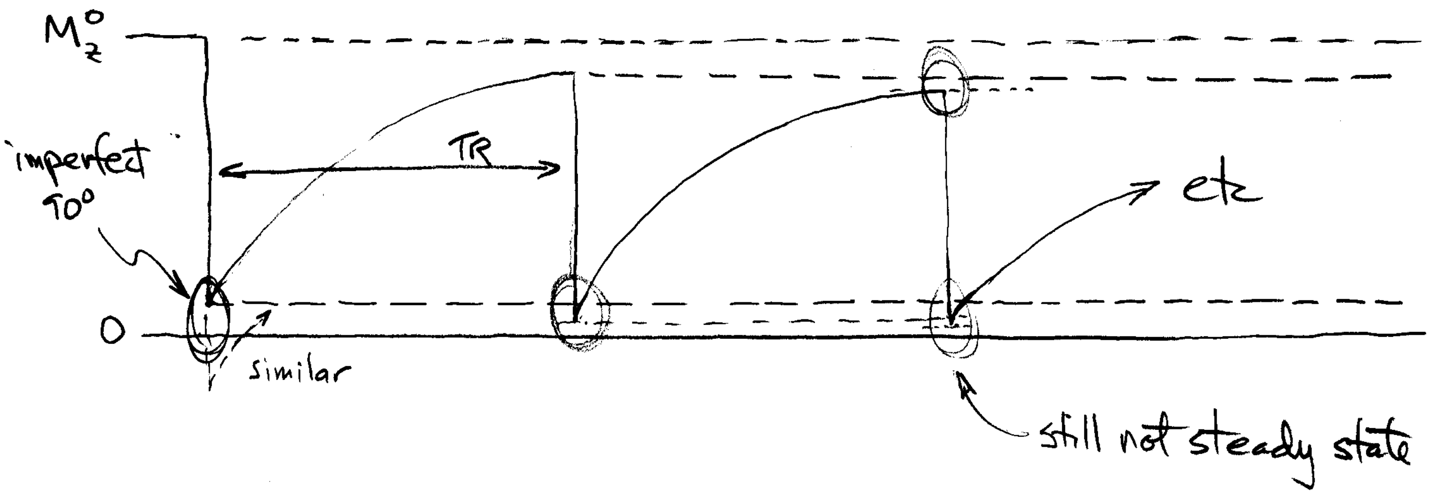
Why imperfect 90° takes multiple flips til steady state

- initial fMRI images are usually discarded (why?)

↳ because they are brighter than all the rest

↳ because multiple flip required before steady state

N.B.: B1 imperfections guarantee this situation will occur (e.g. at 3T, flip angle varies almost 25% across brain)



- at 3T, steady state for typical 1-2 sec TR images reached after ~8 images

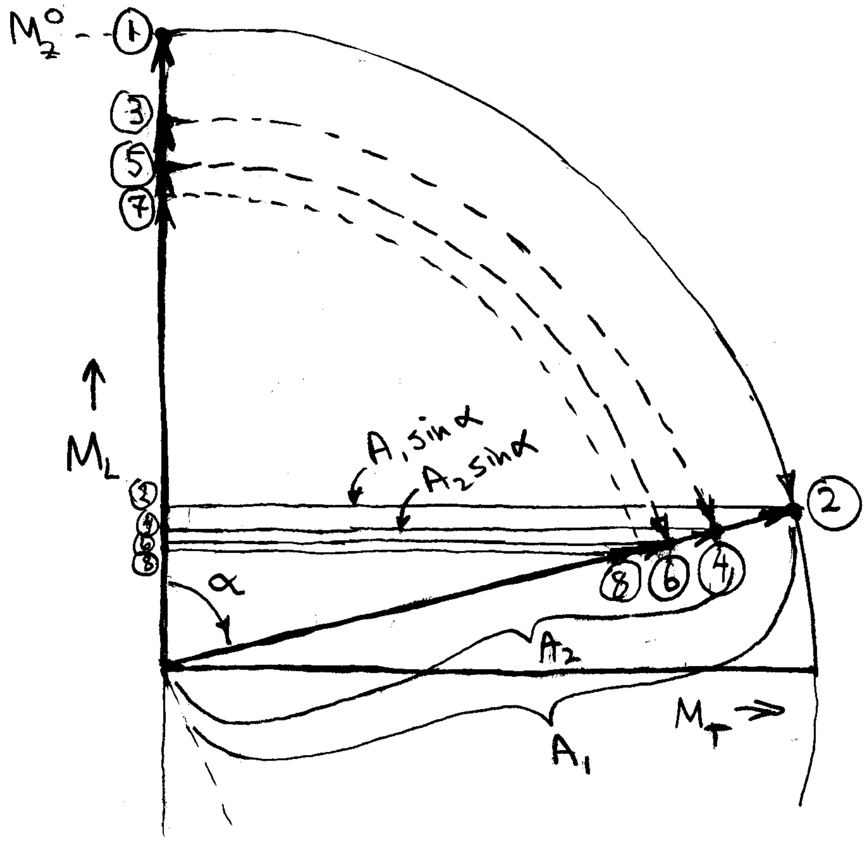
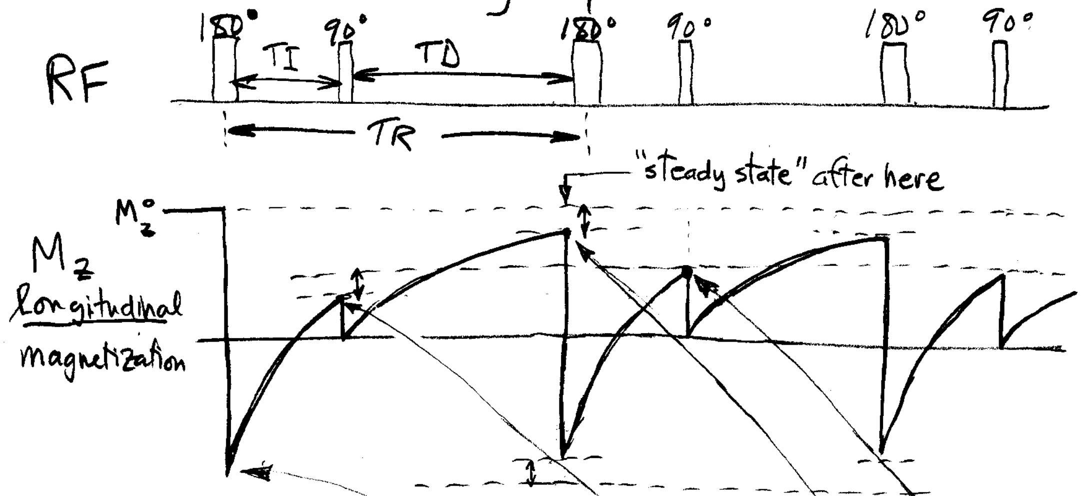


IMAGE CONTRAST

IR (still just saturation-recovery — no echo)

- inversion recovery w/ no echo



- 180 deg pulse reverses longitudinal magnetization

$$M_{z'}^{(0)} = -M_z^0$$

- recovery to end of first TI from long. part of Bloch eq.

$$M_{z'} = M_z^0\left(1 - \underbrace{2}_{\text{from 180°, since } -M_z \text{ in second Bloch term}} e^{-\overset{t}{TI}/T1}\right) \Rightarrow$$

flipped into transverse by second pulse (1st 90) → ignore!

- longitudinal then regrows from zero → (from first Bloch term only)

$$M_{z'} = M_z^0\left(1 - e^{-(TR-TI)/T1}\right)$$

- after second 180°, just change sign again

$$M_{z'} = -M_z^0\left(1 - e^{-(TR-TI)/T1}\right)$$

2nd Bloch term

- apply relaxation eq. again

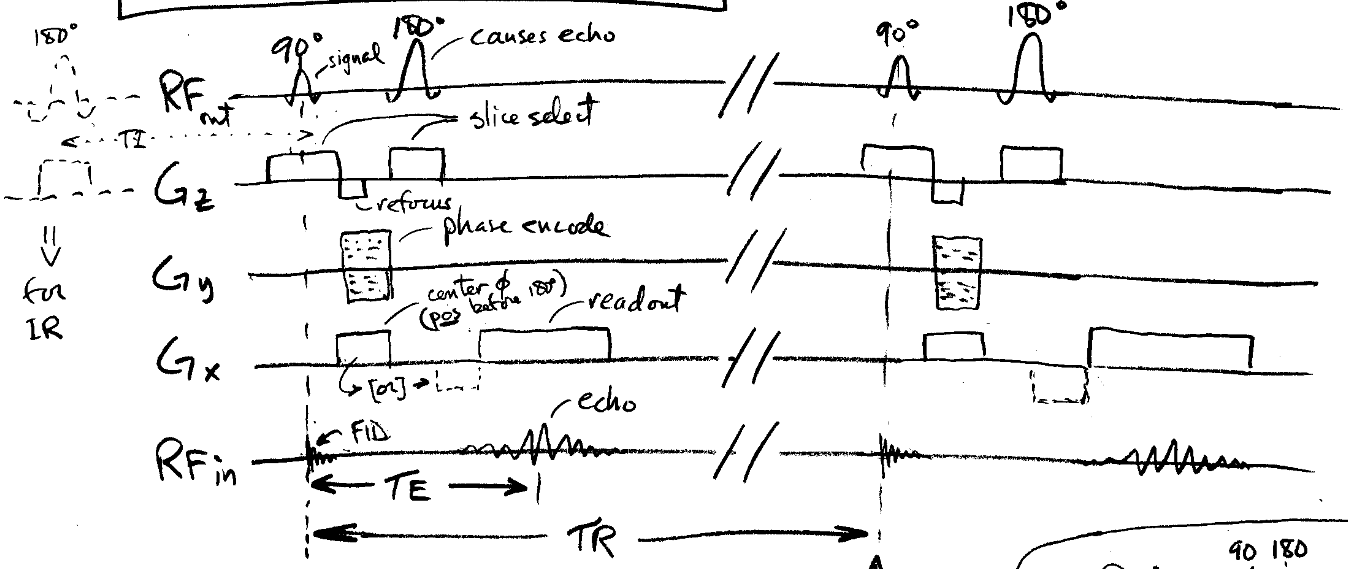
$$M_{z'} = M_z^0\left(1 - e^{-TI/T1}\right) - M_z^0\left(1 - e^{-(TR-TI)/T1}\right)e^{-TI/T1}$$

*
$$\boxed{M_z' = M_z^0\left(1 - 2e^{-TI/T1} + e^{-TR/T1}\right)}$$

↳ this is magnetization flipped to transverse, therefore made recordable

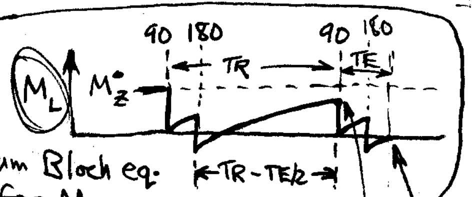
IMAGE CONTRAST

SE, IR-SE



[N.B. centering G_x lobe also causes gradient echo]

- steady state mag (2^{nd} TR) just before 90° ⟹ Sum Bloch eq. for M_L



$$M_{z'}(0_-) = M_z^\circ\left(1 - 2e^{-(TR-TE/2)/T1} + e^{-TR/T1}\right)$$

- the echo signal ($M_{\underline{T}}$) unlike in simple saturation-recovery FID has an additional delay before it is recorded, so we have to take account of transverse mag relaxation

$M_L = 0$ during max M_T (echo)

$$A_E = M_z^\circ\left(1 - 2e^{-\overbrace{(TR-TE/2)}^{\to TR}/T1} + e^{-TR/T1}\right) e^{-TE/T2} \longrightarrow$$

from spin echo equation hell

- if we assume TE much less than TR, then we can simplify:

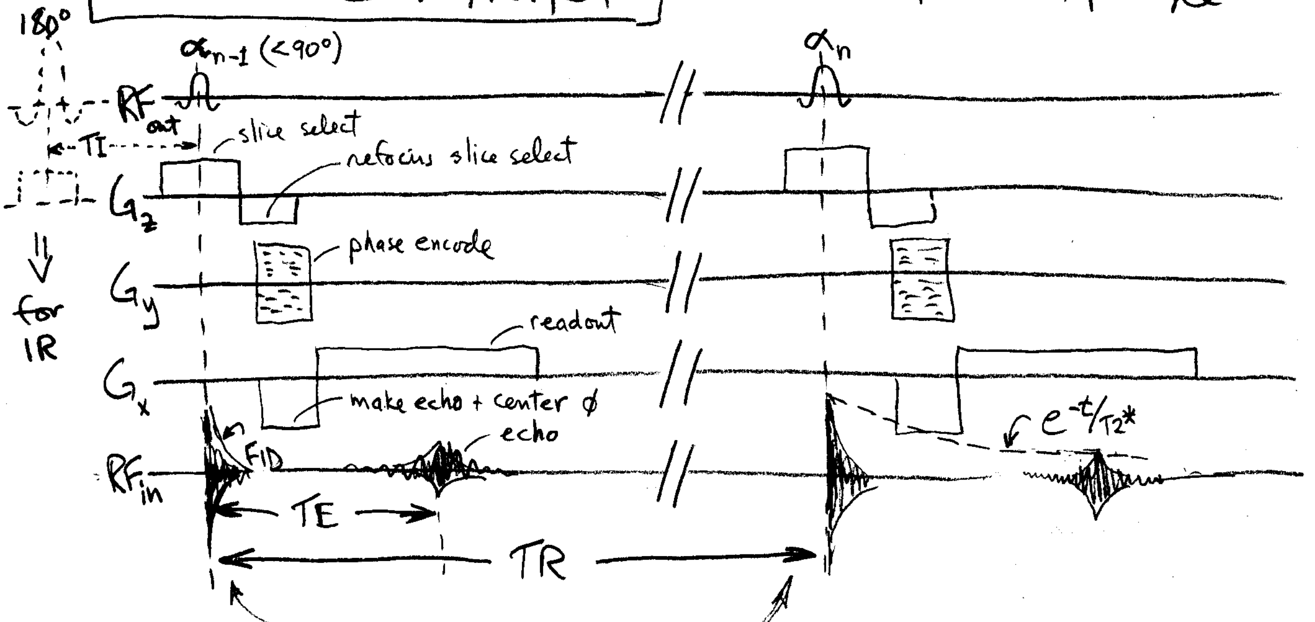
↳ TR-TE ≈ TR

*
$$\boxed{\underbrace{A_E}_{\text{amplitude echo}} = \underbrace{M_z^\circ}_{\text{proton density}} \underbrace{\left(1 - e^{-TR/T1}\right)}_{\text{TR controls T1 contrast}} \underbrace{e^{-TE/T2}}_{\text{TE controls T2 contrast}}}$$

- similar equation for SE-IR

*
$$\boxed{A_E = M_z^\circ\left(1 - 2e^{-TI/T1} + e^{-TR/T1}\right) e^{-TE/T2}}$$

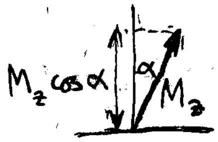
IMAGE CONTRAST GRE w/ small tip angle



- use basic longitudinal relaxation from Bloch eq. again
  - ↳ & assume $M_{x'y'}^{(n)}(0_-) = 0$ → transverse dephased before next pulse (long TR or spoiler)

$$M_{z'}^{(n)}(0_-) = M_z^0\left(1-e^{-TR/T2}\right) + M_{z'}^{(n-1)}(0_+)\,e^{-TR/T1}$$

- assume we have a small tip angle:



$$\Rightarrow M_{z'}^{(n)}(0_+) = M_z^{(n)}(0_-)\cos\alpha$$

$$\underbrace{M_{z'}^{(n)}(0_-)} = \boxed{M_z^0\left(1-e^{-TR/T1}\right)} + \underbrace{M_{z'}^{(n-1)}(0_-)}\cos\alpha\, e^{-TR/T1}$$

- assume we are in dynamic equilibrium: (current = previous)

$$\overbrace{M_{z'}^{(n)}(0_-)} = \overbrace{M_{z'}^{(n-1)}(0_-)} = M_{z'}^{ss}(0_-)$$

steady state:
- subst. into prev. eq. for both $M_z(0_-)$'s
- solve for it

$M_{ss} = M_0 + M_{ss}\cos$
$M_{ss} - M_{ss}\cos = M_0$
$M_{ss}(1-\cos) = M_0$
$M_{ss} = M_0/(1-\cos)$

prepulse steady state longitudinal

$$M_{z'}^{ss}(0_-) = \frac{M_z^0\left(1-e^{-TR/T1}\right)}{1-\cos\alpha\, e^{-TR/T1}}$$

post-pulse transverse magnetization

$$M_{x'y'}^{ss}(t) = \frac{M_z^0\left(1-e^{-TR/T1}\right)}{1-\cos\alpha\, e^{-TR/T1}}\cdot\underbrace{\sin\alpha\, e^{-t/T2^*}}$$

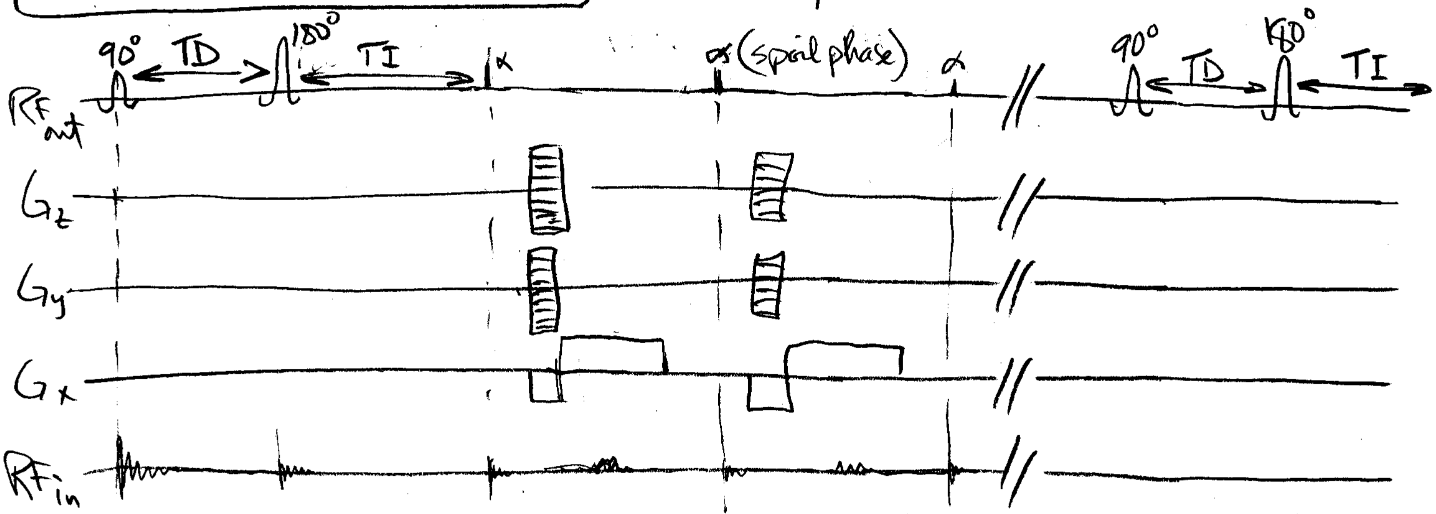
gradient echo amplitude

$$* \boxed{A_E = \frac{M_z^0\left(1-e^{-TR/T1}\right)}{\underbrace{1-\cos\alpha\, e^{-TR/T1}}}\sin\alpha\, e^{-TE/T2^*}}$$

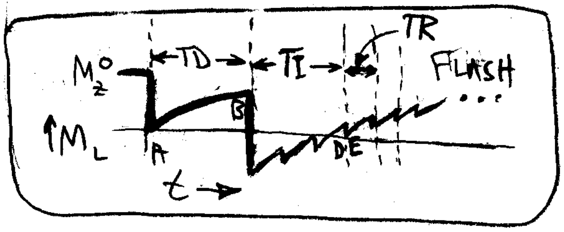
from gradient echo

T1 contrast mainly depends on flip angle, not TR → cos 90° = 1 → eliminates T1 weight since denom = numer

IMAGE CONTRAST MDEFT / 3D FLASH

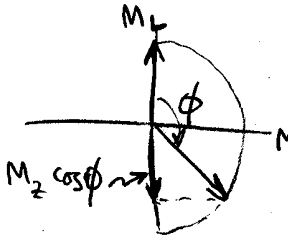


- saturate, wait for contrast_1, invert, wait for contrast_2, FLASH (center out)



A) $M_z'\begin{pmatrix}\text{just}\\ \text{after}\\ 90^\circ\end{pmatrix} = 0$ (perfect 90°)

B) $M_z'\begin{pmatrix}\text{after}\\ \text{TD}\end{pmatrix} = M_z^0\left(1-e^{-TD/T1}\right)$ (Bloch term #1)



C) $M_z'\begin{pmatrix}\text{just}\\ \text{after}\\ \text{invert}\end{pmatrix} = \cos\phi\, M_z^0\left(1-e^{-TD/T1}\right)$

D) $M_z'\begin{pmatrix}\text{after}\\ \text{TI}\end{pmatrix} = M_z^0\left(1-e^{-TI/T1}\right) + \left[\cos\phi\, M_z^0\left(1-{}^{-TD/T1}\right)\right]e^{-TI/T1}$

$$= \boxed{M_z^0\left[1-\left[1-\cos\phi\left(1-e^{-TD/T1}\right)\right]e^{-TI/T1}\right]}$$ after preparation

Special case TI=TD: $= M_z^0\left[1-e^{-TI/T1}\right]^2$

↳ using hard 130° 'inversion' can cancel hard alpha B1 inhomogeneities (Thomas et al. '05)

- after the first α pulse:

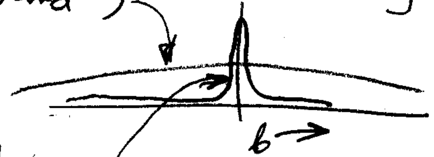
E) $$^{*}\boxed{M_z'\begin{pmatrix}\text{just}\\ \text{after}\\ \text{first }\alpha\end{pmatrix} = M_z^0\left[1-\left[1-\cos\phi\left(1-e^{-TD/T1}\right)\right]e^{-TI/T1}\right]\sin\alpha}$$

contrast-4c

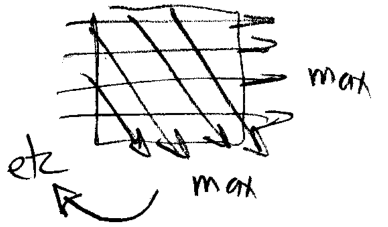
MAGNETIZATION TRANSFER CONTRAST

- protons in macromolecules & bound to membranes have wide range of resonant freqs ("bound") → T2 = 1 msec
  - ↳ i.e. signal but invisible w/ usual TE

vs.



- free protons in blood, CSF, water have narrow range of resonant freqs ("free") → T2 = 50 msec
- mag transfer pulse sequence
  1) off center freq pulse to hit "bound" (but don't hit water too hard)
  2) wait for <u>magnetization</u> <u>transfer</u> from saturated longitudinal M_L of "bound" ⟶ M_L of "free"
  3) result of transfer → <u>attenuation</u>
     - yes: WM, cartilidge, muscle
     - no: blood, CSF, fat
  - → N.B. this always happens a little (cf. T1-<u>weigted</u>, T2-<u>weighted</u>) something to keep in mind if <u>hard pulse</u> (wide freq)
- used to increase contrast in TOF
  - TOF (not explained) bright vessels from inflow fresh spins
  - MT-contrast added: suppress tissue but <u>not</u> blood
- view w/ MIP: maximum intensity projection along lines



⟶ view as movie

contrast-5

SIGNAL-TO-NOISE, CONTRAST-TO-NOISE

Haacke

tissue	T1	T2 (1.5T)	PD
GM	950	100	0.80
WM	600	80	0.65
CSF	4500	2200	1.00 water
blood	1200	100-200	

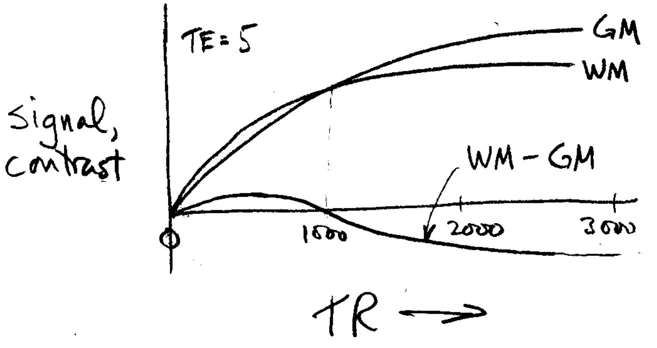
Lauterbur

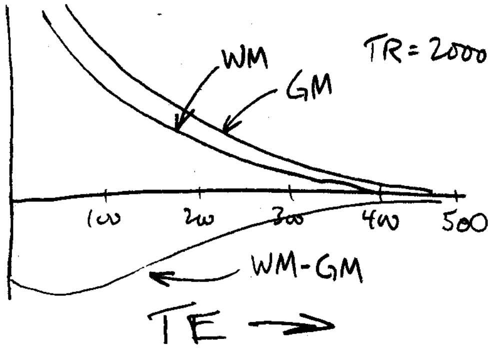
tissue	T1	T2	PD
GM	760	77	0.69
WM	510	67	0.61
CSF	2650	280	1.00

- signal-to-noise defined as: $\mathrm{SNR} \equiv \frac{\overset{\text{avg}}{\bar{S}}}{\sigma_n}$ ← avg obj signal; ← s.d. non-object region
- temporal SNR: $t\mathrm{SNR} \equiv \frac{\bar{S}_t}{\sigma_t}$
- "contrast" is a difference → $\bar{C}_{AB} \equiv \bar{S}_A - \bar{S}_B$ ↳ e.g., [WM-GM, activated-rest]
- contrast-to-noise ratio:

$$\mathrm{CNR}_{AB} \equiv \frac{\bar{C}_{AB}}{\sigma_n} = \frac{\bar{S}_A - \bar{S}_B}{\sigma_n} = \mathrm{SNR}_A - \mathrm{SNR}_B$$

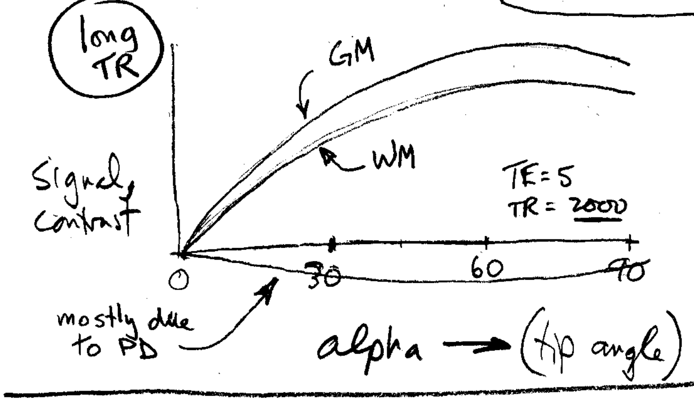
- spin-echo: $\boxed{A_E = M_z^0 \left(1 - e^{-TR/T1}\right) e^{-TE/T2}}$

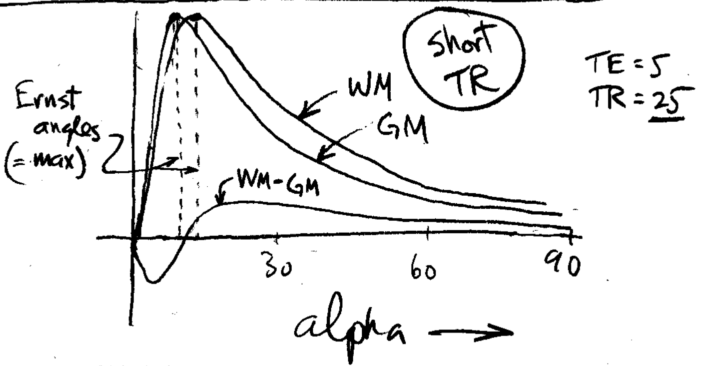




- gradient echo: $\boxed{A_E = \frac{M_z^{\circ}\left(1 - e^{-TR/T1}\right)}{1 - \cos\alpha\, e^{-TR/T1}} \sin\alpha\, e^{-TE/T2*}}$

1.5T [T2*$_{GM}$ = 50, T2*$_{WM}$ = 40]





- general rules: spin-echo, long TR GE

proton-density weighted	TR ↑↑ (no T1 diffs)	TE ↓↓ (no T2 diffs)
T1-weighted	TR ≈ T1 (big T1 diffs)	TE ↓↓ (no T2 diffs)
T2-weighted	TR ↑↑ (no T1 diffs)	TE ≈ T2 (big T2 diffs)

SIGNAL-TO-NOISE S/N

- generalized dependence of SNR on 3D imaging parameters

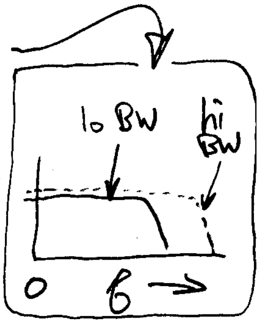
$$\text{SNR/voxel} \propto \underbrace{\Delta x \Delta y \Delta z}_{\text{voxel size (of same number)}} \sqrt{\underbrace{N_{acq}}_{\text{num repeats}} \underbrace{N_x N_y N_z}_{\text{number of voxels (of same size)}} \underbrace{\Delta t}_{\text{read timestep}}}$$

- size (volume) of voxels (with the number of voxels held constant), linear effect on S/N
  - ↳ e.g., $\boxed{3 \times 3 \times 3 \text{ mm} \rightarrow 4 \times 4 \times 4 \text{ mm}} \rightarrow \frac{64}{27} = 2.37$ times better S/N
- more voxels (with size of voxels, Δt per read step constant) $\sqrt{n}$ effect on S/N
  - ↳ e.g., $\boxed{64 \times 64 \rightarrow 128 \times 128} \rightarrow \frac{\sqrt{128 \times 128}}{\sqrt{64 \times 64}} = 2$ times better S/N
- # acquisitions $\sqrt{n}$ better S/N
  - ↳ e.g. $\boxed{1 \text{ acq} \rightarrow 2 \text{ acq}} \rightarrow \frac{\sqrt{2}}{\sqrt{1}} = 1.41$ times better S/N

- longer timestep during readout, $\sqrt{\Delta t}$ better S/N

$$\Delta t = \frac{1}{BW_{read}},$$ digitization timestep during echo acquisition

- BW_{read} determined by cutoff freq. analog lowpass filter
- Δt controls BW because lowpass cutoff has to be set higher for smaller (higher freq-detecting) Δt
- must filter out freq's $> f_{max} = \frac{1}{2\Delta t}$ because they alias



fourier-0

COMPLEX ALGEBRA

- don't confuse freq, angle!

$$\underbrace{e^{\underbrace{-i\omega t}_{\text{freq.}\times\text{current time} = \text{angle (lin. var.)}}}}_{\text{complex/circular var.}}$$

makes linear var. into circular variable

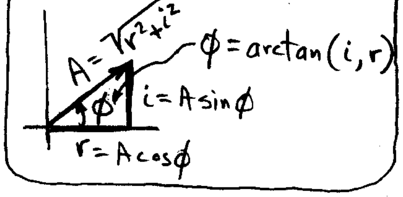
freq. (lin. var.) x current time

angle (lin. var.)

complex/circular var.

- how to convert:

$(r, i) \Leftrightarrow (A, \phi)$



real/imaginary

add : $(r_1, i_1) + (r_2, i_2) = (r_1 + r_2,\ i_1 + i_2)$

mult : $(r_1, i_1) \times (r_2, i_2) = (r_1 r_2 - i_1 i_2,\ r_1 i_2 + i_1 r_2)$

ampl./phase

add: $(A_1, \phi_1) + (A_2, \phi_2) = (A_1\cos\phi_1 + A_2\cos\phi_2,\ A_1\sin\phi_1 + A_2\sin\phi_2)$

multiply: (commutative) $(A_1, \phi_1) \times (A_2, \phi_2) = (A_1 A_2,\ \phi_1 + \phi_2)$ →

divide: $(A_1, \phi_1) \div (A_2, \phi_2) = (A_1/A_2,\ \phi_1 - \phi_2)$

N.B.: 3rd kind of vector mult. different than dot product and cross product (and G.A. non-commutative pseudoscalar multiply)

complex to real power: $(A, \phi)^n = (A^n, n\phi)$

$e^{i\phi}$ = [expand as series recognize cos, sin series

the real "e" to "purely imaginary" power

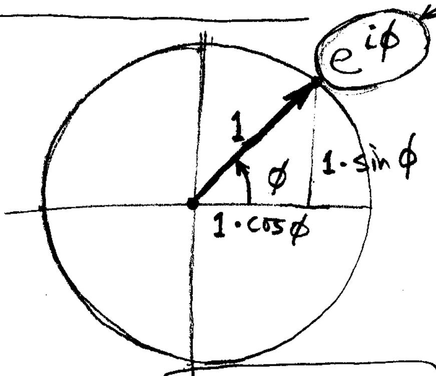
$= \cos\phi + i\sin\phi$

$= \cos\phi,\ \sin\phi$

= vector on unit circle

$e^{i\phi n} = (\cos\phi + i\sin\phi)^n$

$= \cos n\phi + i\sin n\phi$



shorthand for a unit vector (2D) pointing in the direction of ϕ

- for arbitrary amplitude, multiply

$A e^{i\phi}$

- change in phase is freq

$\frac{d\phi}{dt} = \omega$ ⟺

- phase is integral of freq. variable

$\phi = \int \omega\, dt$

Fourier transform

$H(f) = \int_t h(t)\, e^{-i2\pi ft}\, dt$

$H(-f) = \int_t h(t)\, e^{i2\pi ft}\, dt$

Convolution

$f(x) = g(x) * h(x) = \int_z g(z)\cdot h(x-z)\, dz$

Cross-correlation

$f(x) = g(x) \otimes h(x) = \int_z g(z)\cdot h(x+z)\, dz$

Convolution Theorem

$\mathbf{F}[g(x)\cdot h(x)] = G(k) * H(k)$ →

[the Fourier transform of two functions multiplied by each other equals the convolution of the Fourier transform of each funct

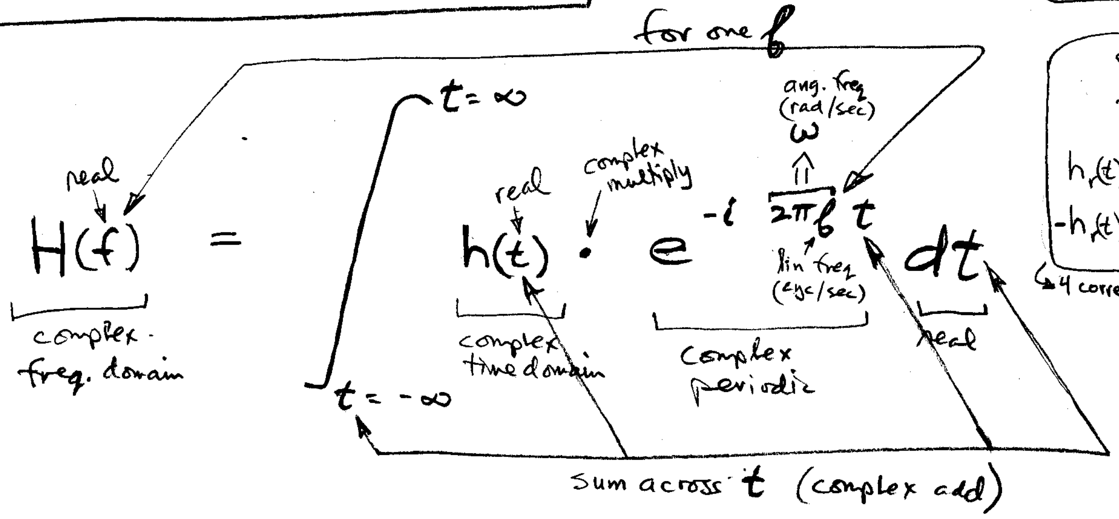
because of FFT, faster if kernel not small

Fourier transform (1)

ft is a phase angle

$f \cdot t = ft$

cyc/sec · sec = cyc



$$H(f) = \int_{t=-\infty}^{t=\infty} h(t) \cdot e^{-i\,2\pi f t}\, dt$$

Fourier integral terms written out

$h_r(t)\cos(2\pi ft) + h_i(t)\sin(2\pi ft)$,

$-h_r(t)\sin(2\pi ft) + h_i(t)\cos(2\pi ft)$

→ 4 correlations

Fourier integral w/real signal

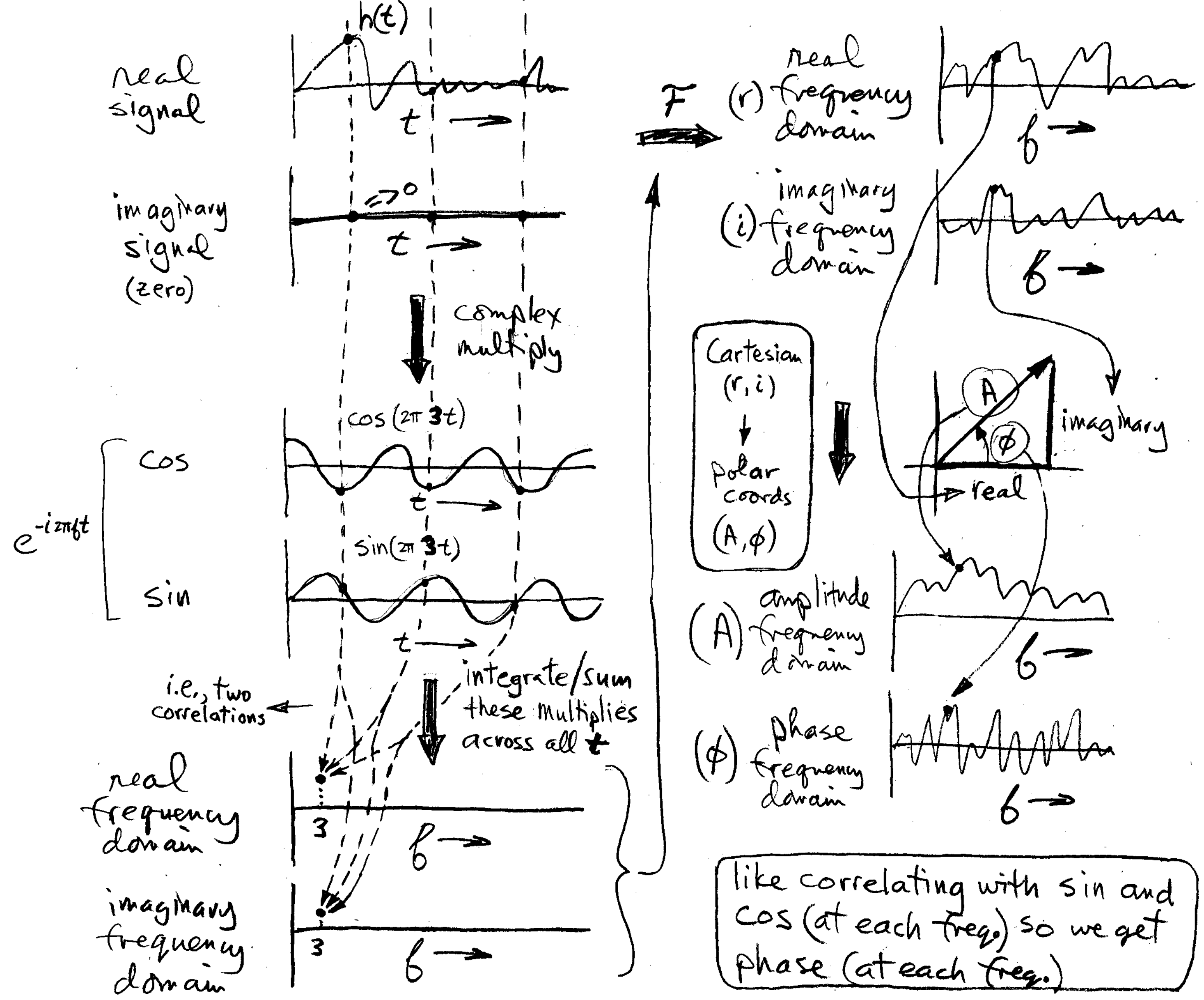
$h_r(t)\cos(2\pi ft)$,

$-h_r(t)\sin(2\pi ft)$

→ 2 correlations

- How to calculate H(f) for one f (f=3):

(real signal: only need 2 correlations)



like correlating with sin and cos (at each freq.) so we get phase (at each freq.)

Fourier transform (1b)

- neg complex exponents/freq
- orthogonality

$$e^{i\phi} = \boxed{\cos\phi + i\sin\phi}$$

$$e^{-i\phi} = e^{i(-\phi)}$$

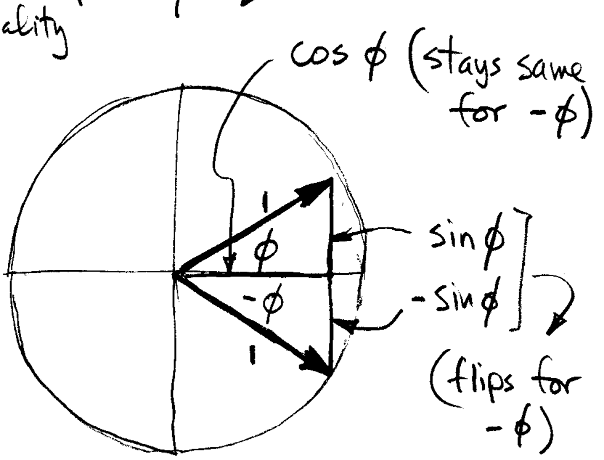
no effect (see below)

move outside

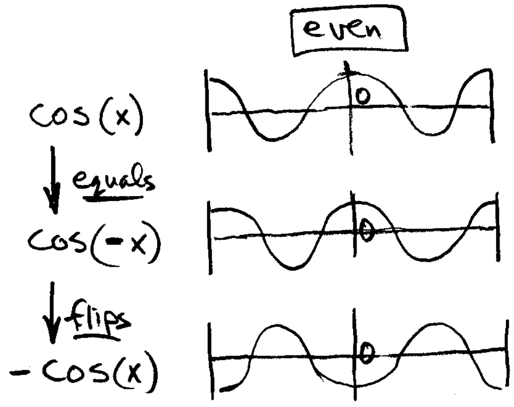
$$= \cos(-\phi) + i\sin(-\phi)$$

$$= \boxed{\cos\phi - i\sin\phi}$$

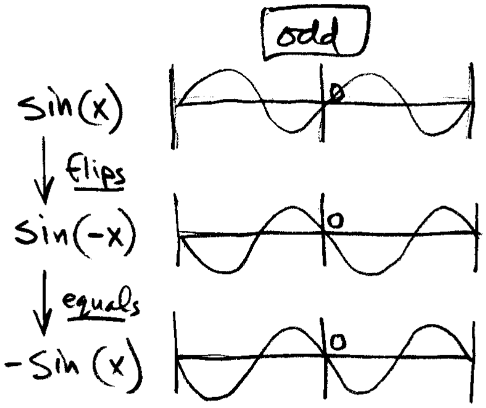
graphically ⟹



- **cos** is an "even" function, **sin** is an "odd" function



if a function is mirror-symmetric along the x-axis around zero: $f(x) = f(-x)$ it is even

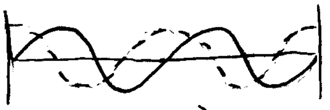


An orthogonal decomposition

- think of discretely sampled $\sin(fx)$, $\cos(fx)$ as vectors
- $\mathrm{Corr}(\vec{v}_1, \vec{v}_2) \equiv$ projection of v_1 onto v_2 $\equiv \vec{v}_1 \cdot \vec{v}_2$

$$\mathrm{Corr}(\cos f_1 x, \sin f_1 x) = 0$$

= sin & cos of same frequency are orthogonal



$$\mathrm{Corr}(\sin f_1 x, \sin f_2 x) = 0$$

= different integer freqs of sin (or cos) are orthogonal

$$\mathrm{Corr}(\cos f_1 x, \sin f_2 x) = 0$$

[as above]

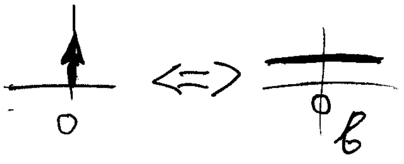
- in the continuous case, orthogonal functions defined as: $$\int_{x=lo}^{x=hi} f(x)\, g(x)\, dx = 0$$

fourier-1c

UNDERSTANDING INVERSE FOURIER TRANSFORM AS ANOTHER CASE OF CORR W/ COS, SIN

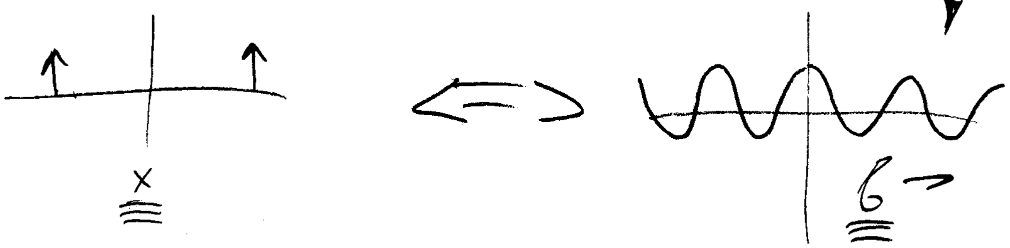
- start with spike in image domain
- take example of spike at $x=0$

[$\cos(x)$, $\cos(2x)$, $\cos(kx)$ all equal 1 there
all freq's correlate w/ spike at $x=0$

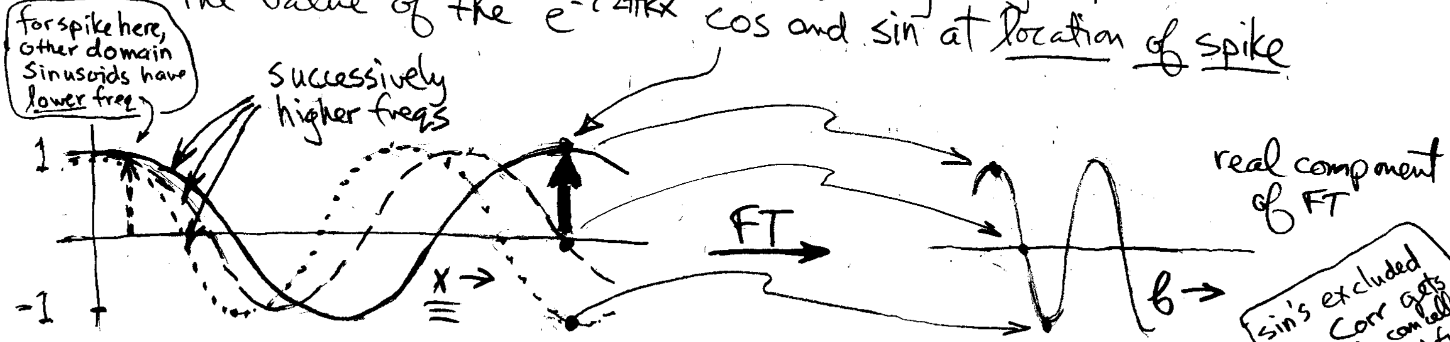


- if spike is moved away from zero, the frequency spectrum obtained by correlating cos with spikes oscillates

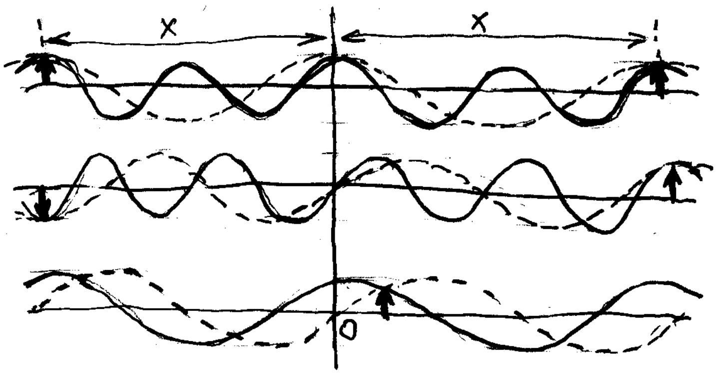
N.B. opposite direction sin spike are on imaginary axis



- to see why this is, since spike is all zero except at spike, the dot product for a given frequency only depends on the value of the $e^{-i2\pi kx}$ cos and sin at location of spike



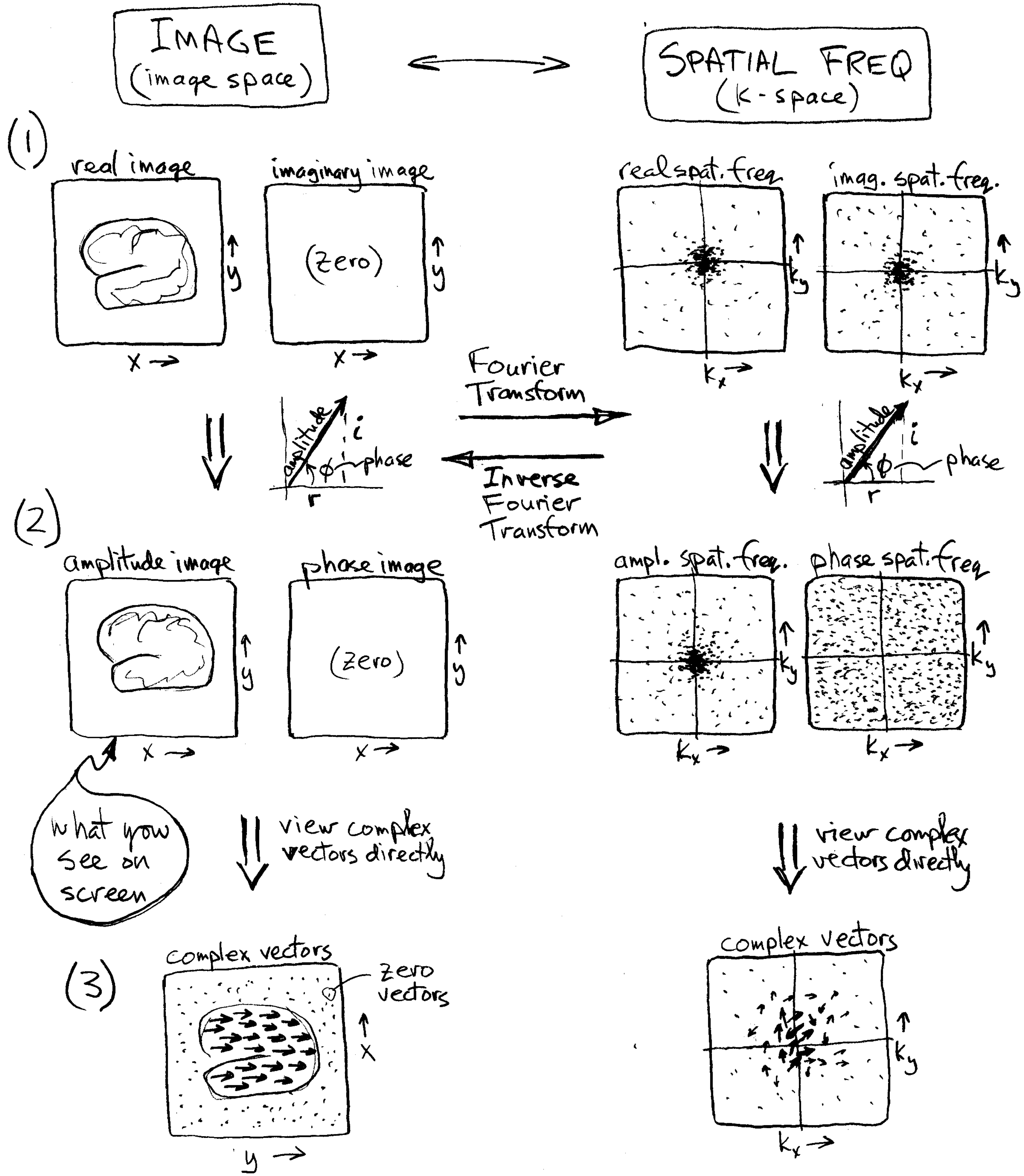
sin's excluded sin corr gets cancelled exactly w/ odd funct



- pos. pair (real) spikes same dist. from origin → pick just "even" cos's (multiple freqs!)
- pos./neg. pair (imaginary) spikes, samed dist orig. → pick just "odd" sin's (muliple freqs!) (now cancels even funct)
- one spike at distance from origin → pick both sin's and cos's (again oscillates in other domain as above)

→ this is one way of thinking about what one point in k-space means, via correlating it w/ cos's, sin's to get periodic result in image space (inverse FT)

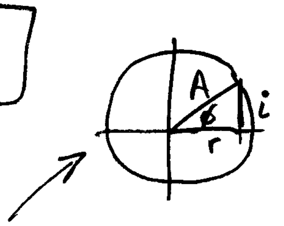
FOURIER TRANSFORM OF AN IMAGE (2)



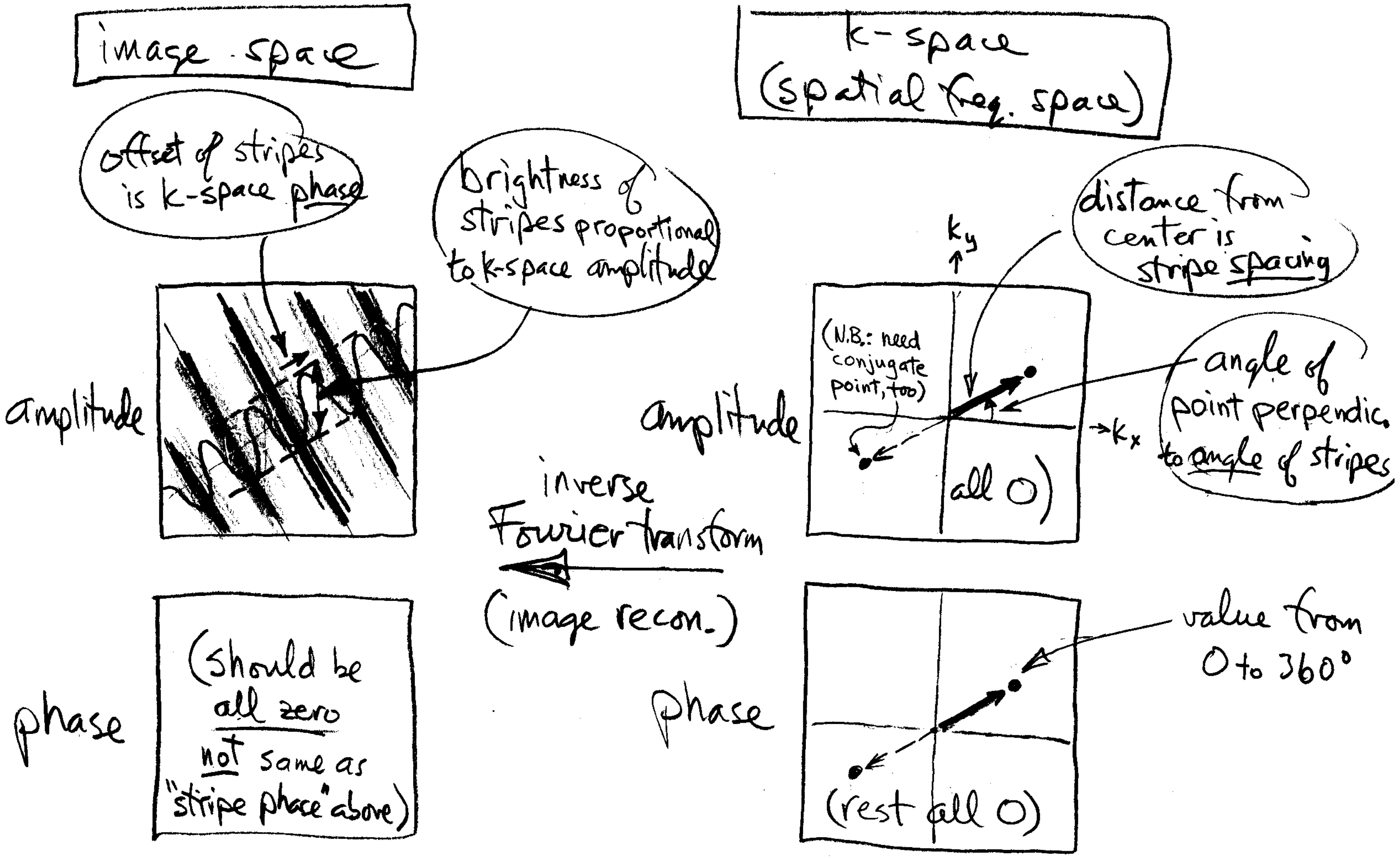
–3 equivalent representations of image & spat. freq. space

Fourier-3

FOURIER TRANSFORM OF REAL IMAGE (3)

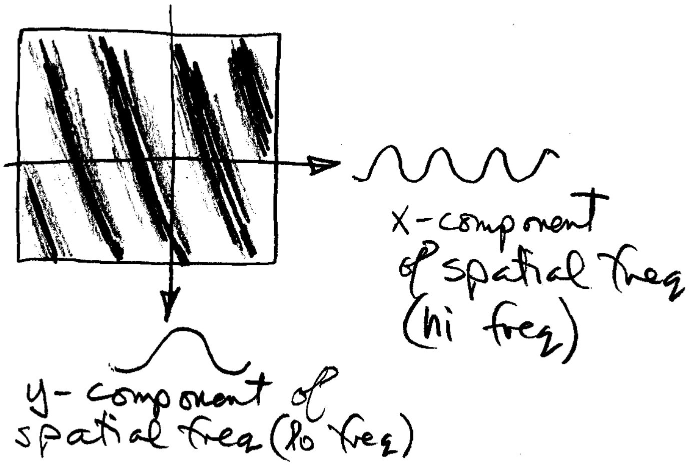


– what a single k-space point (actually pair of points) looks like for real image (polar coordinates A, φ instead of r, i)



– Cartesian dimensions of k-space — x- and y- spatial freq

N.B.: each dimension of spatial freq. space (k-space) from correlation w/ sin & cos — don't confuse k_x, k_y w/ sin, cos!



N.B. – increasing one 1D component increases the spatial freq of the 2D wave and rotates it

FOURIER TRANSFORM OF IMAGE (4)

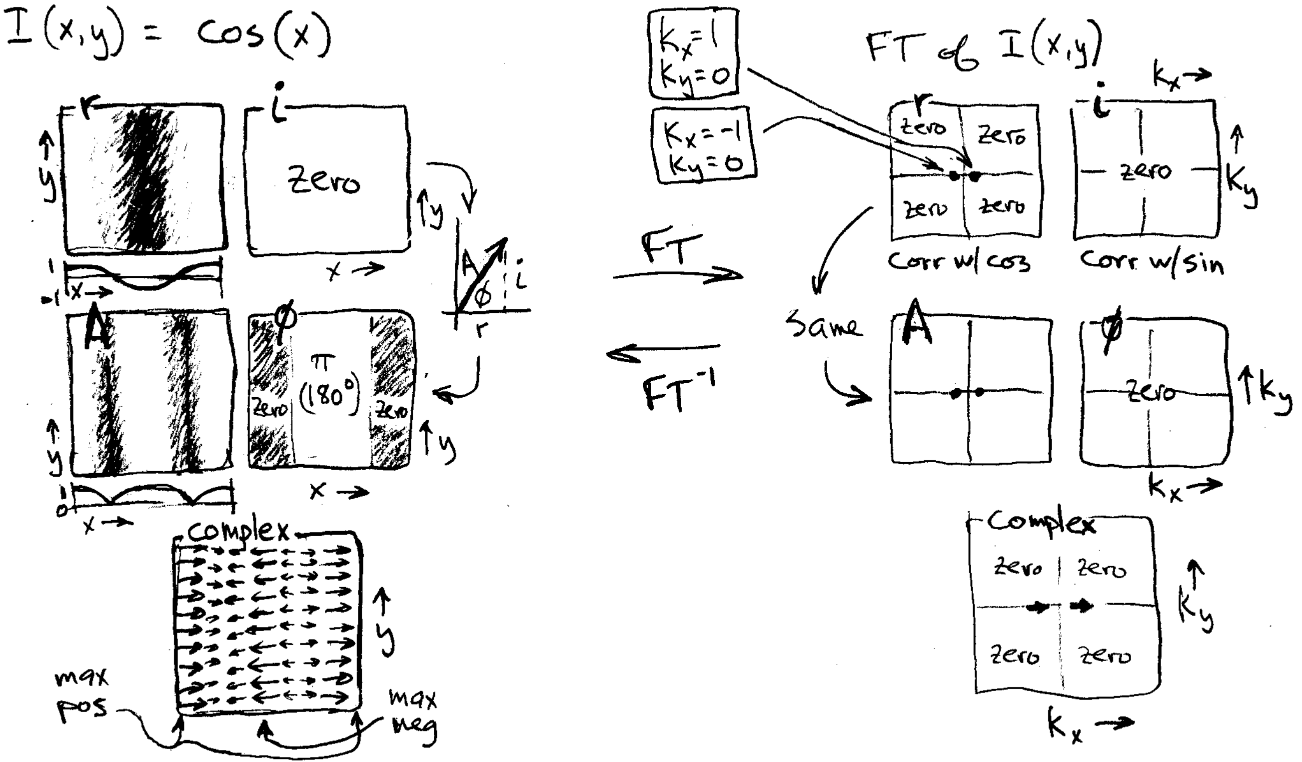
- 3 equivalent representations of complex numbers in image space and spatial-freq. space (k-space)
- example: cosinusoid in image space, then shifted in x-dir

REAL IMAGE

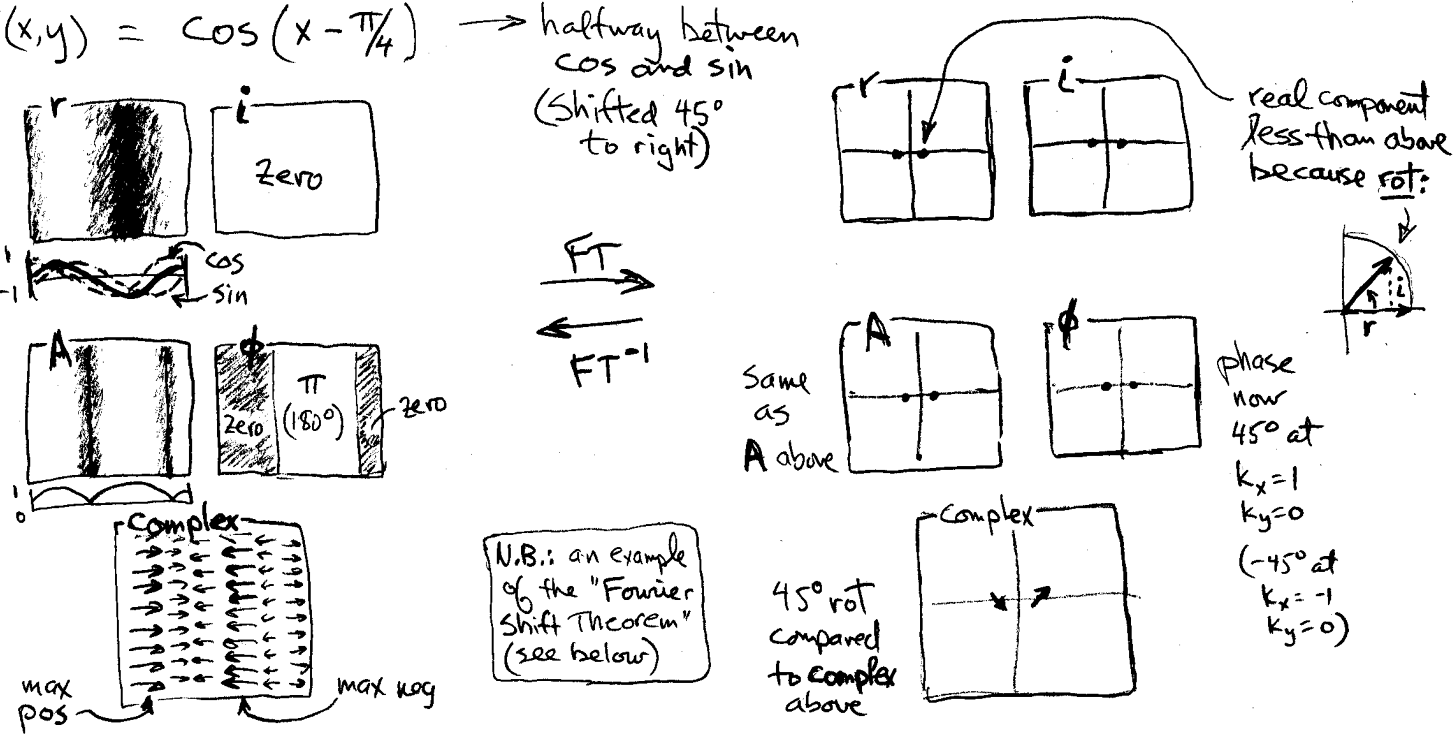
large values only here:

FT OF REAL IMAGE

I(x,y) = cos(x)



I(x,y) = cos(x − π/4) → halfway between cos and sin (Shifted 45° to right)



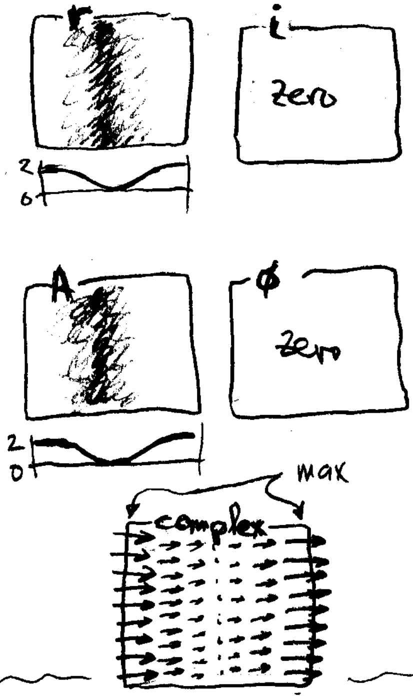
N.B.: an example of the "Fourier Shift Theorem" (see below) fourier-3c

FOURIER TRANSFORM OF IMAGE (5)

- (cont.) center of k-space (real image)
- complex image

REAL IMAGE

$I(x,y) = \underline{\underline{1}} + \cos(x)$



center of k-space:

$$H(\overset{0}{k}) = \int_x h(x)\cdot \underbrace{e^{-i2\pi kx}}\,dx$$

$\Downarrow$ avg image brightness $\Leftarrow$ 1 (real)

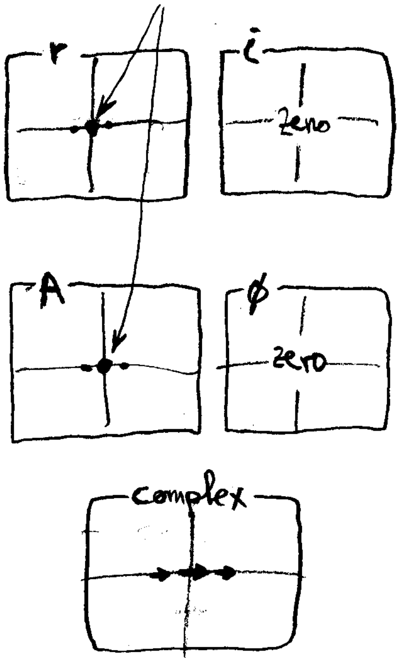
$\xrightarrow{FT}$

$\xleftarrow[FT^{-1}]{}$

[the center of k-space is zero w/ pure sin or cos image b/c avg. brightness = 0]

FT OF REAL IMAGE

positive center k-space



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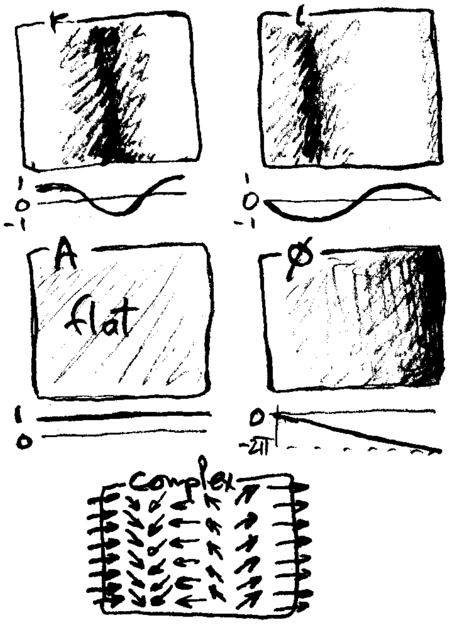
*COMPLEX IMAGE

$I(x,y) = \cos(x) - i\sin(x)$
$= e^{-ix}$



$\xleftrightarrow{FT,\ FT^{-1}}$

("missing" spike results in single spike correlating with cos and sin)

N.B.: this k-space is non-Hermitian:

k-space will only have Hermitian symmetry if image is real:

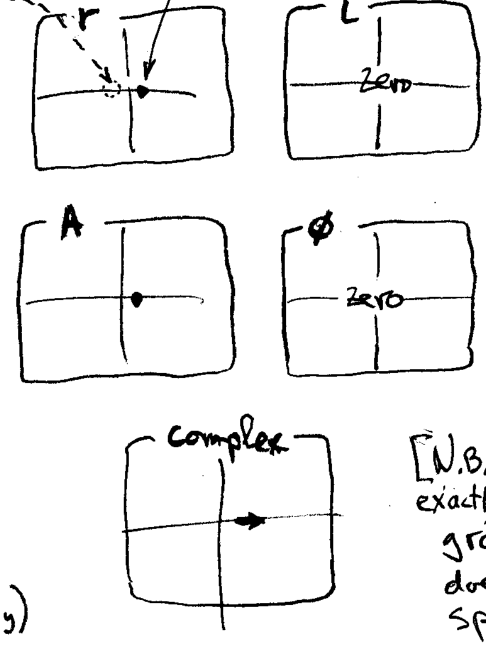
Hermitian symm. when complex conjugate (= complex num w/ sign flipped in imag. part) is equal to funct value w/ neg arg:

1D: $H(-k) = H^*(k)$

2D: $H(-k_x,-k_y) = H^*(k_x,k_y)$

FT OF COMPLEX IMAGE

spike only on one side of k-space



[N.B. this is like what an artifact "spike" does tho it would have rand. phase]

[N.B. this is also exactly what a gradient does to image space!]

FOURIER TRANSFORM OF IMAGE (6)

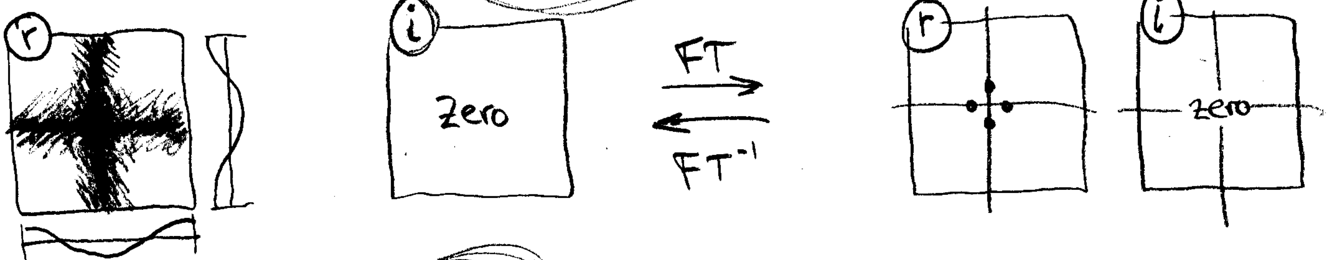
- (cont.) x- and y-spatial freqs.
- special case: real image from sum of reals

REAL IMAGE

FT OF REAL IMAGE

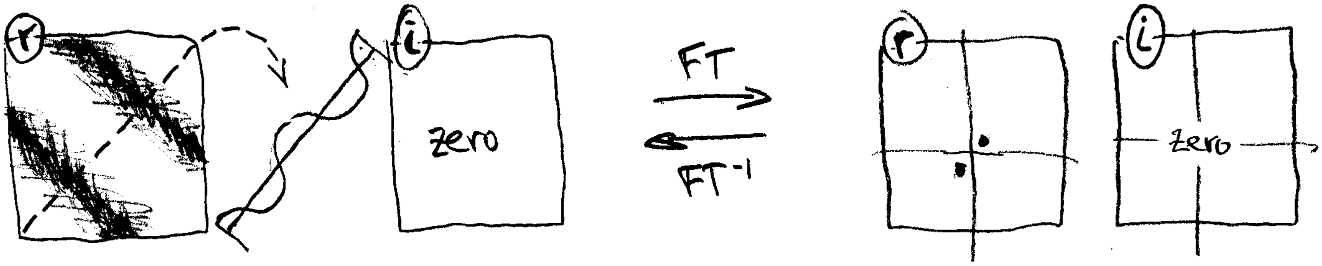
$I(x,y) = \cos(x) + \cos(y)$

N.B. adds but doesn't rotate stripes

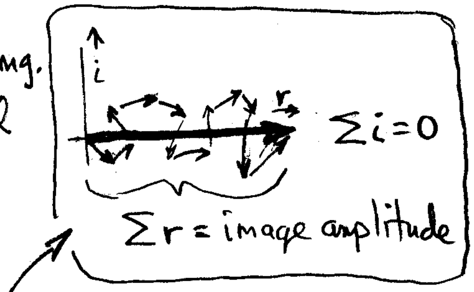


$I(x,y) = \cos(x+y)$

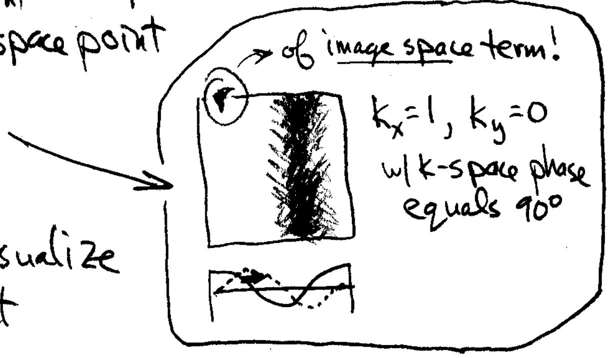
rotates stripes!



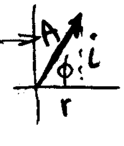
- remember, single k-space point transforms to complex img.
- but if Hermitian symmetry, imaginary components cancel
- Since all we want in image space reconstruction is real component, can just add real components of complex vectors at each image space point for every complex image corresponding to each k-space point



- N.B.: the k-spase phase will affect offset of real valued image space cosinusoid



- therfore for real-valued image, we can visualize inverse FT as real-valued sum of offset real-valued cosinusoids
- N.B. cannot do this with MRI k-space data since phase errors (incl. multiple wraps) mess up real component → must use amplitude img



GRADIENT COILS

- gradient coils for x, y, z generate approximately * a linear gradient in the strength of the $\underline{\underline{z}}$-component of the magnetic field B_z ⟹ all effects of gradients discussed before were in z-direction (parallel to B_0)

- for example, the $\underline{x}$ gradient coil induces a ramp in <u>z-component</u> of the magnetic field when moving in the <u>x-direction</u>:

$$B_{G,\underline{\underline{z}}} = G_x x$$

*- since a pure linear gradient of $B_{G,\underline{\underline{z}}}$ in <u>only</u> the x, y, or z directions is not possible according to Maxwell equations, each gradient coil also induces a magnetic field that has components in the x- and y-direction ($B_{G,\underline{\underline{x}}}$ and $B_{G,\underline{\underline{y}}}$)

- the other magnetic field components are (2 to 3 orders of mag. less) usu. ignored because they are so small relative to $B_{G,z}$, since $B_{G,z}$ is <u>added</u> to B_0, and since B_0 is <u>much</u> stronger than $B_{G,z}$, $B_{G,y}$, & $B_{G,z}$

- since standard reconstruction methods assume the existence of "non-Maxwellian" gradient fields, spatial distortion is introduced

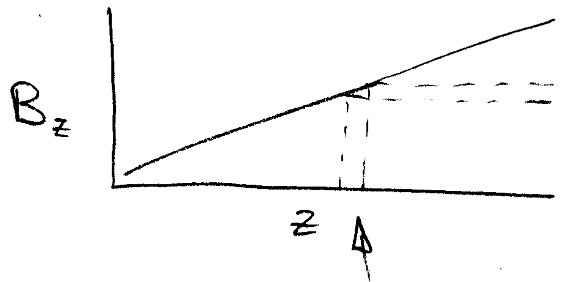
- the Maxwellian terms $B_{G,x}$ $B_{G,y}$ are known; <u>can</u> be included in the re'con. process →

e.g., extra phase in z-dir ⊥ to x-grad

$$\Delta\phi\, G_x(\vec{x}) \approx \frac{-\gamma z^2 G_0^2 t}{2B_0}$$

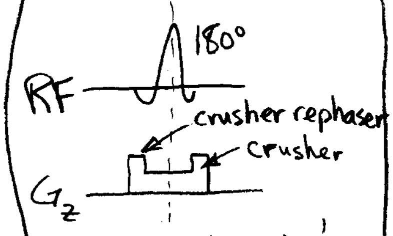
SLICE SELECTION (G_z)

- slice select gradient on <u>during RF stim</u>



$$f = \overset{\frac{\gamma}{2\pi}}{\gamma}\left(B_0 + \overset{G_z z}{B_{G,z}}\right)$$

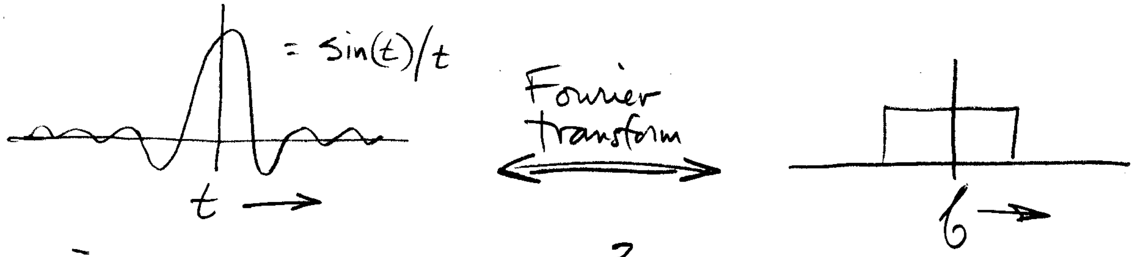
- protons here can only be excited by a narrow band of radio frequencies



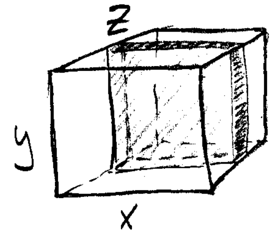
- 180° pulses don't need rephasing (self-rephasing)
- crusher lobes added to kill spurious transvese

- to apply a pulse containing a narrow band of frequencies, we use a <u>sinc</u> pulse envelope (Fourier transform of a narrow freq. band)

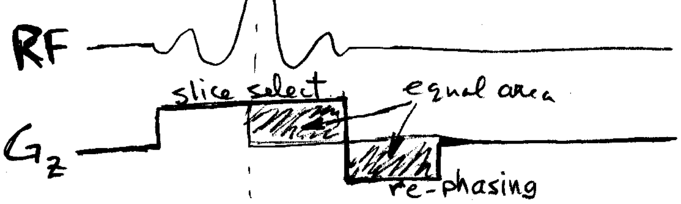
in practice, Gaussian pulse envelope good, too



- this excites protons in a narrow slab



- since the slice-selection gradient introduces (space-dependent) phase shifts (see freq encode) these have to be removed by a post-excitation rephasing z-gradient



- approximation from assuming tip occurs instantaneously in middle
- valid for small tip: 90° → 52%
- in practice: adjust to max, use crusher to kill spurious transverse <u>on 180°</u>

slice-3

PULSES FOR SLICE SELECTION

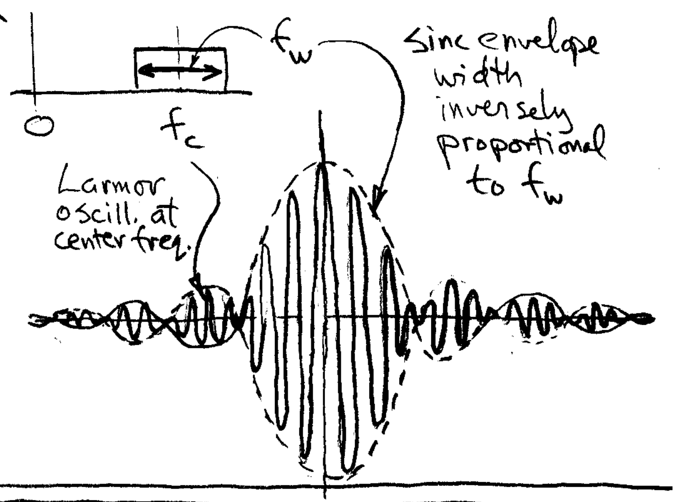
Fourier approach details

- Fourier transform approach to slice-selective pulse (linear approx. even tho tipping is non-linear)

$$\underbrace{\vec{B1}(t)}_{\substack{\text{time dependent}\\\text{RF stimulation}\\\text{(complex)}}} \propto \int_{f=-\infty}^{f=\infty} \underbrace{p(f)}_{\substack{\text{frequency}\\\text{selection}\\\text{function}}} \cdot e^{-i2\pi t f} df$$

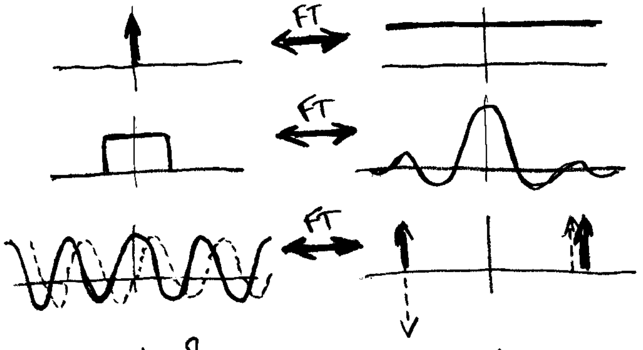
i.e., time-dependent complex (=quadrature) pulse waveform in Fourier transform of frequency spectrum of RF pulse

Solve with: p(f) = frequency band:



$$\vec{B1}(t) = \underbrace{A}_{\substack{\text{amplitude}\\\text{controlling}\\\text{flip angle}}} \cdot \underbrace{f_w}_{\substack{\text{freq}\\\text{width}\\\text{(controls}\\\text{slice width)}}} \cdot \underbrace{\text{sinc}(\pi f_w t)}_{\substack{\text{sinc envelope}\\\text{determined by}\\\text{freq. width, } f_w\\\text{(N.B. wider } f_w\\\text{is narrower sinc)}}} \cdot \underbrace{e^{-i2\pi t f_c}}_{\substack{\text{modulation}\\\text{(complex)}\\\text{at center}\\\text{freq., } f_c}}$$

Fourier Transform Pairs, Rules



- multiplication in one domain equals convolution in other:

$$\mathbf{F}[g(t) \cdot h(t)] = G(f) \circledast H(f)$$

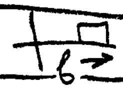
means do FT

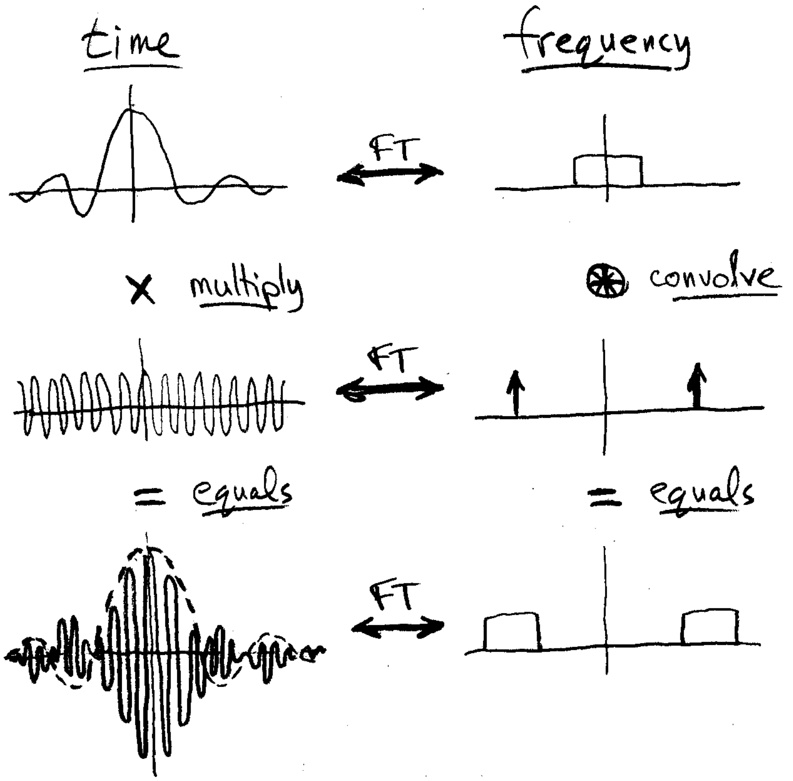
FT

FT

- convolution with delta funct impulse moves other function to impulse center

Fourier Transform Solution to:





slice-4

SLICE SELECT RF PULSES

Interleaved Acquisition

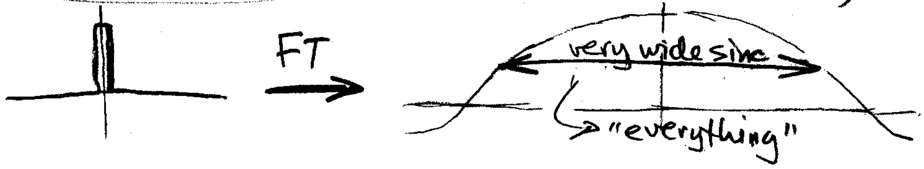
⇒ better S/N b/c imperfect slice profile
→ spin history prob if motion

Common RF pulses

pulse envelope

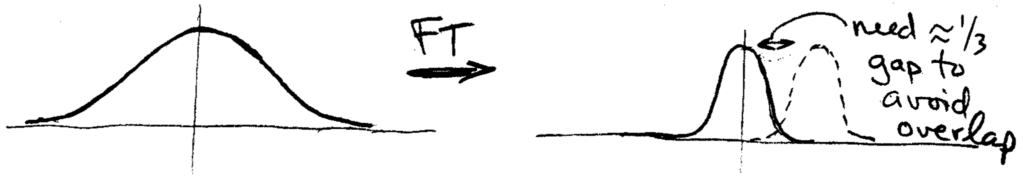
freq spectrum (≈ slice profile)

non-selective pulse ("hard" pulse)

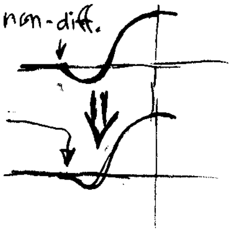


standard slice select sinc

Gaussian



↳ pulses need to be "apodized" (have "foot" removed)
↳ multiply by function so begin/end of pulse is differentiable

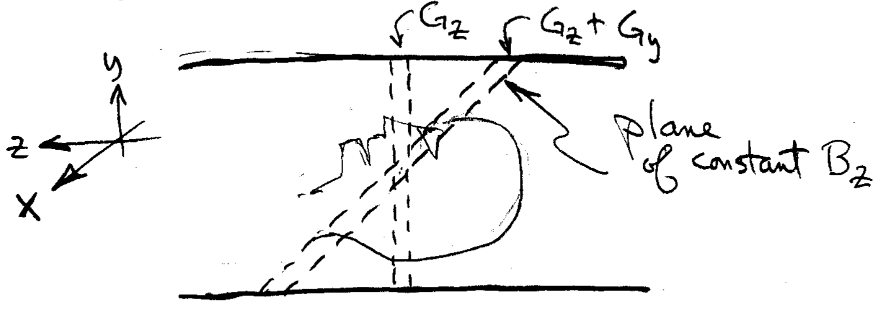


Fat Saturation

- fat protons have chemical shift causing resonant freq offset
- add phase offset not due to gradients, RF
- fix by off-water-resonance 90° (saturation) pre-pulse centered on fat freq
  ↳ need high quality (narrow freq) pulse to avoid saturate water!

HOWTO
1) fat sat pulse
2) wait T2 so fat signal decays, but no T1 regrowth of fat
3) RF stim for water "protons-of-interest"

Adding Another Gradient Tilts Slice



- with 3 gradients on, can excite arbitrary angle plane
- translate plane by changing either gradient amplitude or RF freq band:

freq.encode-0

WHY "FREQUENCY-ENCODING" IS A MISNOMER

- comes from original analogy in Lauterbur (1973):

Spectroscopy	Imaging
1) <u>chemical shift</u> change freq →	<u>gradient</u> changes freq.
2) stimulate w/ broadband RF →	same
3) time-sample FID containing multiple freqs →	same
4) FT of FID to get spectrum of Δf offsets →	FT of FID to get Δx offsets

- this is techically correct (FT of FID) but <u>highly</u> misleading
  - e.g., phase-encoding (turning a different gradient ON and OFF <u>before</u> recording FID) seems to be something completely different since OFF gradient can't affect <u>freqs</u> in FID
- the "k-space" perspective is a "Copernican turn"
  - idea is that data is <u>not</u> a set of samples of a <u>time</u> <u>domain</u> signal generated by multiple chemical-shift like frequencies
  - → rather, it is a set of samples of a <u>frequency</u>-<u>domain</u> signal, each sample generated by multiple spatial locations, (which are analogous to multiple time points)

N.B.: sometimes useful to go back to orig spectroscopic analogy for debugging noise sources since freq encode lines still FT (or FT^{-1}) to spectra

- i.e., the '<u>direction</u>' of the FT (Fourier transform) is <u>reversed</u>:

	signal	result
spectroscopy	samples of oscillations in <u>time</u>-<u>domain</u> $\xrightarrow{FT}$	<u>frequency</u>-<u>domain</u> spectrum of shifts
MRI	samples of spat. freq. in <u>freq.</u> <u>domain</u> $\xrightarrow{FT^{-1}}$	spatial object (like a <u>time</u>-<u>domain</u> signal)

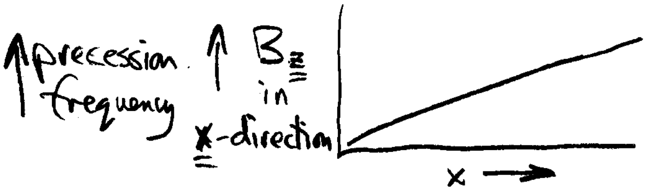
N.B. not full set all freqs like gradient makes!

- the original analogy only 'works' because $FT \approx FT^{-1}$ (except sign change) freq encode-1

FREQUENCY ENCODING (1)

avoid this mistaken intuition!

- frequency encode gradient (G_x) causes precession rates to vary linearly in x-direction



↳ correct (remember that strength of G_x causes variation of slope of B_z in x-direction)

- different frequency signals are mixed together and recorded as a 1-D signal over time

  ↳ correct, but remember, we are recording summed local magnetization vectors after de-modulation

- a Fourier transform, which can convert back and forth between x-position (cf. time) and spatial frequency (cf. temporal freq) is done on signal

  ↳ correct

- spatial frequencies get confused/conflated with precession frequencies

  ↳ wrong !!

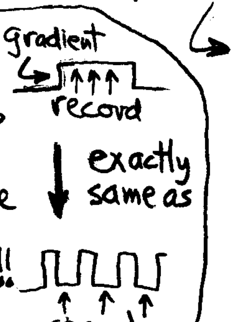
- therefore, the Fourier transform is used to convert position-dependent precession frequencies into spatial position

  ↳ conceptually wrong !! → FT^{-1} (inverse) actually converts spatial frequencies to spatial position

  → the spatial frequency increases for each time point in the readout

  → the precession freq ramp is constant each timestep

** N.B.: freq ramp does not need to be on during recording !!

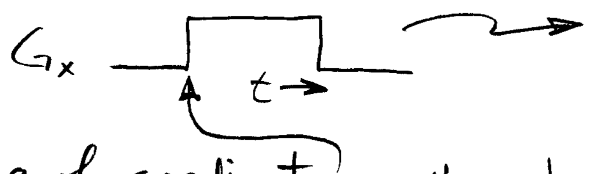


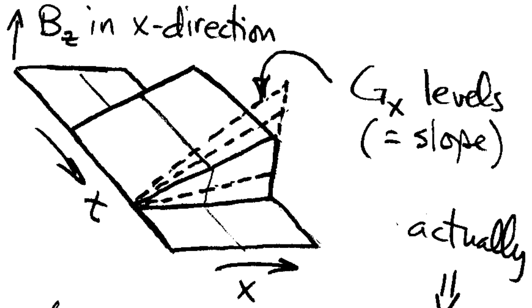
freq encode - 2

FREQUENCY ENCODING (2)

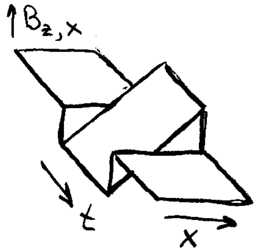
correct intuition — why phase critical

- "frequency"-encode gradient (G_x) turned on during during echo causes precession rates to immediately vary with x-position





actually ⇓

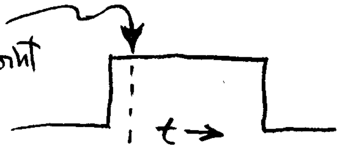


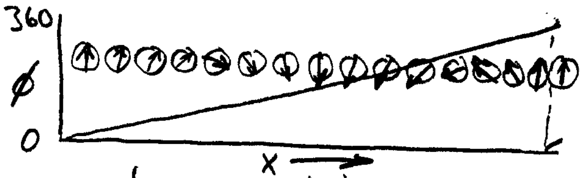
- at beginning of gradient on, the phase of signal coming from each x-position is the same

$\boxed{\text{summed phase angle is what we measure}}$

- early after gradient on, phase advances (because of faster precession frequency) arise with greatest phase advance at largest x-position

single time point early

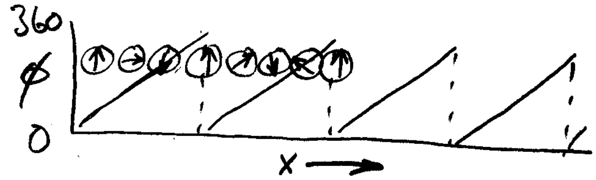




⇒ one cycle of spatial frequency of phase angle (= low spatial freq)

- later during gradient on, phase advances cause multiple wraparounds of phase angle across space

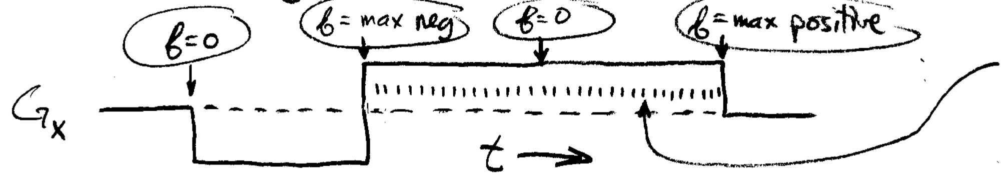
single time point later



⇒ multiple cycles of spatial frequency of phase angle (= hi spatial freq)

- in practice, the lowest spatial frequency (= 0) occurs in the middle of the gradient on time because the phase is "wound" negatively by a preparatory gradient (to the highest negative spatial frequency) before data collection occurs

φ is spatial frequency



individual RF data samples (after demodulation)

FREQUENCY ENCODING (3)

why each datapoint is 1 spatial freq

Standard Fourier transform : (Temporal freq ⟷ time)

$$\underbrace{H(f)}_{\text{freq domain}} = \int_{t=-\infty}^{t=\infty} \underbrace{h(t)}_{\text{time domain}} \cdot \underbrace{e^{-i\,2\pi f t}} \, \underbrace{dt}$$

time domain ↳ r, i=0

↳ [cos 2πft, -sin 2πft

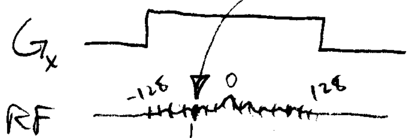
sum across all time

"k" is often used instead of "f" for the frequency variable

Imaging equation : (spatial freq. ⟷ space)

$$\underbrace{S(f)} = \int_{x=-\infty}^{x=\infty} \underbrace{I(x)} \cdot \underbrace{e^{-i\,2\pi f x}} \, \underbrace{dx}$$

$S(f)$ → one data point (i.e., one <u>spatial freq</u>) during readout (2 components)



get this single readout point by summing signal across x-position (RF coil records sum)

even though variable is f, it represents one <u>time point</u> during readout

$I(x)$ → signal strength at one x-position (brightness of image point)

spin density (spectral dens.)

to make image, do inverse Fourier transform of recorded signal $S(f)$

$e^{-i\,2\pi f x}$ → cos $(2\pi f x)$, -sin $(2\pi f x)$

oscillations come from readout phase wrapping, where **f** is single spatial freq (e.g. 5) and **x** goes across object

$f = G_x t$, that is, <u>spatial</u> freq depends on amount of time gradient was on (<u>this</u> **f** <u>increases</u> with time!)

[FID: $G_x t$
SE: $G_x (t-TE)$

<u>don't</u> confuse with instantaneously changed <u>precession</u> freqs which are <u>constant</u> across <u>entire</u> readout time (for each **x** position)

dx → Sum across x of object (x = ∞)

this is done by RF coil recording sum

ALTERNATE DERIVATION (incl. effects of G_x) SIGNAL EQ

- oscillators at $\omega = \gamma B$ at each position (just x for now)

$$S(t) = m(x)\, e^{-i\phi(x)}\, dx$$

- by definition, freq, ω is rate of change of phase, φ

$$\frac{d\phi(x,t)}{dt} = \omega(x,t) = \gamma B(x,t) \quad \underline{\text{and}}_{\text{integrating}} \quad \phi(x,t) = \int_0^t \omega(x,t)\, dt = \gamma \int_0^t B(x,t)\, dt$$

- assuming phase initially 0, B affected by gradients

$$B(x,t) = B_0 + G_x(t) \cdot x$$

$$\phi(x,t) \overset{\text{so}}{=} \gamma \int_0^t B_0\, dt + \left[\gamma \int_0^t G_x(t)\, dt\right] x$$

$$= \omega_0 t + 2\pi k_x(t)\, x$$

k is time integral of gradient waveform

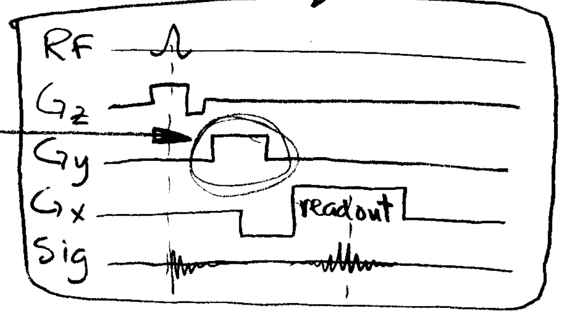
- demodulation removes the B_0-caused carrier frequency $e^{-i\omega_0 t}$ from the first equation

$$S(t) = \int_x \underbrace{m(x)}_{\substack{\text{amplitude}\\ \text{of each}\\ \text{oscillator}}} \underbrace{e^{-i 2\pi k_x(t) x}}_{\substack{\text{gradient-}\\ \text{controlled}\\ \text{phase}}}\, dx$$

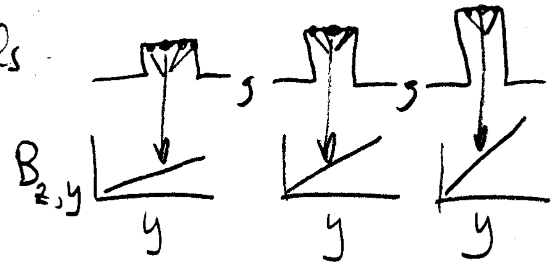
phase-1

PHASE-ENCODE GRADIENT G_y

Pulse Sequence



- Turn on gradient after excitation but before readout
- different levels of G_y



- higher levels of G_y (slope of B_z in y-direction!)

  ↳ higher spatial freq. (more phase wraps) in y-direction
- phase wraps persist after phase-encode gradient off
- read-out gradient (G_x) phase wraps then add to phase-encode phase

2D Imaging Equation

$$k_x = \overset{\text{fixed strength}}{G_x}\,\overset{\text{readout time}}{t} \qquad k_y = \overset{\text{increases each TR}}{G_y}$$

$$\underbrace{S(k_x,k_y)}_{\substack{\text{signal recorded}\\ \text{at single time point}\\ \text{(one readout point)}}} = \underbrace{\int_y\!\!\int_x}_{\substack{\text{sum across}\\ \text{x-y plane}}} \underbrace{I(x,y)}_{\substack{\text{image}\\ \text{(= strength of}\\ \text{magnetization}\\ \text{at each x-y pnt)}}} \cdot \underbrace{e^{-i2\pi(k_x x \boxed{+} k_y y)}}_{\substack{\text{phase (vector of}\\ \text{unit length and particular}\\ \text{angle which is function}\\ \text{of } G_x \text{ and } G_y)}} \, dx\,dy$$

⇓ complex signal (from phase-sensitive detection)

⇓ done by RF coil

⇓ scalar (what we try to reconstruct)

⇓ phase angle (of scalar magnetization!) in rotating frame, set by gradients

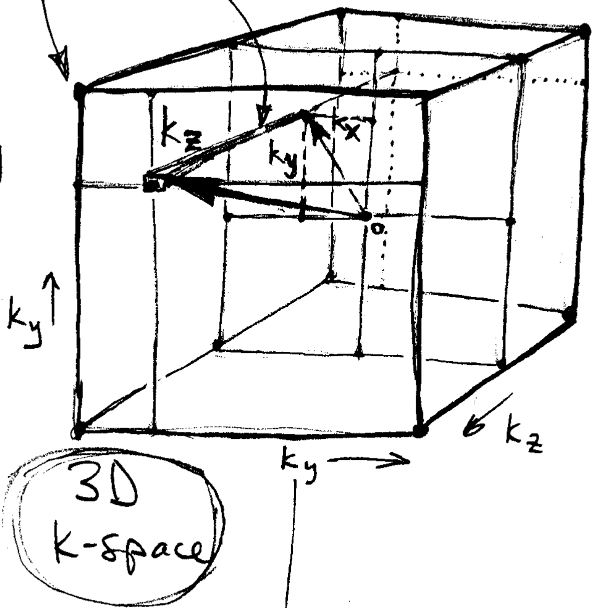
- ignoring relaxation, spatial frequency k_x and k_y have no "inertia" — they stay wherever the gradients last left them phase -2

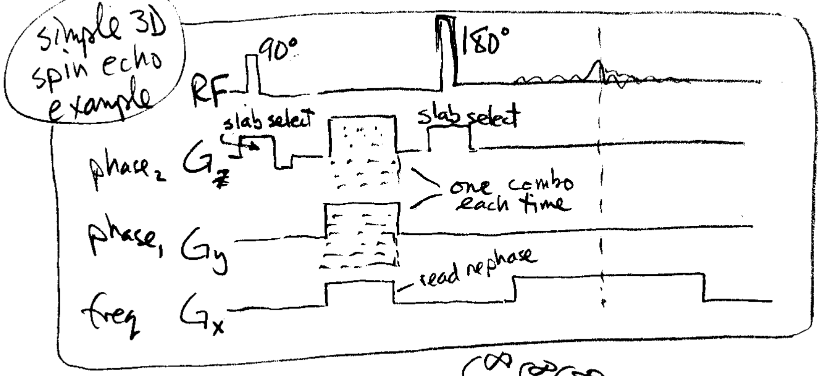
3-D IMAGING – two phase-encode gradients

- use z-gradient for 2nd phase-encoding instead of slice selection
- excitation of whole slab (slice-select is whole brain)
- simple spin echo example (in real life, usu. done with echo trains [FSE] or small flip angle to allow short TR [SPGR]

total spatial freq. greater than in any 2-D k-space point (really a sphere is optimal)

2nd phase encode add (onto orig freq encode phase)





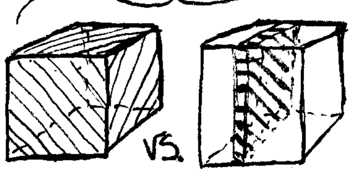
$$\underbrace{S(k_x, k_y, k_z)}_{\text{signal recorded at single point of readout}} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I(x,y,z)\, e^{-i2\pi(k_x x + k_y y \boxed{+} k_z z)}\, dx\, dy\, dz$$

- i.e., freq-encode phase, first phase-encode phase, and second phase-encode phase all just add (= 3D rotation of phase stripes)

N.B, this ignores relaxation effects for now

- S/N much better than 2-D because each entire excited volume contributes to signal from each pulse instead of just slice
  ↳ phase stripes created throughout volume vs. slice :

one readout point in image space

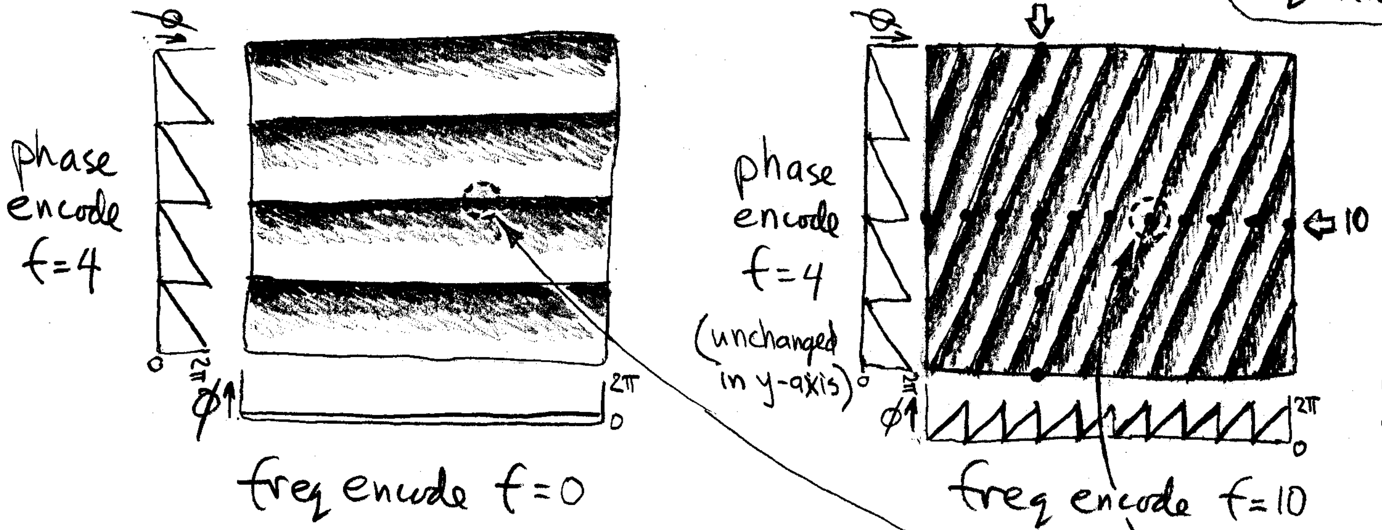


PHASE & FREQ, 2D & 3D

pictures of phase in <u>image</u> space

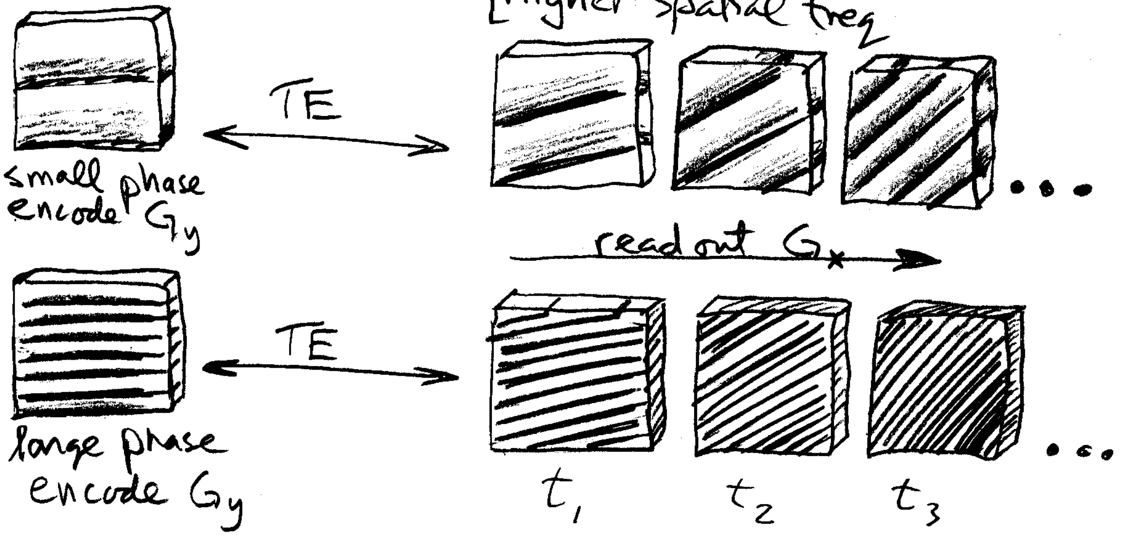
N.B.: stripes have sharp edges from phase wrap (<u>not</u> sinusoid since ϕ from 2-comp quadrature!)

- since the phase-encode gradient and the freq encode gradient both affect <u>phase</u> the result is a <u>rotation</u> of phase "stripes" when the two add



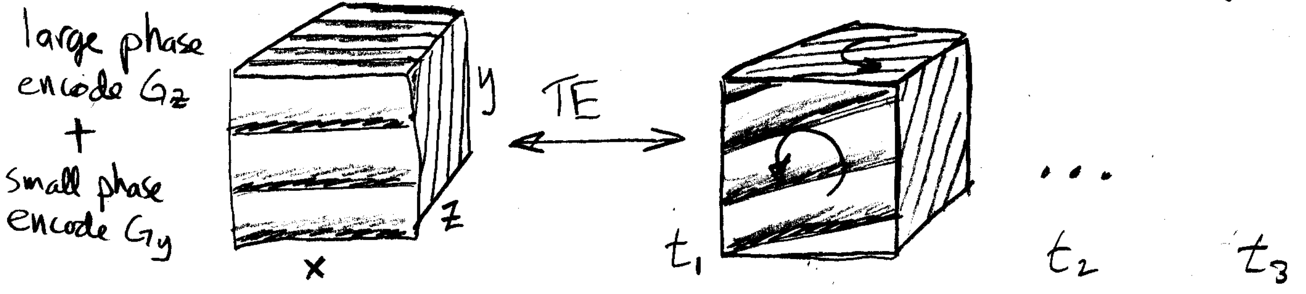
stripes here represent complex value
↓
phase of whole image summed to one (complex) number by RF coils

- successive read out steps: [more rotation [higher spatial freq



e.g., after y-gradient, spins at a point might be 2 cyc ahead while after x-gradient spins at <u>same</u> pnt 8 cyc ahead: <u>but counting wraps</u> in y-direction, still only 2 ahead

- 3D phase encode w/ G_y <u>and</u> G_z starts rotated in y-z plane

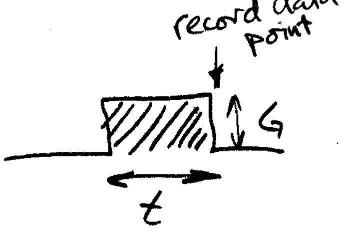


GRADIENTS MOVE K-SPACE LOCATION OF DATA POINT

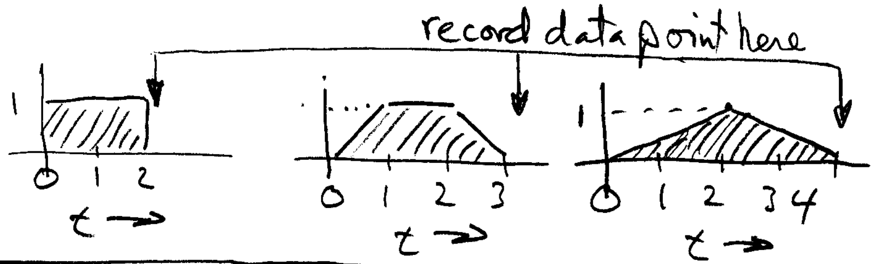
- k-space (spatial-frequency space) location is set by integral of gradient over time up to recording point

$$\underbrace{k}_{\substack{\text{spatial freq}\\ \text{recorded at}\\ t=\text{recordtime}}} = \gamma \int_0^{t=\text{recordtime}} \underbrace{G(t)}_{\substack{\text{gradient}\\ \text{strength as}\\ \text{function of } t}}\, dt \xrightarrow[\substack{\text{form of}\\ \text{integral}\\ \text{w/ boxcar}\\ \text{gradient}}]{\text{simple}} k = Gt \longrightarrow$$

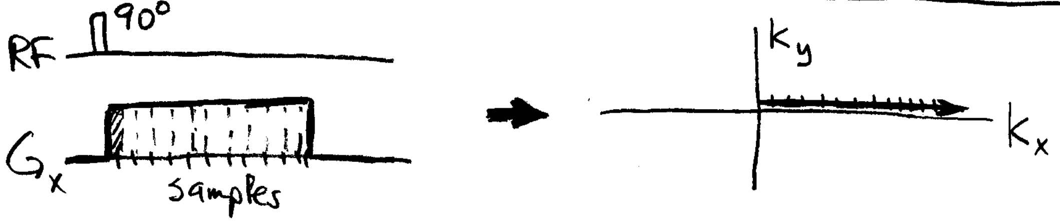
(k is area under curve)



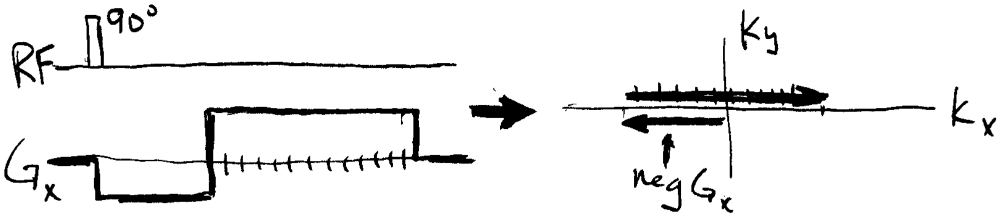
- all of the following gradients end up at the same point in k-space:



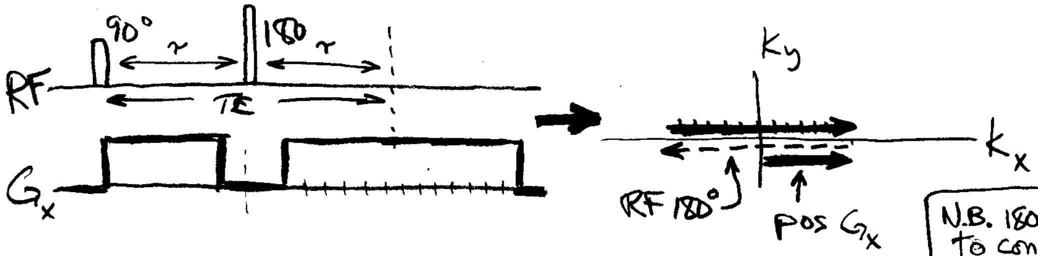
Frequency-encode FID



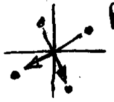
Frequency-encode gradient echo



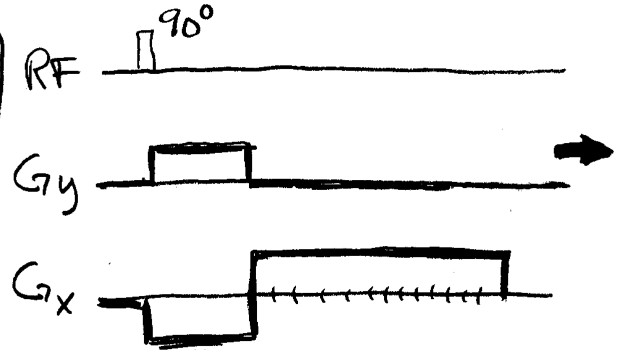
Frequency-encode spin-echo (plus gradient echo!!)



N.B. 180° moves to conjugate pnt.



Phase-encode then frequency encode gradient echo



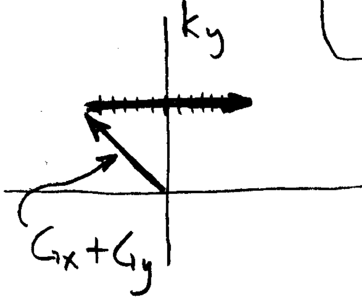


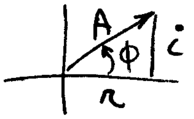
IMAGE RECONSTRUCTION

$$\underbrace{S(k_x,k_y)}_{\text{signal (complex)}} = \underbrace{\iint_{y\;x}}_{\text{RF coil sums}} \underbrace{I(x,y)}_{\substack{\text{brain spin}\\ \text{density (real)}}} \underbrace{e^{\ominus i2\pi(k_x x + k_y y)}}_{\substack{\text{gradient-caused phase wraps (complex)}\\ \text{in transverse magnetization}}} dx\,dy$$

↳ N.B.: assumes perfect sinusoids! (they're not)

$$\underbrace{I(x,y)}_{\text{screen image}} = \iint_{k_y\;k_x} \underbrace{S(k_x,k_y)}_{\text{signal (complex)}} \underbrace{e^{\oplus i2\pi(xk_x + yk_y)}}_{\sin,\ \cos} dk_x\,dk_y$$

ideally → image is real
in practice → complex
↳ use amplitude image:



adding exponents same as multiplying two $e^{i2\pi kx\text{'s}}$

$$= \iint_{k_y\;k_x} S(k_x,k_y)\, e^{i2\pi x k_x} e^{i2\pi y k_y} \boxed{dk_x}\, dk_y$$

same as two sequential 1D FFT's (actual code)

$$= \int_{k_y} \left[\int_{k_x} S(k_x,k_y)\, e^{i2\pi x k_x} dk_x \right] e^{i2\pi y k_y} dk_y$$

- in practice, finite number of samples, N and M, are collected k_x and k_y directions of k-space (integral → discrete sum)

$$\underbrace{I(x,y)}_{\substack{|x| < \frac{1}{\Delta k_x}\\ |y| < \frac{1}{\Delta k_y}}} = \sum_{m=-M/2}^{\overset{\text{b/c M/2 belongs to next replica}}{M/2-1}} \left[\sum_{n=-N/2}^{N/2-1} S(n,m)\, e^{i2\pi \overbrace{n\,\underbrace{\Delta k_x}_{\substack{\text{sampling}\\ \text{interval}\\ \text{in k-space}}}}^{\text{was } k_x} x} \Delta k_x \right] e^{i2\pi m \Delta k_y y} \Delta k_y$$

SAMPLING

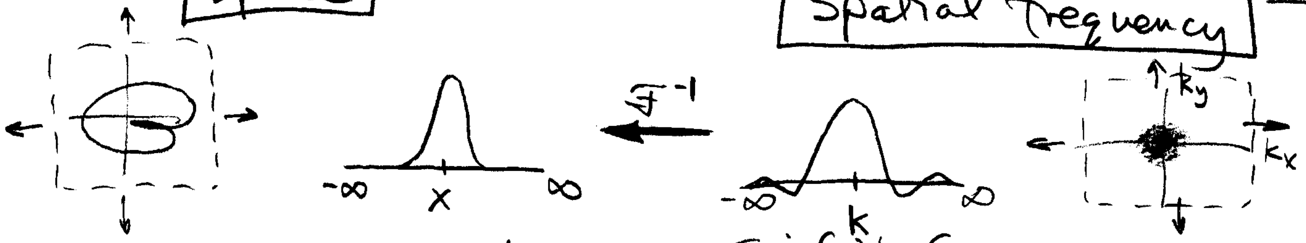
aliasing, FOV

aliasing from insufficient samples in time domain

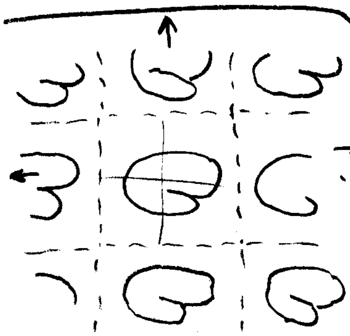
- must consider effects sampling limited points in k-space
  [limited in range of frequencies sampled ($k_{min} \rightarrow k_{max}$)
  limited in rate of sampling (Δk)
- N.B. aliasing less familiar when result of limited frequency domain sampling than limited space or time domain sampling

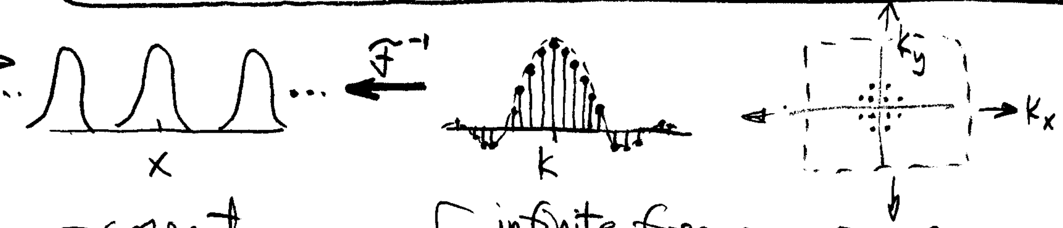
space

spatial frequency



- correct reconstruction ← [- infinite frequency range
- infinitely fine sampling

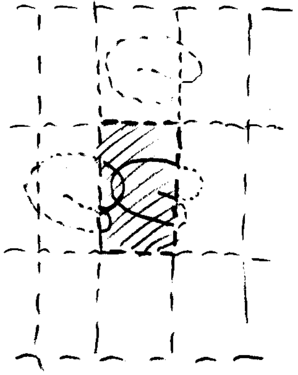


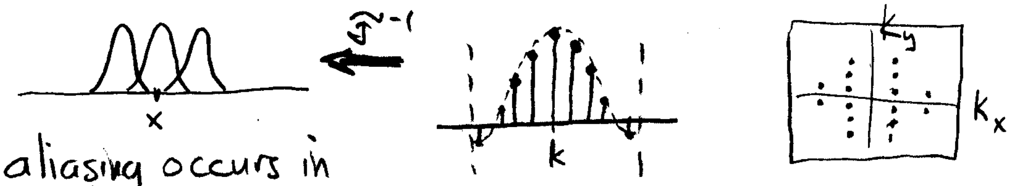


- correct plus replicas ← [- infinite frequency range
- finite spacing of samples

[as above w/ blurring, ringing]

- as above w/ blurring, ringing ← [- finite frequency range
- finite spacing of samples





- aliasing occurs in spatial domain
- replicas overlap, causing wraparound ← [- finite frequency range
- too-wide spacing of samples

thus, finer sampling of same range of spat. freqs increases FOV recon2b

UNDER/OVER SAMPLE

more examples

$$FOV_x = \frac{1}{\Delta k_x}$$

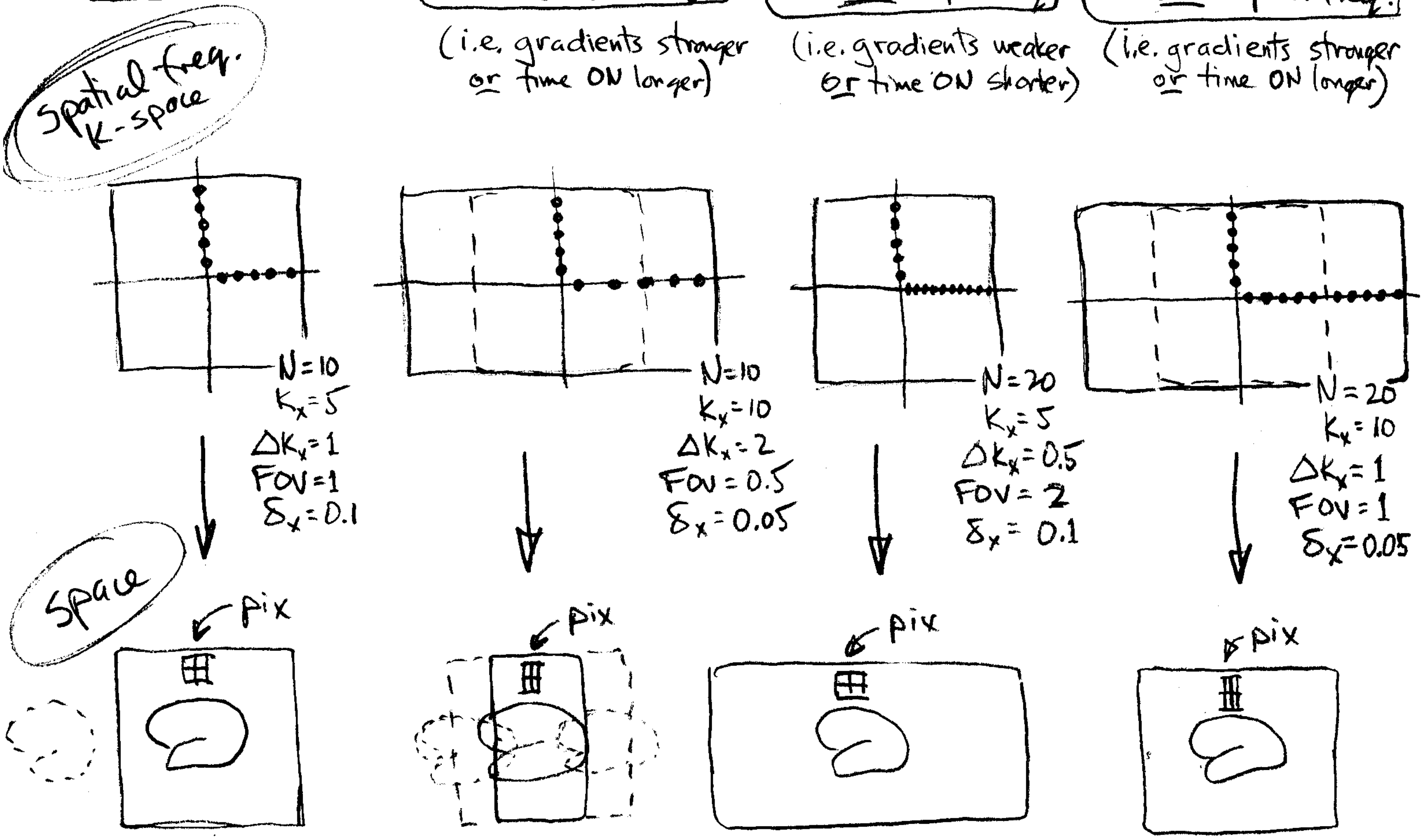
FOV (distance to repeat) is reciprocal of spatial frequency sampling interval

$$\delta_x = \frac{FOV_x}{N} = \frac{1}{N \Delta k_x}$$

pixel size is FOV divided by K-space sample count

3 more examples (not incl. less samples to same spat. freq [bottom last page])

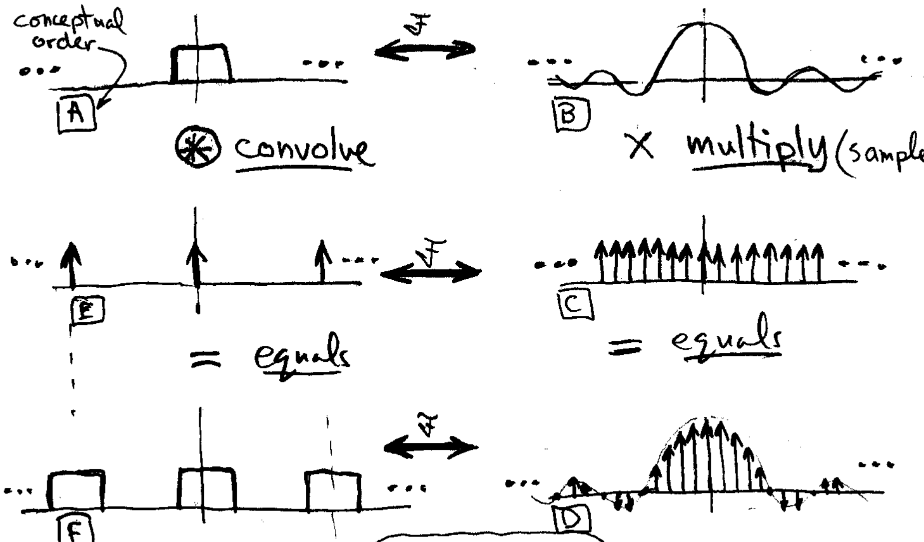
Basic Image	Same num samp. to 2X spat. freq.	2X num. samples to same spat. freq	2X number samples to 2X spat. freq.
	(i.e. gradients stronger or time ON longer)	(i.e. gradients weaker or time ON shorter)	(i.e. gradients stronger or time ON longer)



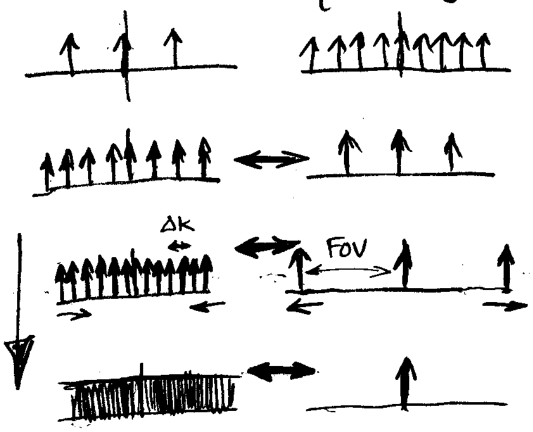
- basic image - square pix	- x-pix half width - replicas intrude [scanner makes square image "wrap" occurs	- square pix - twice x-pix count so FOV = 2X - this is "phase oversamp" [scanner crops to square replicas move out	- X-pix half width - twice X-pix count - same FOV - this is decrease pixel size w/o change FOV

Fourier Transform Solution to Replicas

① image/brain **space** ② sampled data **spatial frequency**



- limit approach to Fourier transform of comb



$$\text{FOV} = 1/\Delta k$$

$$\Delta k = 1/\text{FOV}$$

Useful FTs

See also normalized sinc w/ zero crossings at 1, 2, 3 ...
$= \frac{\sin(\pi k)}{\pi k}$

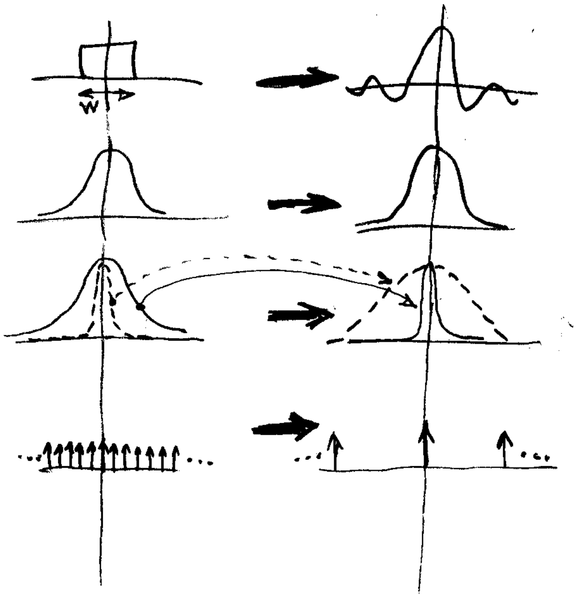
$$\text{sinc}(k) = \begin{cases} 1 \text{ at } k=0 \\ \frac{\sin(k)}{k} \end{cases}$$

rect $\quad \text{rect}\left(\frac{x}{w}\right) \xrightarrow{\mathcal{F}} w \cdot \text{sinc}(\pi W k)$

Gaussian (special case) $\quad e^{-\pi x^2} \xrightarrow{\mathcal{F}} e^{-\pi k^2}$

Gaussian (adj width) $\quad e^{\overset{\text{larger} \Rightarrow \text{narrower}}{-ax^2}} \xrightarrow{\mathcal{F}} \sqrt{\frac{\pi}{a}}\, e^{\frac{-\pi^2 k^2}{a}}$

comb $\quad \sum_{n=-\infty}^{\infty} \delta\left(x - \frac{n}{\Delta k}\right) \xrightarrow{\mathcal{F}} \Delta k \sum_{p=-\infty}^{\infty} \delta(k - p\Delta k)$



recon-4

POINT-SPREAD FUNCTION

$$\hat{I}(x) = \Delta k \sum_{n:\{N/2,\, N/2\}} S(n\Delta k)\, e^{i 2\pi n \Delta k x}$$

- set true image to δ-function, then measured signal is:

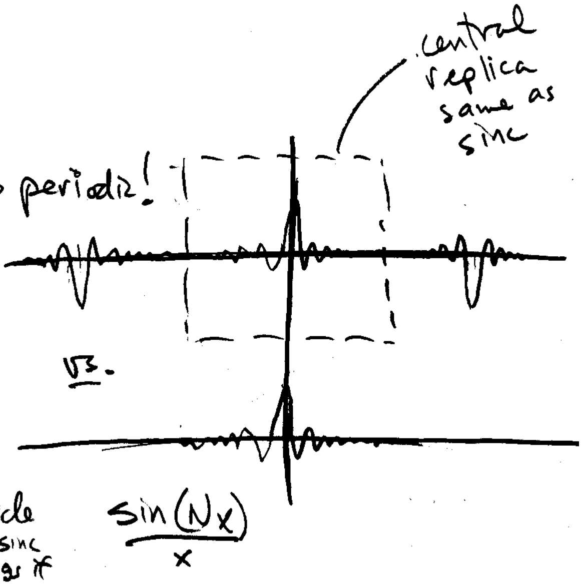
$$S(m\Delta k) = 1$$

- substitute into recon to get PSF:

$$h(x) = \Delta k \sum_{n:\{-N/2,\, N/2\}} e^{i 2\pi n \Delta k x}$$

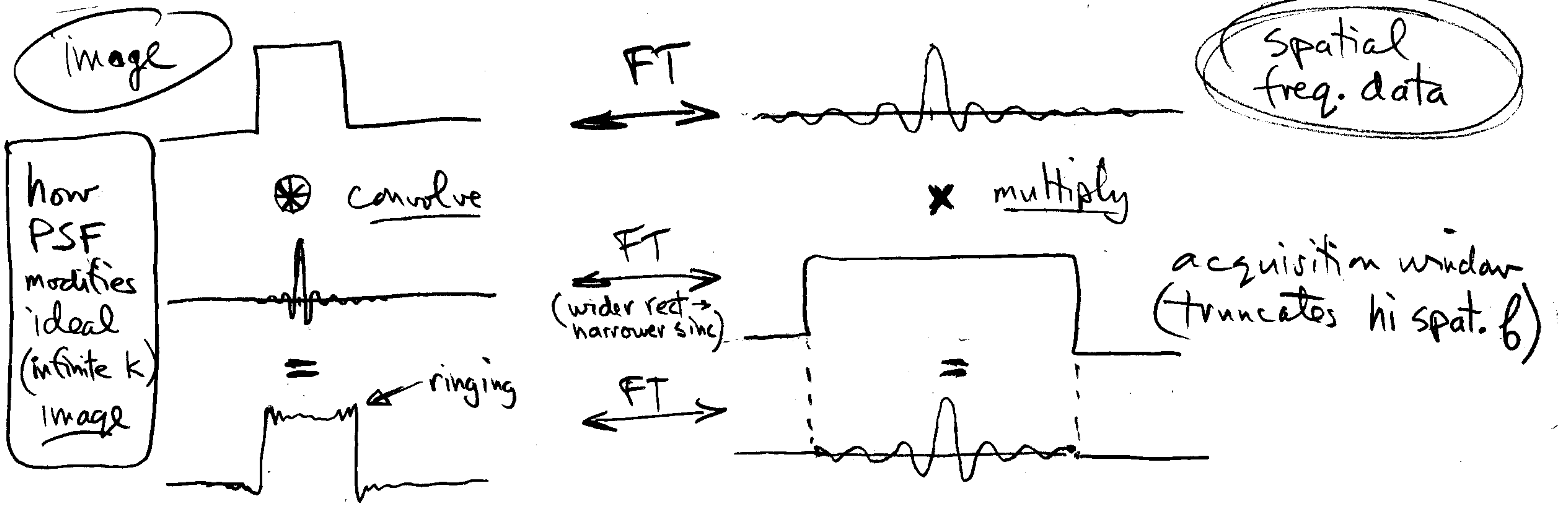
- simplify

$$h(x) = \Delta k \frac{\sin(\pi N \Delta k x)}{\sin(\pi \Delta k x)} \Rightarrow \text{periodic!}$$



- that is, image is reconstructed from a sum of sinc's, because the FT of a boxcar pixel in <u>k-space</u> is an <u>image</u> sinc

all outside center at sinc zero-crossings if $k_{max} = x_{max}$



recon-5

GENERAL LINEAR INVERSE RECON FOR MRI

$$S(k_x) = \int_x I(x)\, \underbrace{e^{-i\,2\pi\, k_x x}}_{\text{perfect spin phase "stripes" modeling real life imperfect ones}} dx$$

Signal eq. → fwd problem

$$I(x) = \int_{k_x} S(k_x)\, \underbrace{e^{+i\,2\pi x\, k_x}}_{\text{perfect "stripes"}} dk_x$$

recon eq. → inv. problem

$$\vec{\mathbf{s}} = \overset{\square}{\mathbf{F}}\vec{\mathbf{i}}$$

$$\begin{bmatrix} \mathbf{s} \end{bmatrix} = \begin{array}{c|c} & x,y \rightarrow \\ \hline k_{xy} \downarrow & \begin{bmatrix} \mathbf{F} \quad \mathbf{i} \end{bmatrix} \end{array} \begin{bmatrix} \mathbf{i} \end{bmatrix}$$

linear "forward solution"
matrix & vectors have complex entries
can build in any measurable priors

$$F_{x,y,t} = \underbrace{g(x,y)}_{\substack{\text{coil gain}\\ \text{at this}\\ \text{location}}}\ \underbrace{e^{-i\phi(x,y)}}_{\substack{\text{coil}\\ \text{phase}}}\ \underbrace{e^{-(nT \pm m\Delta t + TE)/T2}}_{\text{T2 decay}}\ \underbrace{e^{-i\gamma\Delta B(x,y,nT\pm m\Delta t)}}_{\substack{B\phi \text{ error}\\ (x,y \text{ dep.})}}\ \underbrace{e^{-i2\pi(m\Delta k_x x + n\Delta k_y y)}}_{\substack{\text{freq + phase}\\ \text{(complex)}}}\ \Delta x \Delta y$$

one readout (m Δt); EPI (nT); time 1st readout (TE)

could insert x-y-dependent gradient non.-lin.

<u>multi-coil</u>

$$\begin{bmatrix} \mathbf{s} \\ 1 \\ \hdashline 2 \end{bmatrix} = \begin{array}{c|c} & x,y \rightarrow \\ \hline k_{xy}\downarrow & \mathbf{F}\ \text{coil 1} \\ \hdashline k_{xy}\downarrow & \text{coil 2} \end{array} \begin{bmatrix} \mathbf{i} \end{bmatrix}$$

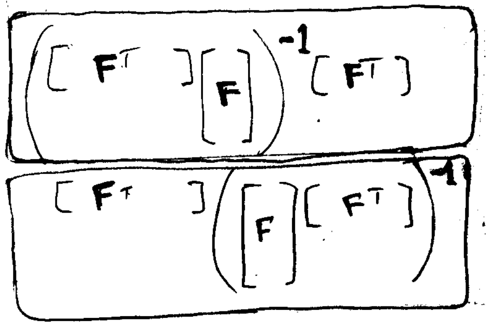
naturally incorporates undistorted field map
different sensitivity function for each coil
contains additional info about source loc.
But, need reference scan, <u>lo-res OK</u>
(need phase corrections for each coil)

$$\mathbf{i} = \mathbf{F}^{+}\mathbf{s}$$ <u>over</u>-determined

Moore Penrose inverse

$$\mathbf{F}^{+} = (\mathbf{F}^T\mathbf{F})^{-1}\mathbf{F}^T \qquad (x\,y)^2 \rightarrow \text{"small"}$$

$$= \mathbf{F}^T(\mathbf{F}\mathbf{F}^T)^{-1} \qquad (x\cdot y\cdot \text{coils})^2 \rightarrow \text{16x bigger for 4 coils}$$



$$\mathbf{i} = \left[(\mathbf{F}^T\mathbf{F})^{-1}\mathbf{F}^T\right]\mathbf{s}$$

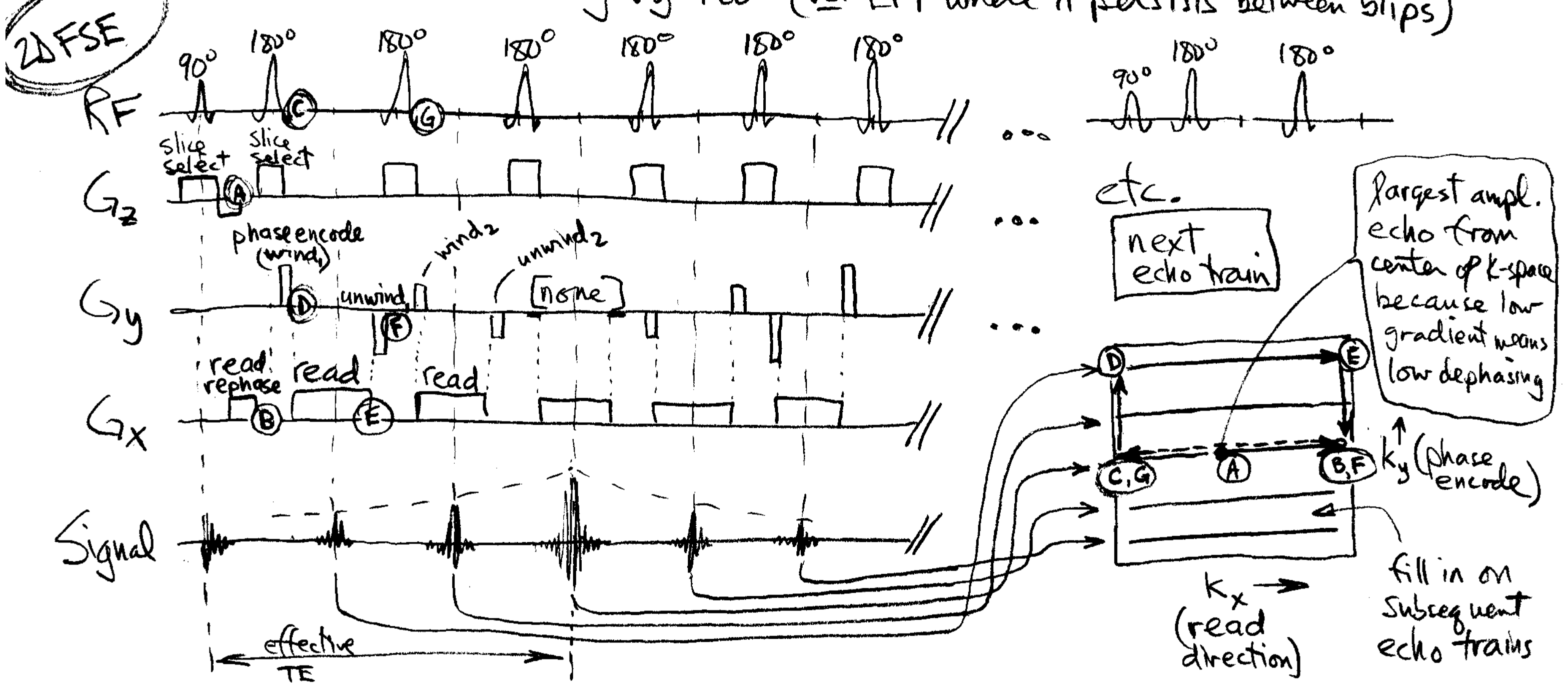
slice-by-slice
assume slice select swamps ΔBφ

FAST SPIN ECHO (FSE)

RARE, FSE, 3DFSE

"echo train"

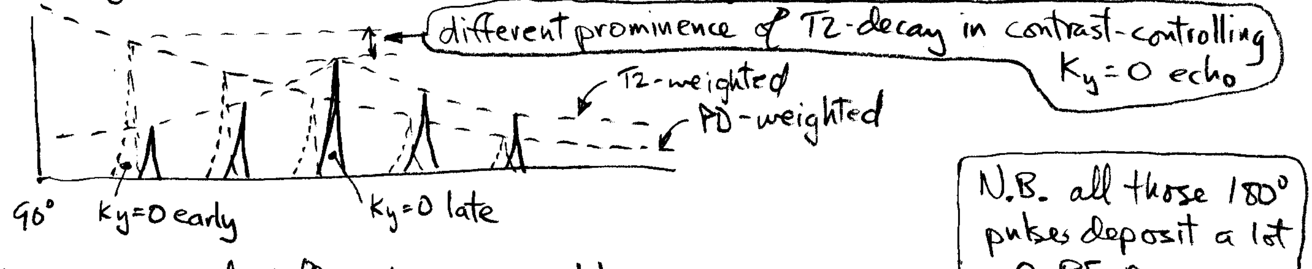
- one 90° pulse followed by *multiple* 180° pulses (e.g., 8) each with a different phase-encode gradient
- each phase "winder" is "unwound" because leftover phase would be re-focused away by 180° (*vs.* EPI where it persists between blips)



- the "effective TE" is the TE when center of k-space is collected (largest effect *on contrast*, *largest echo*)

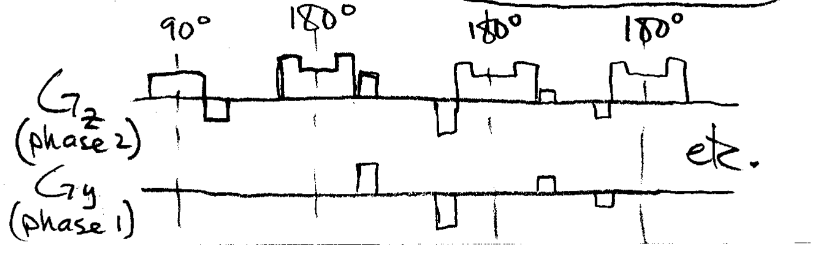
N.B.: only *one* read rephase subsequent 180's reset from right to left

- each subsequent echo has more T2 decay: $E_n = e^{-nTE/T2}$ $n = 1, 2, \ldots M$
- by arranging to collect $k_y = 0$ early, PD-weighted instead of T2-weighted

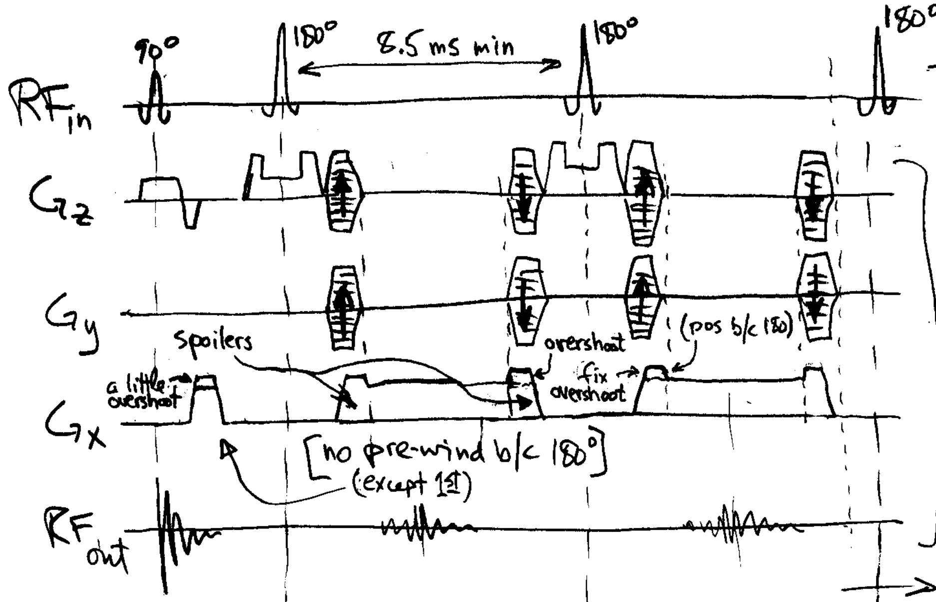


N.B. all those 180° pulses deposit a lot of RF power: 90°+180° = *45x* power 30°

- possible to collect *different* T2-weighting of echoes by estimating T2 curve from $G_y = 0$ echo train
- 3DFSE — like 2D except wind/unwind added to thick slice select (w/crushers on 180°)



MULTI-SLAB 3DFSE, PROBLEMS



→ echo train e.g. 20

→ etc to fill 3D k-space

G_z is "partition"

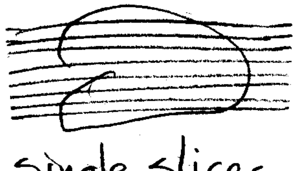
G_y is "phase encode"

G_x readout needs no pre-wind since 180° does it

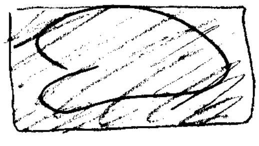
→ TE_{eff} is time from 90° to echo thru center of k-space

- echoes die out quickly by e^{-t/T_2}
- since echoes after 90° limited to < 30, can't fill 3-D k-space in a reasonable time
- SAR constraint $\quad SAR \propto B_0^2 \theta^2 \Delta f$ $\quad$ rough approx. for non-adiabatic standard pulses

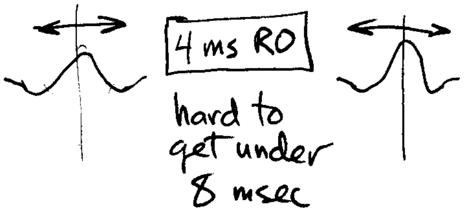
  ↳ 180° pulses deposit 4-6X power of 90°
- "multi-slab" is halfway between slices and single-slab


single slices


multi-slab


single slab

- problem at slice boundaries — esp. movement
- multislab requires slice selective RF pulses → longer than non-selective 'hard' pulses

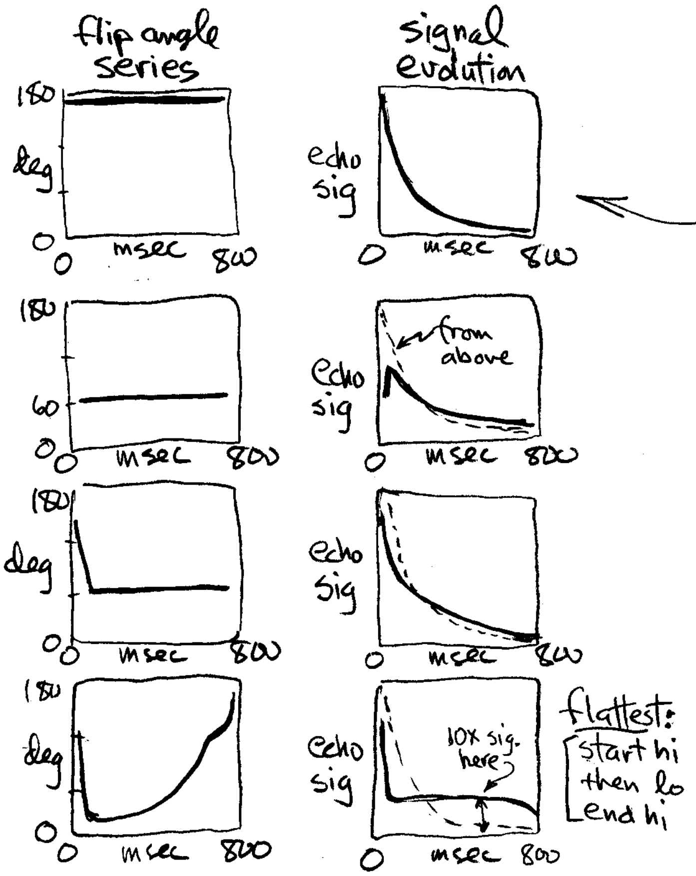


hard to get under 8 msec inter-echo spacing

limits speed of covering k-space

pulse-1c

SINGLE-SLAB 3D FSE (SPACE)



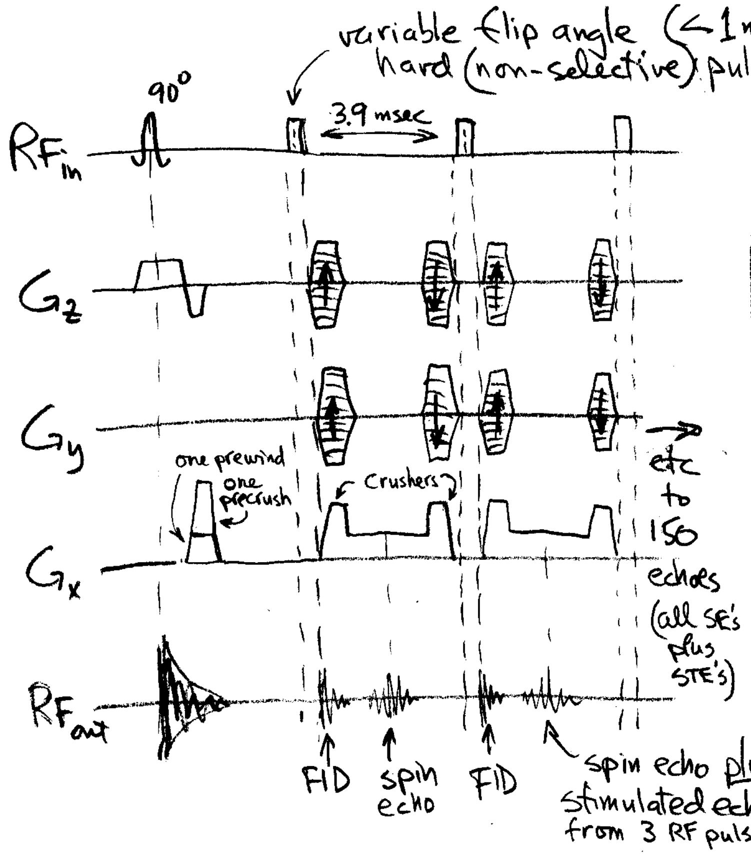
from Mugler (2014) _J Mag Res Img_

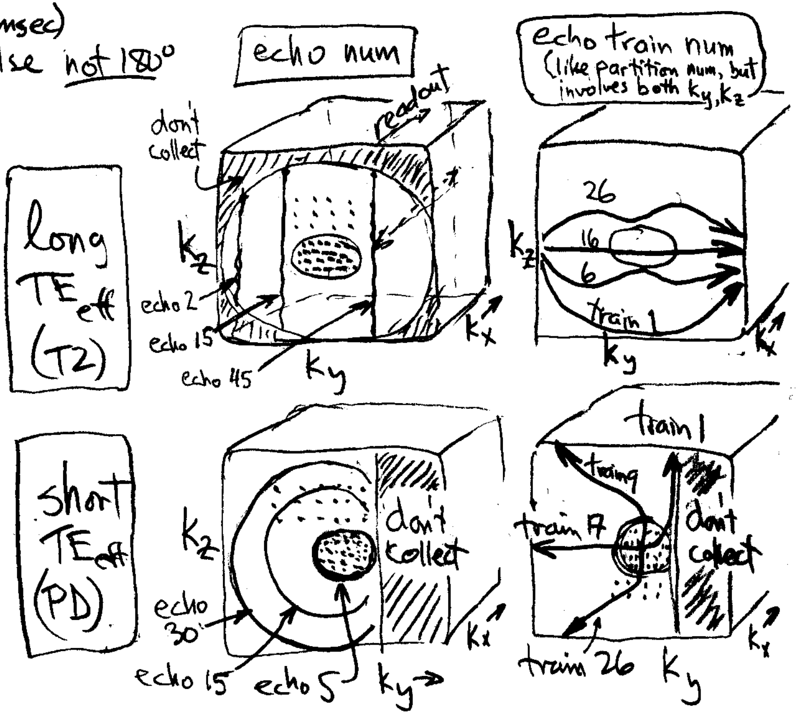
- regular FSE (180° pulse train)
- Sub 180° pulses cause each successive pulse to _also_ generate a _stimulated_ echo (STE)
  - → this "storage" in z-axis preserves magnetization for longer time
- smaller flip angles allow much longer echo trains
  - → enough to collect whole plane of 3-D k-space +
- different than hyperechoes (not symmetric)
- contrast must consider STE

N.B.: T1!!

$$\boxed{\begin{aligned} SE &= \sin\alpha_1 \sin^2\frac{\alpha_2}{2} e^{-2\tau_1/T2} \\ STE &= 1/2 \sin\alpha_1 \sin\alpha_2 \sin\alpha_3 e^{-\tau_2/T1} e^{-2\tau_1/T2} \end{aligned}}$$

- _single-slab 3DFSE pulse seq._





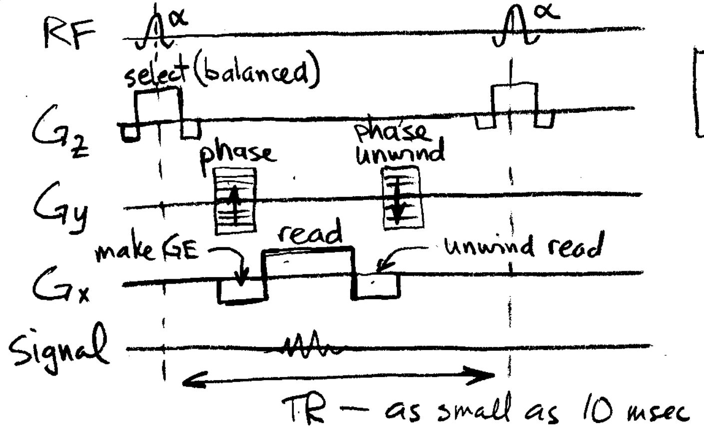
NB: time to cent k-space is ≈ 5X apparent contrast time b/c of "storage" (e.g. TEeff = 585 ms looks like FSE TE=140 ms)

Pulse-2

FAST GRADIENT ECHO

(GRASS FISP | FLASH SPGR | MPRAGE)

- small tip so TR can be greatly reduced (e.g. 10 msec, less than T2)
- 'leftover' undecayed transverse magnetization ["unwound" and re-used / "spoiled" before next shot



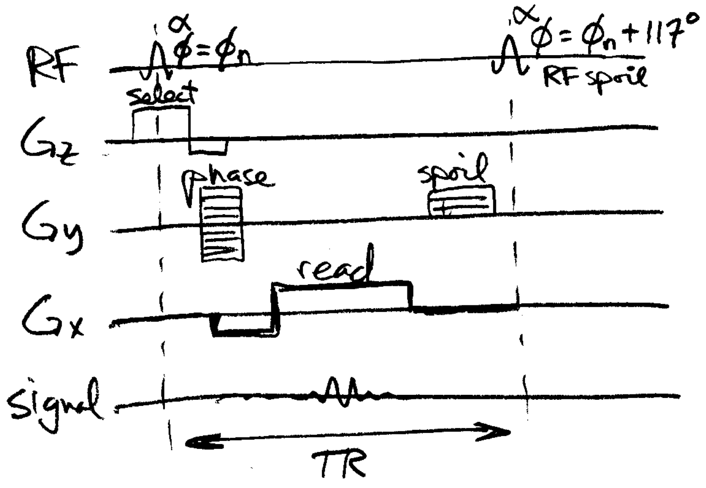
STEADY-STATE COHERENT (GRASS, FISP)

- unwind phase from phase-encode M_T before next pulse (there because TR < TE)
- unwind read gradient, too

$$S = k \sin\alpha \left[\frac{1}{1+\cos\alpha+(1-\cos\alpha)\,T1/T2}\right] e^{-TE/T2}$$

↳ T2/T1-weighted contrast (bright CSF)

tissue	T2/T1
brain	0.11
fat	0.3
CSF	0.7



STEADY-STATE, SPOILED (SPGR, FLASH)

- spoil with random gradient (but this still allows some α refocusing)
- spoil with gradient plus incremented phase of RF pulses (RF spoiling)
- good gray-white contrast (T1-weighted)

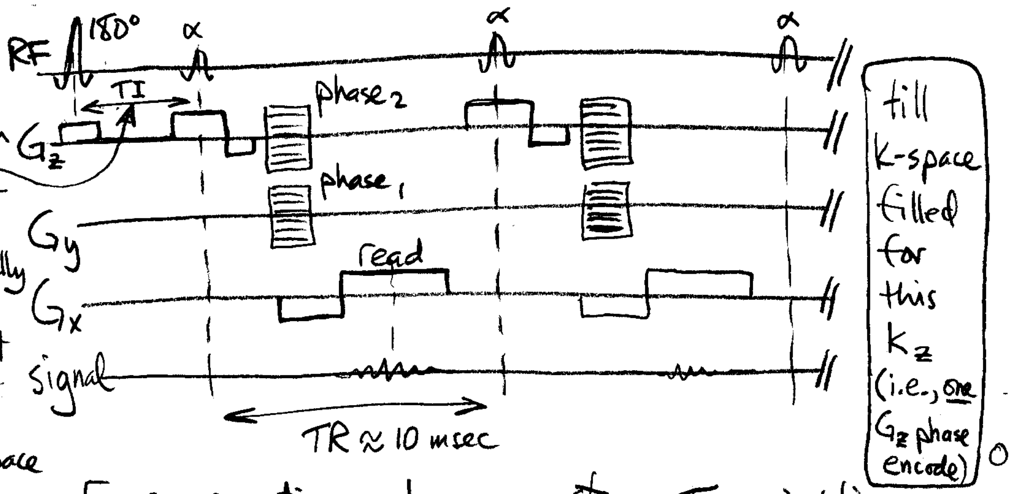
(shown as 3D sequence — possible with ones above, too)

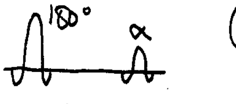
NON-STEADY STATE, MAGNETIZATION-PREP.

(MP-RAGE)

nominal inversion time

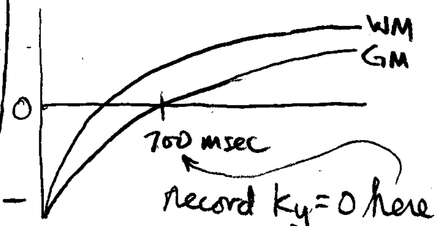
effective TI actually time to TR that records center of k-space ($k_y=0$)





etc. →

- longitudinal mag. not affect much by low angle pulses



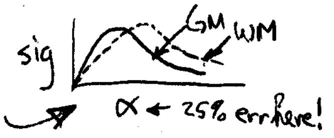
[- preparation-pulse → strong T1-weighting
- contrast varies in spatial-freq-dependent way

QUANTITATIVE T1 — INTRO, METHODS

Motivation

HOW TO FIX: collect more than one copy of vol.

- image values are arbitrary/relative (diff seq's, manufacturers)
- uncorrected coil fall-off (receive inhomogeneity) can result in 2-3x differences in voxel brightness
- uncorrected variation in local B1 field can cause contrast variation
  - at 3T, B1 can vary by 25% across the brain
  - this can invert contrast in a fast gradient echo



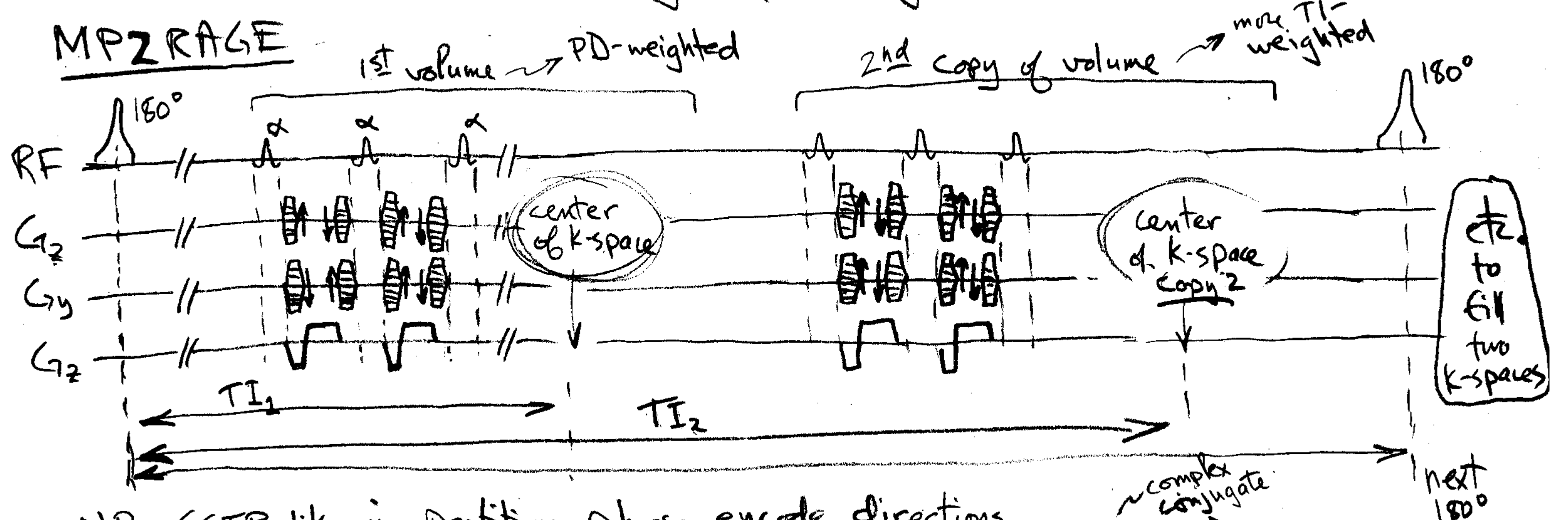
Pre-scan normalise

- collect lo-res GE image, receive w/ body coil (no coil fall-off)
- set parms. to get low GM/WM contrast
- collect data scan (e.g. MPRAGE) w/ surface coils, strong GM/WM
- use ratio between scans to generate smooth correction field

T1 divided by T2

- MPRAGE → strong T1-contrast
- SPACE → T2-weighted (no T1 weighting)
- T1/T2 removes coil fall-off
- problems
  - distortion different in GE (MPRAGE) and SE (SPACE)
  - noise in regions of low signal

MP2RAGE



- N.B. SSFP-like in partition, phase-encode directions
- convert to -0.5 to 0.5 image:

$$S = \text{real}\left[\frac{\vec{S}^{*}_{TI_1}\cdot\vec{S}_{TI_2}}{\|\vec{S}_{TI_1}\|^2+\|\vec{S}_{TI_2}\|^2}\right]$$

complex conjugate

- calc. PD & T1 from above

or

cf. 2 flip angles

pulse-2c

QUANTITATIVE T1 — HELMS 2-FLIP ANGLE METHOD

- start w/ gradient echo signal eq., dropping T2-decay → $e^{-TE/T2}$

$$\boxed{S_{Ernst} = A \cdot \sin\alpha \cdot \frac{1 - e^{-TR/T1}}{1 - \cos\alpha \cdot e^{-TR/T1}}}$$

Ernst eq.

($\max$: $\cos\alpha_E = e^{-TR/T1}$)

"Ernst angle"

($\alpha_E = \cos^{-1}(e^{-TR/T1})$)

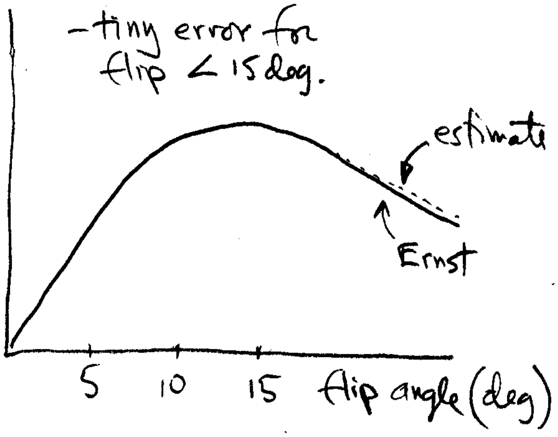
- Simplify/linearize/estimate
  - TR ≪ T1
  - linear approx. of exponentials
  - Taylor expansion simplification of sin, cos, drop small terms

Helms et al. (2008)

$$\boxed{S \cong A \cdot \alpha \cdot \frac{TR/T1}{\alpha^2/2 + TR/T1}}$$

($\max$: $\alpha^2/2 = TR/T1$)

- tiny error for flip < 15 deg.



- solve for TD and A (proton-density) given signals from 2 diff flip angles

2 flip angles

$$T1_{est.} = 2TR \frac{S_1/\alpha_1 - S_2/\alpha_2}{S_2\alpha_2 - S_1\alpha_1}$$

$$A_{est.} = \frac{S_1 S_2 (\alpha_2/\alpha_1 - \alpha_1/\alpha_2)}{S_2\alpha_2 - S_1\alpha_1}$$

- problem: flip angle varies a lot at 3T (e.g. 25%) from nominal/requested (e.g., flip series)

B1 map

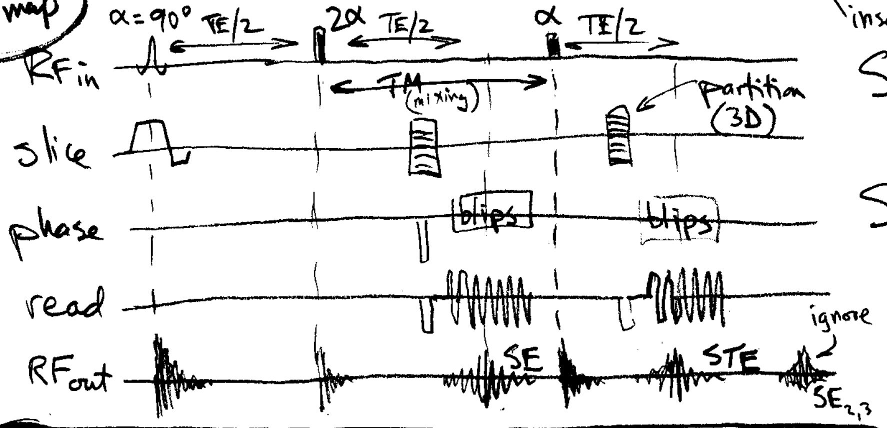
- acq. spin-echo and stimulated echo (EVI)

estimate T1, T2

solve for 2 α's

insert

$(\sin\alpha \cdot \sin^2(\frac{2\alpha}{2}))$



$$S_{SE} = K \cdot \sin^3\alpha \; e^{-TE/T2}$$

$$S_{STE} = K/2 \cdot \sin^2\alpha \cdot \sin 2\alpha \cdot e^{-TE/T2} \cdot e^{-TM/T1}$$

$$\alpha = \cos^{-1}\left(\frac{S_{STE} \cdot e^{-TM/T1}}{S_{SE}}\right)$$

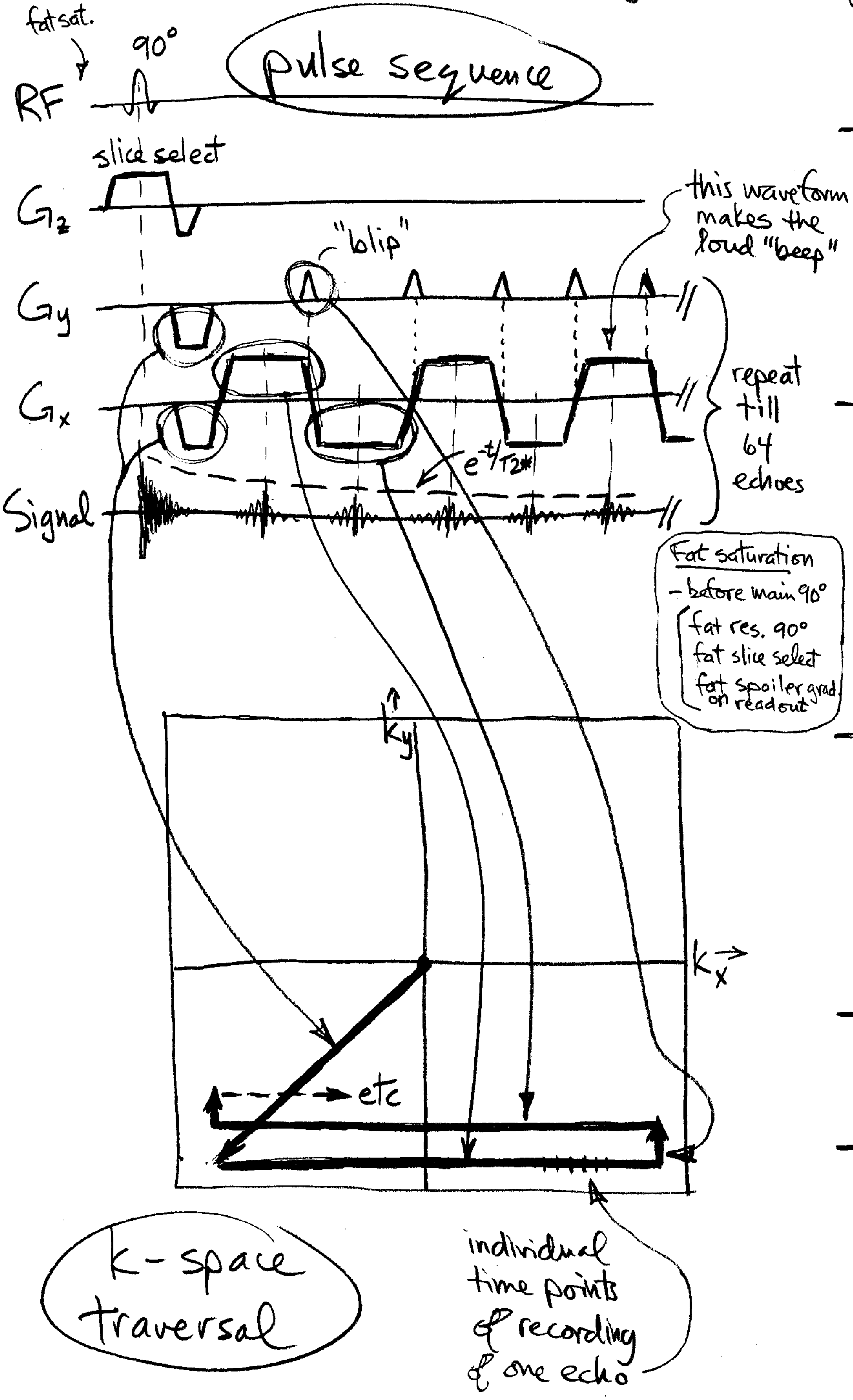
Jiru & Klose (2006)

$T2^*_{est}$ — add EPI-like echo train to each FLASH excit.

ECHO PLANAR IMAGING EPI

(another fast gradient echo)

- single shot EPI collects <u>all</u> k-space lines (e.g. 64) after a 90° RF pulse using a train of <u>gradient</u> echoes



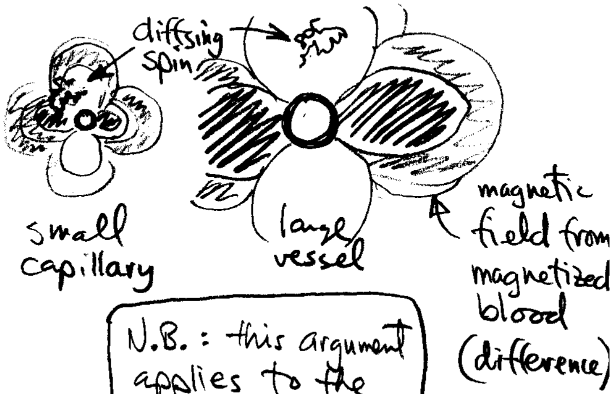
Fat saturation
- before main 90°
  - fat res. 90°
  - fat slice select
  - fat spoiler grad on readout

- since there is only one RF pulse per slice, spins never get reset to all-the-same (= zero freq, center of k-space)
- therefore, the recording point (Δt) in k-space (= spin phase stripe pattern) stays wherever the x and y gradients last left it
- that explains why successive y phase-encode steps are achieved without changing the size of the G_y "blips"
- echoes are T2*-weighted (gradient echo)
- contrast mainly determined by echoes near center of k-space, which are only recorded after about 32 echoes

SPIN ECHO EPI

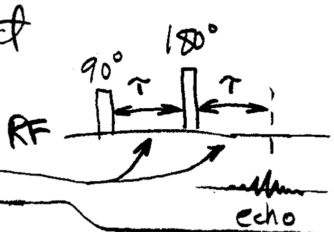
why SE-BOLD may be selective for capillary bed

- standard EPI is a <u>gradient echo</u> method, which results in <u>T2*-weighting</u>
- deoxyhemoglobin is paramagnetic, which reduces signal in a T2*-weighted image due to greater dephasing
- the excess of <u>oxy</u>hemoglobin (probably the result of the need to drive O_2 into tissue, which requires more O_2 in the blood than is actually used) leads to the <u>positive</u> BOLD effect
- spin echo corrects (cancels) static T2* (T2') dephasing, <u>incl.</u> deoxy
- if all spins stayed in the same position, spin echo would eliminate the BOLD effect by eliminating dephasing
- diffusion exposes spins to different fields (<u>reducing</u> gradient echo dephasing!)
- magnetic field gradients produced by large vessels are smoother across space than those produced by small vessels



N.B.: this argument applies to the <u>extra</u>vascular signal

- for TE ≈ 100 ms, spins diffuse 10's of μm, which is larger than diameter of small capillary, meaning that spins will likely experience different fields over time
- therefore, spin echo will be <u>less</u> successful at <u>canceling</u> BOLD effect <u>near</u> small vessels (BOLD effect will be reduced <u>near</u> large vessels where diffusion less likely to expose spin to different fields here)

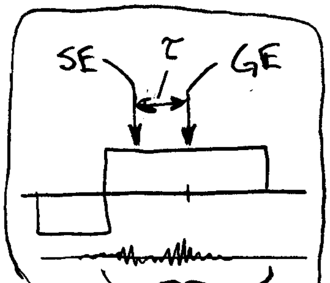


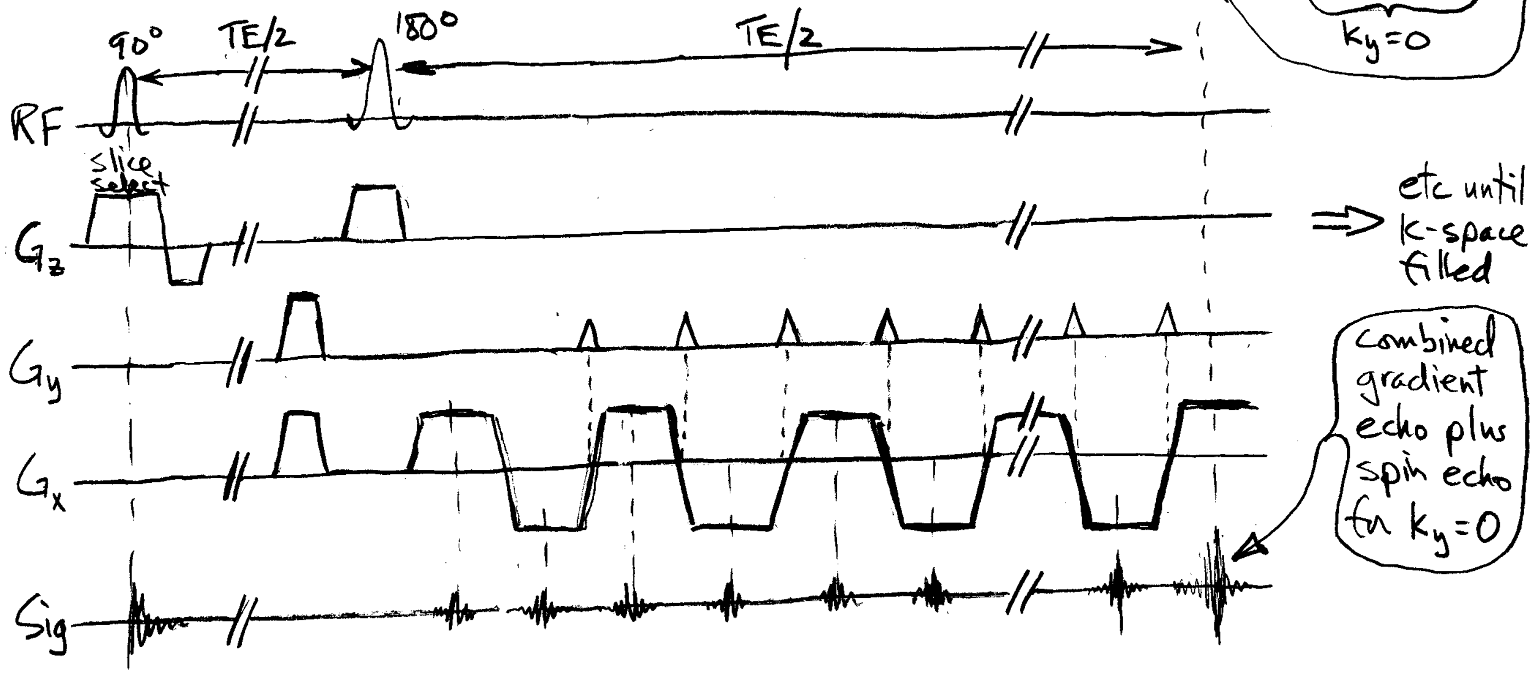
- this argument only works for <u>extra</u>vascular spins — <u>intra</u>vascular signal in BOLD is large (despite being only 4% by volume) because large gradient produced around red blood cells

→ measure intra/extra w/ bipolar pulse which kills signal in faster moving blood in moderate and larger vessels ⇒ over half of SE-BOLD at 1.5T is venous...

SPIN ECHO EPI

- EPI is a multi-gradient echo pulse sequence
- "spin-echo EPI" uses a 180° pulse to add a single spin echo to the contrast-controlling gradient echo through the center of k-space
- "asymmetric spin-echo EPI" arranges for the spin echo to occur τ msec before the gradient echo, which gives more $T2^*$-weighting (for $k_y=0$ echo)

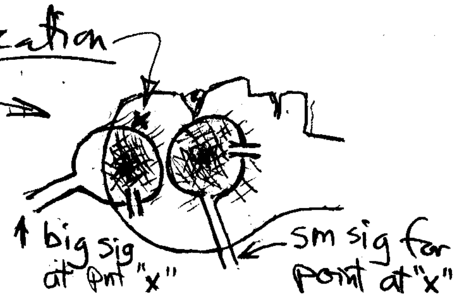


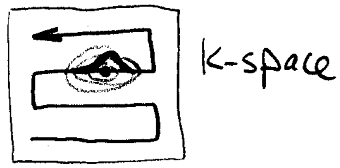


- the 180°-pulse rephasing reduces the $T2^*$ signal, which is why the partially rephased asymmetric spin echo has been more commonly used
- at higher fields, spin echo EPI is more promising
  - signal to noise is higher so we can take spin echo hit
  - contribution from venous blood is reduced, since blood T2 is so short, we can let it decay away before recording

COIL FALL-OFF / UNDERSAMPLE / GRAPPA / SENSE

- coil fall-off intuitively contains info about location if same brain location imaged by different coils w/ diff. fall-offs
  - → but what does this look like in k-space?

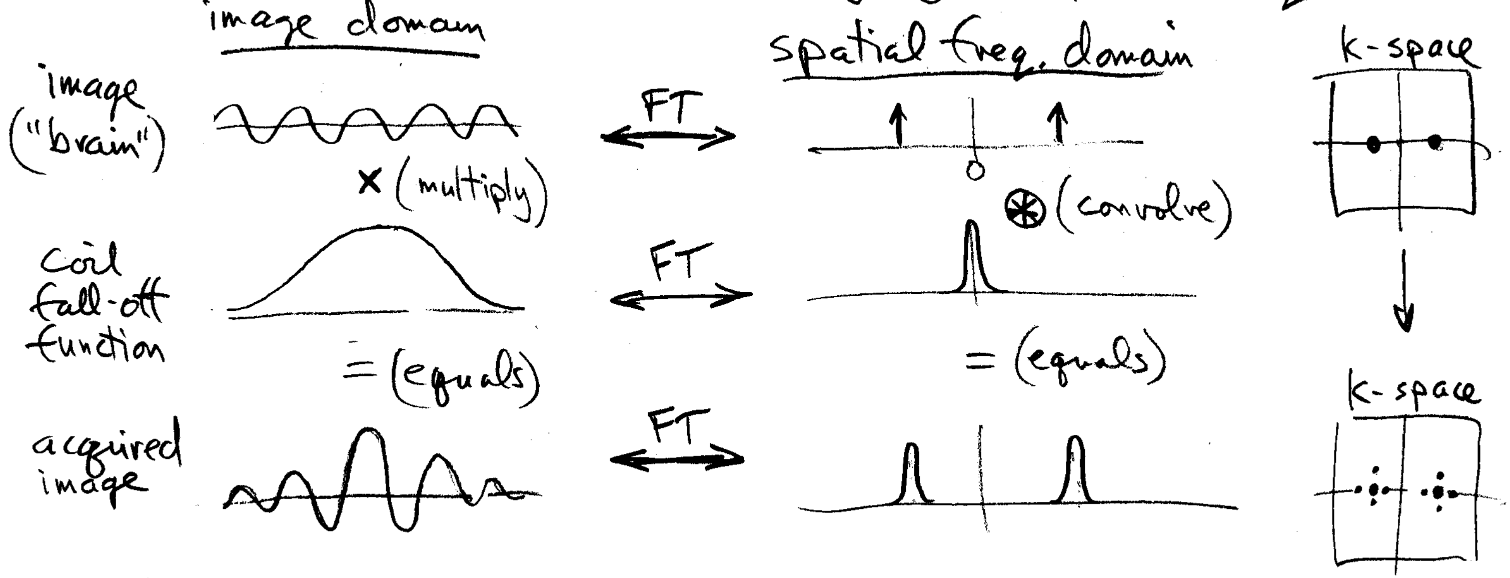




- slow variation in RF-field fall-off (e.g. 1-4 cyc/FOV) causes a blur in acquired data in k-space

- to see this, consider multiplication (N.B. not addition!) by coil fall-off function in image space, which equals convolution (w/ FT of that function) in k-space – at all spatial frequencies !!

- simple example w/ "brain" consisting of one spatial freq.

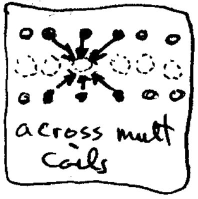


- N.B. inverse FT of k-space data "smeared" in spat freq space is sharp image w/ fall-off (not blurred img.)

So can undersample k-space!!

- "smear" means normally orthogonal spat. freq's "leak" to adj. freqs.

GRAPPA – construct k-space "kernel" to fill in missing k-space lines by training on fully-sampled data from near k-space center →

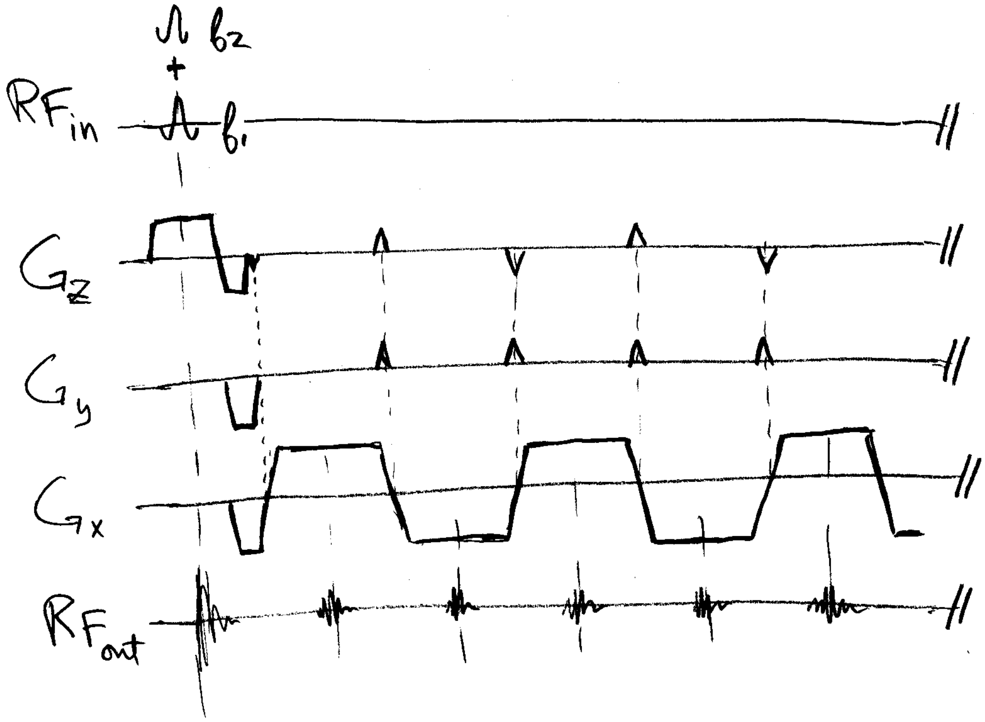


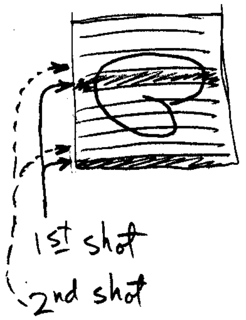
SENSE – general linear inverse approach

- N.B.: neither would work unless normally orthog. spat. freqs. blurred!

pulse-5c

SMS / MULTIBAND / blipped CAIPI

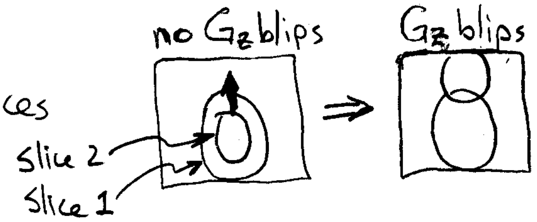




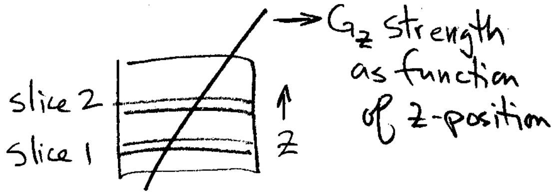
etc.

(or interleaved)

- excite multiple slices at once
- function of G_z blips is to shift slices in $G_{\underline{\underline{y}}}$ direction



- this occurs because for given slice, a phase pedestal is added to the entire slice
  - ↳ this "Fourier Shift Theorem"
  - ↳ [N.B.: different than B0 defect-induced incremented phase errors]

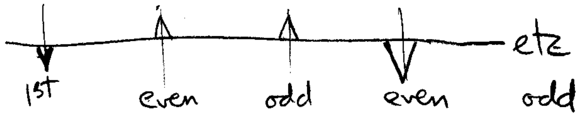


- problem w/ all up G_z blips → phase error builds up

trick #1 – start w/ 2 slices, one at z=0, other above
  - ↳ if π (180°) phase shift used, blip up/down same! (no effect at z=0)
  - ↳ i.e., move top on bottom replica

trick #2 – for multiple slices not all at z=0, phase no longer same for even/odd
  - ↳ but can add phase to equilibrate to k-space before recon.

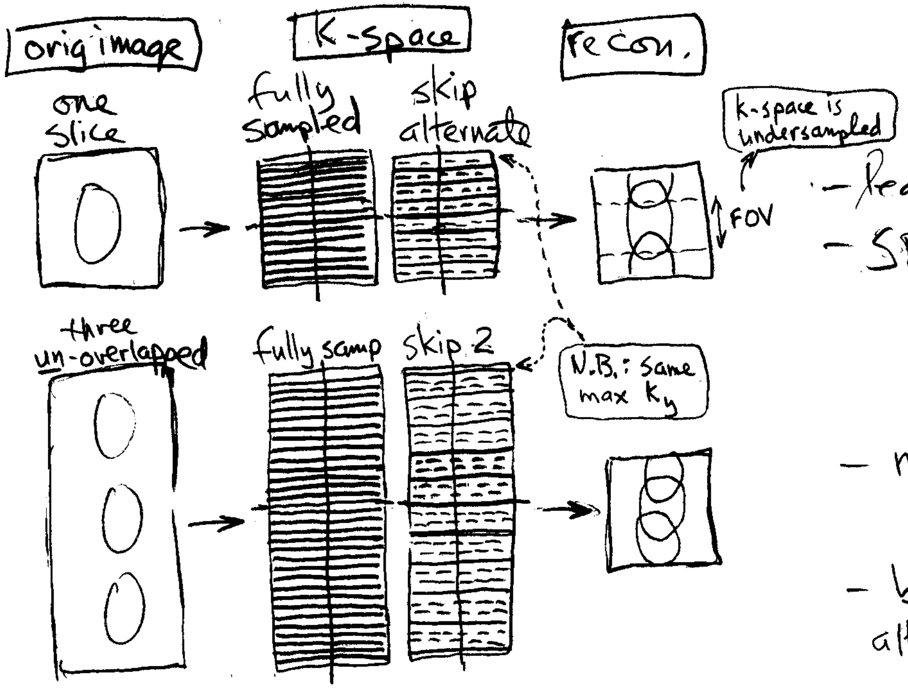
trick #3 – for more than 2 slices :



pulse-5d

MULTI BAND/BLIPPED CAIPI (cont.)

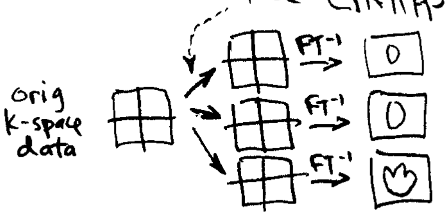
- relation between leave-one-out aliasing and nominally fully-sampled SMS



- leave alternate lines out wraps image
- SENSE/GRAPPA can fix (fill in) b/c local coil view smears k-space data to adjacent spatial frequencies
- nominally, w/ SMS we record every line of k-space
- but equivalent to leave alternate out b/c our <u>multi-slice</u> FOV was not big enough

- slice-GRAPPA
  - reg GRAPPA → fill in missing lines
  - slice GRAPPA → fill in multiple k-spaces for each overlapped slices by training on fully-sampled data at beginning of scan

i.e. <u>not</u> SMS

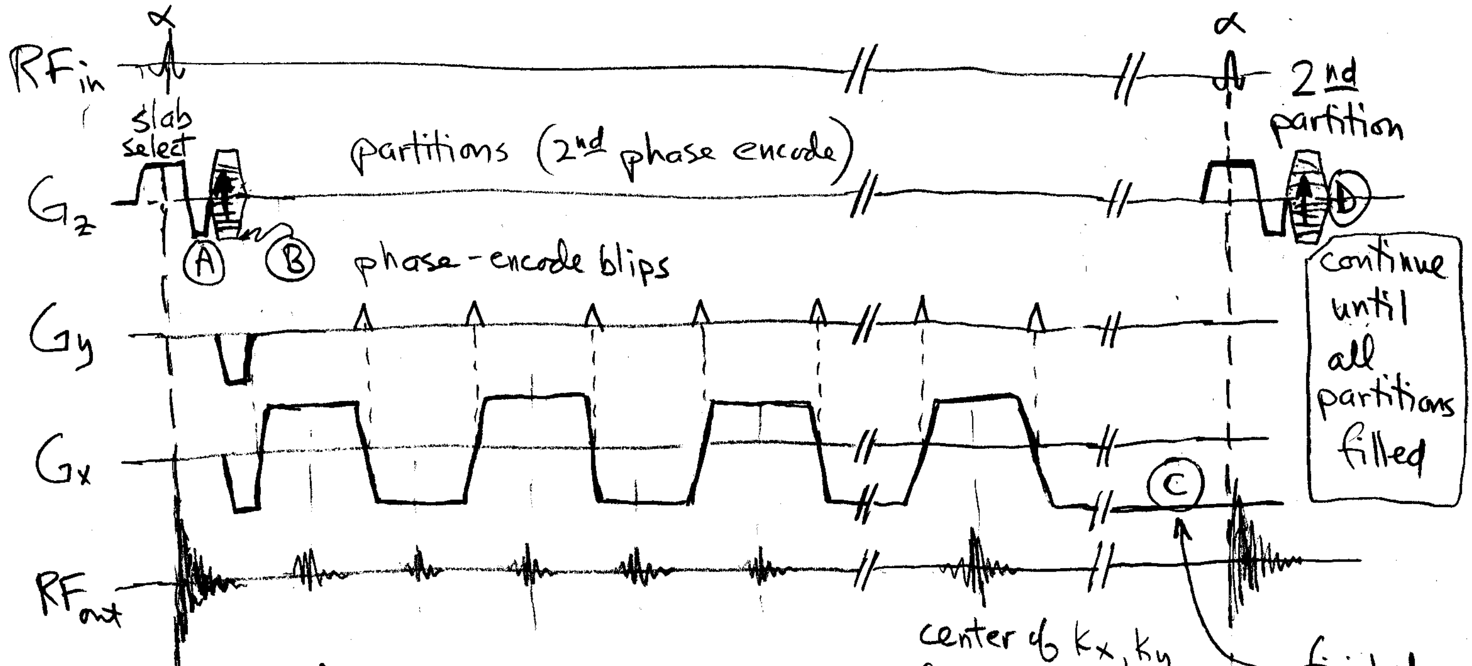


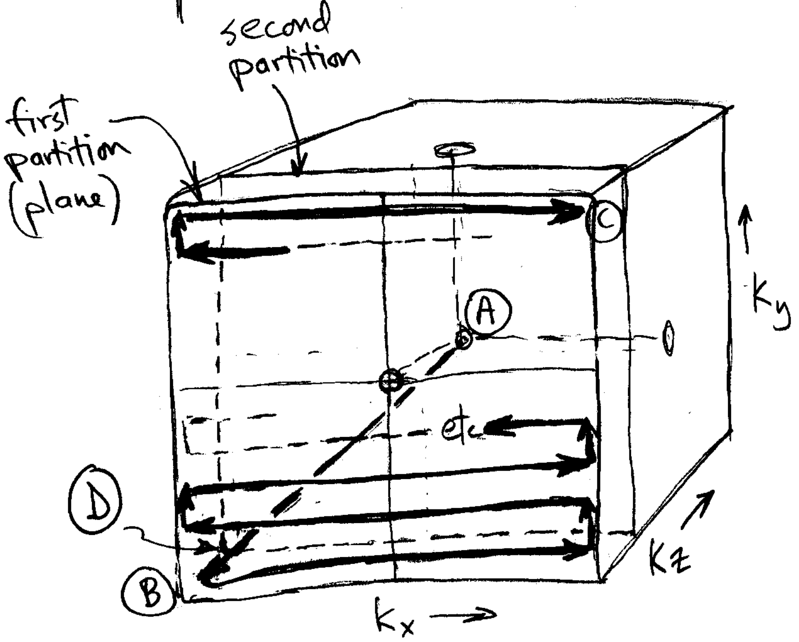
- interslice "leakage block"
  - when training GRAPPA kernel on fully-sampled data, also mimize interslice leakage (split-slice-GRAPPA)
  - can also do regular GRAPPA on top of this
  - reason: <u>for diffusion</u>, loss in S/N from undersample <u>cancelled</u> by shorter TE readout (bigger signal)
    - also, gain from reduced image distortion from shorter k_y readout

pulse-se

ECHO-VOLUME IMAGING EVI

- multi-shot (like FLASH) but acquiring one plane of 3-D k-space per shot (can do spin echo, too)





- entire k-space must be filled before 3D image is reconstructed

- since entire volume is excited each shot, potentially higher S/N

- must use smaller flip angle to avoid killing M_L since entire volume excited every partition (e.g every 80 msec)

- main issue is movement artifact since data assembled from many shots over several secs

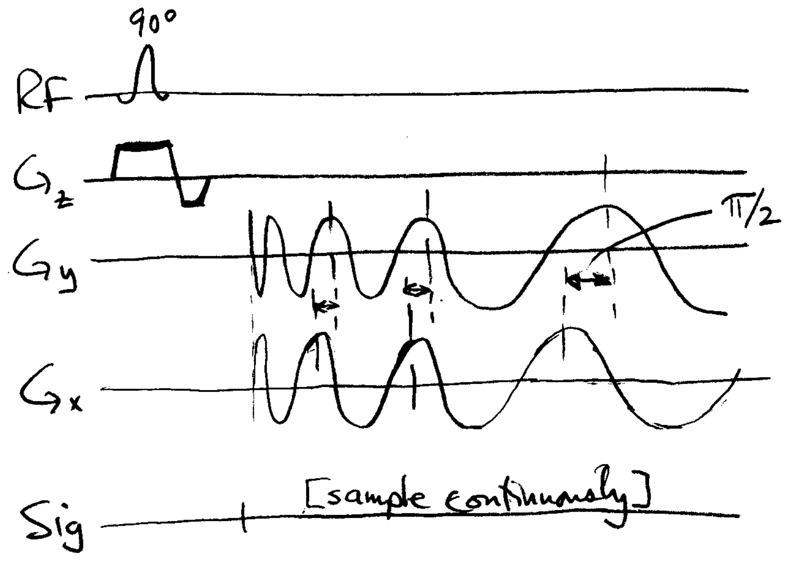
- breathing-induced B_0 problems in different partitions may cause blur

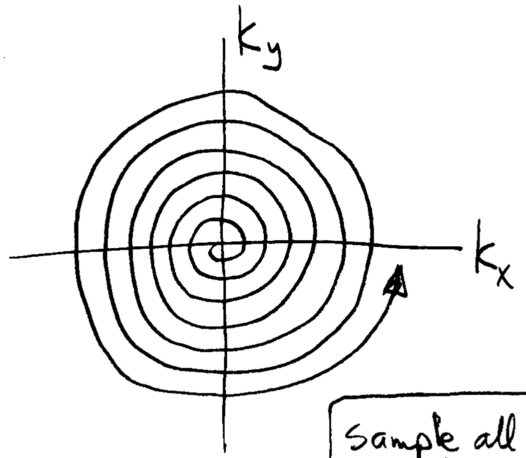
SPIRAL IMAGING

- by using smoothly changing gradients (sinusoids) less gradient power required than w/trapezoids (less eddy currents)

earlier EPI hardware like this: sinusoidal gradient waveform from resonant circuit w/non-uniform sampling to get constant Δk_x

- sinusoids in both G_x and G_y give spiral k-space trajectory





Sample all orientations of each spatial frequency while slowly increasing spatial freq.

- constant angular velocity goes too fast at large k_x, k_y
- constant linear velocity better but impractical near $k_x = 0$, $k_y = 0$
- compromise: start constant angular, end constant linear → 

Constant angular velocity

$$\omega(t) = \omega_0 t$$

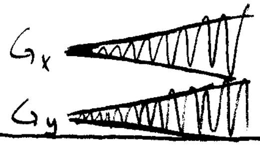
$$\mathbf{k}(t) = A t e^{i\omega_0 t}$$

$$\mathbf{G}(t) = \frac{1}{\gamma}\frac{d}{dt}\mathbf{k}(t)$$

$$= A e^{i\omega_0 t} + i A t \omega_0 e^{i\omega_0 t}$$

$$G_x(t) = A\cos\omega_0 t - A t \omega_0 \sin\omega_0 t$$

$$G_y(t) = A\sin\omega_0 t + A t \omega_0 \cos\omega_0 t$$



Constant linear velocity

$$\omega(t) = \omega_0 \sqrt{t}$$

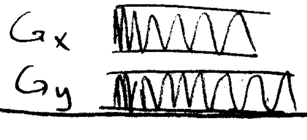
$$\mathbf{k}(t) = A\sqrt{t}\, e^{i\omega_0\sqrt{t}}$$

$$\mathbf{G}(t) = \frac{1}{\gamma}\frac{d}{dt}\mathbf{k}(t)$$

$$= \frac{A}{2t} e^{i\omega_0\sqrt{t}} + \frac{A}{2}\omega_0 e^{i\omega_0\sqrt{t}}$$

$$G_x(t) = \frac{A}{2t}\cos\omega_0\sqrt{t} + \frac{A}{2}\omega_0\cos\omega_0\sqrt{t}$$

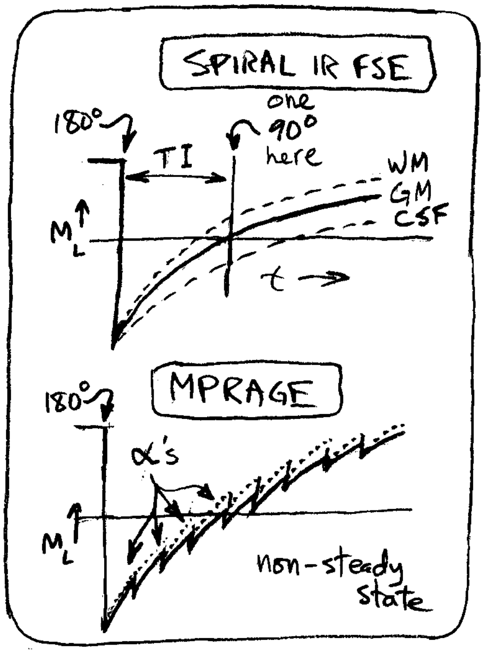
$$G_y(t) = \frac{A}{2t}\sin\omega_0\sqrt{t} + \frac{A}{2}\omega_0\sin\omega_0\sqrt{t}$$

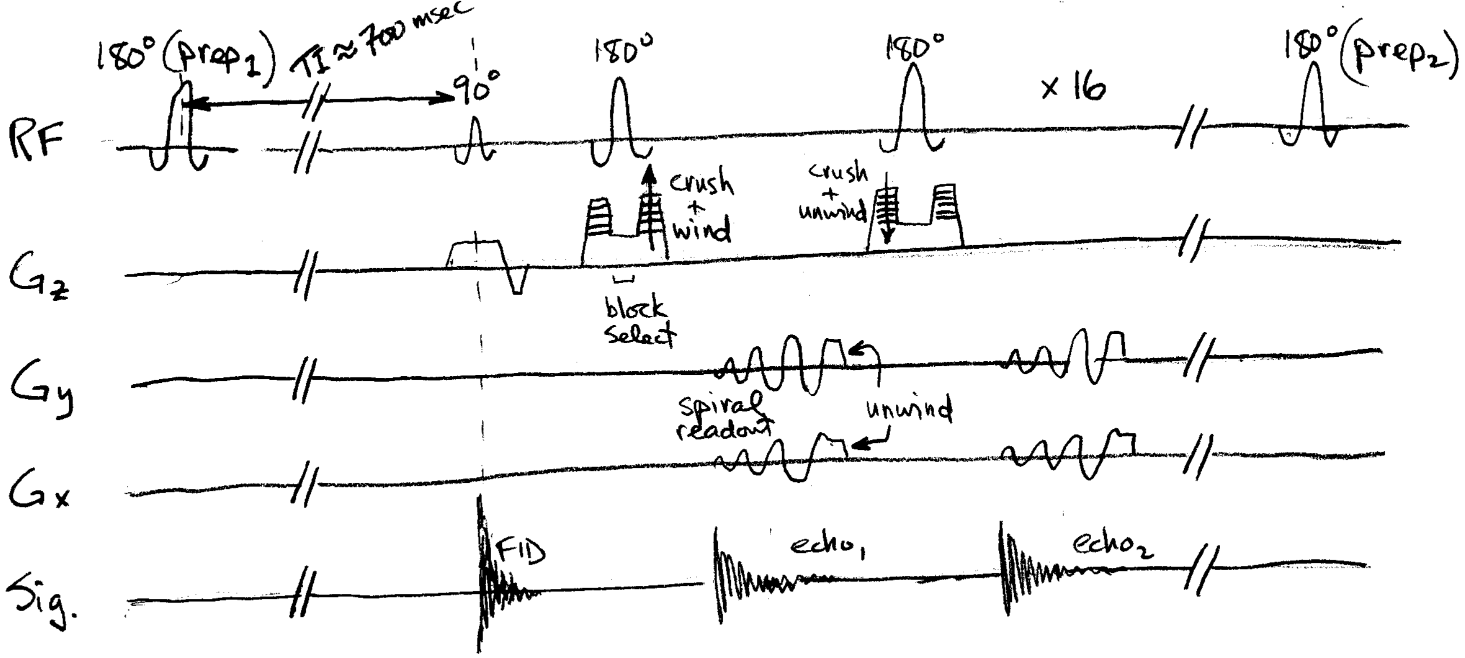


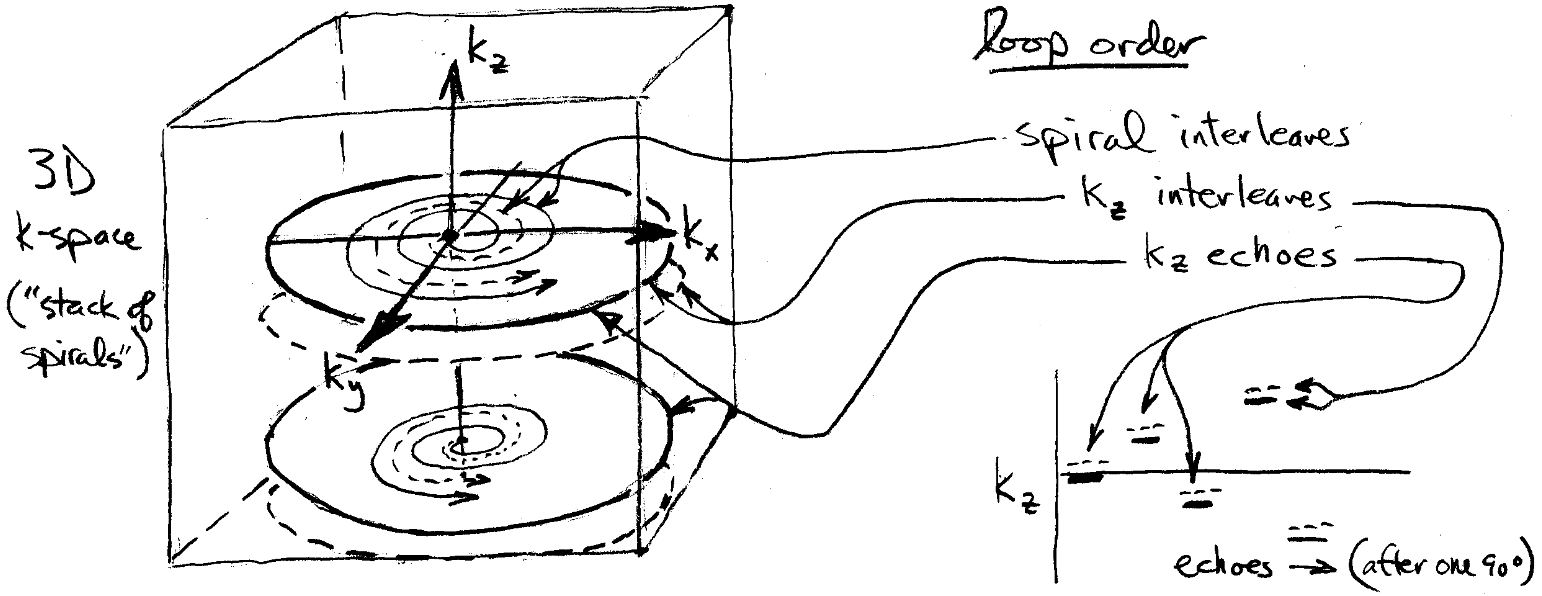
pulse – 7

SPIRAL 3D IR FSE (from Eric Wong)

- 3D: block select vs. slice select
- FSE: multiple echoes from one 90°
- spiral: multiple spirals vs. multiple lines
- interleaved spirals (like FSE interleaves)
- true IR (vs. MPRAGE) → all echoes after 90° derive from mag w/ same T1 contrast (vs. non-steady-state)
- possible to preserve sign
- high, uniform contrast, but lots of waiting (TI), high BW





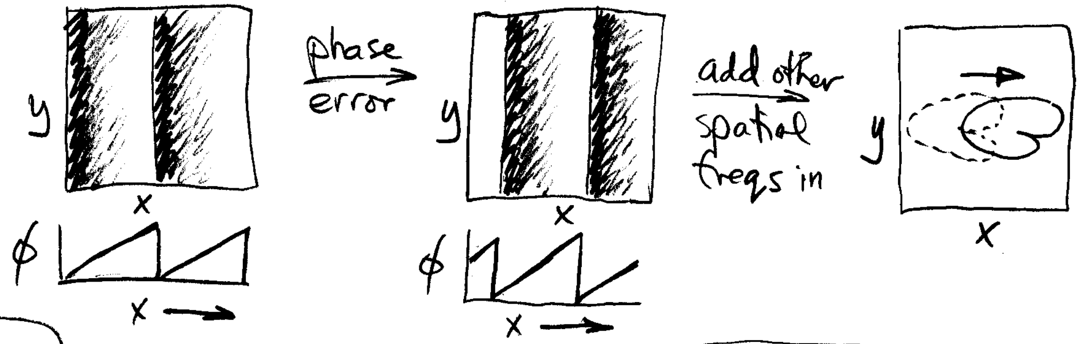


PHASE ERRORS & ECHO-CENTERING ERRORS

Phase Errors

- anything that causes a deviation of the B_z field strength from the expected value $(B_{0,z} + G_{x,z}x + G_{y,z}y + G_{z,z}z)$ changes precession frequency and therefore, expected phase angle
- incorrect phase of spatial frequency stripes results in a shift in space in the magnitude image after reconstruction

phase stripes in image domain (one k-space point)



Fourier (phase) shift theorem

phase shift in spatial freq. domain causes spatial shift in image domain

multiply by complex same as add to exponent: $x - x_0$

$$\underbrace{I(x-x_0)}_{\text{x-offset in image}} = \int_{k_x} \underbrace{e^{-i2\pi k_x x_0}}_{\text{phase shift in signal}} \underbrace{S(k_x)}_{\text{k-space signal}} e^{i2\pi k_x x}\, dk_x$$

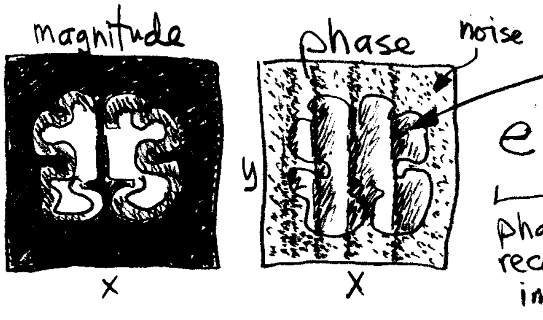
N.B.: this is a "pedestal" of phase, not a gradient

- first defense: freq. prescan
- refine w/ shimming and B_0-mapping/phase unwrapping before reconstruction

---

Echo Centering Error

- if realignment of all spins $(k_x = k_y = 0)$ doesn't occur at the middle of read gradient, echo is shifted
- since echo is in spatial frequency domain, this is frequency shift
- spatial frequency shift results in wrapping in phase image after reconstruction
  ↳ magnitude image unchanged



Fourier freq. shift theorem

freq. shift in freq. domain causes phase shift in spatial

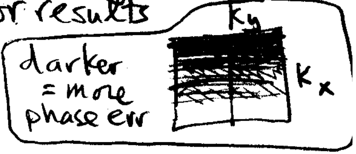
$$\underbrace{e^{i2\pi k_{x_0} x}}_{\text{phase shift in reconstructed image}} \underbrace{I(x)}_{\text{image (complex)}} = \int_{k_x} \underbrace{S(k_x - k_{x_0})}_{\text{offset in spatial freq. space}} e^{i2\pi k_x x}\, dk_x$$

FAST SCAN ARTIFACTS EPI vs. Spiral

brain-induced field defects lead to phase errors

EPI

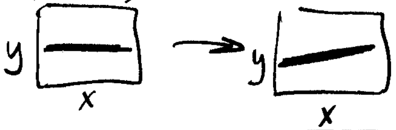
- G_x readout gradient strong → [field defects smaller percentage; less deformation of k_x (vertical stripe components)]
- G_y "blips" are weak and total G_y record time much longer (5 times) than standard readout (50 ms vs. 10 ms)
- an extra gradient in the x-direction, for example, wraps and unwraps phase as a function of x-position
- but G_x big, so effect on freq.-encode direction is much less than on phase-encode direction, where error accumulates
- for a given x-position, the strength of the spurious gradient is constant, so the accumulation of phase error results shift in the y-direction (=k-space spin-stripe displ.)



- the phase error causes a shift in the y-direction proportional to x-gradient strength (= shear) but no blurring

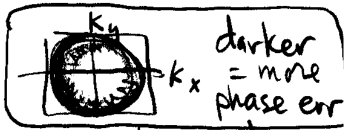
the lack of blurring has lead to a preference for EPI, despite the substantial image shifts

(N.B. shift varies w/x-position)

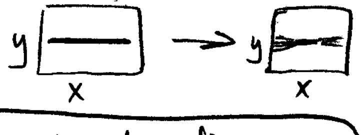


Spiral

- with center-out spirals phase errors accumulate in a radial direction →



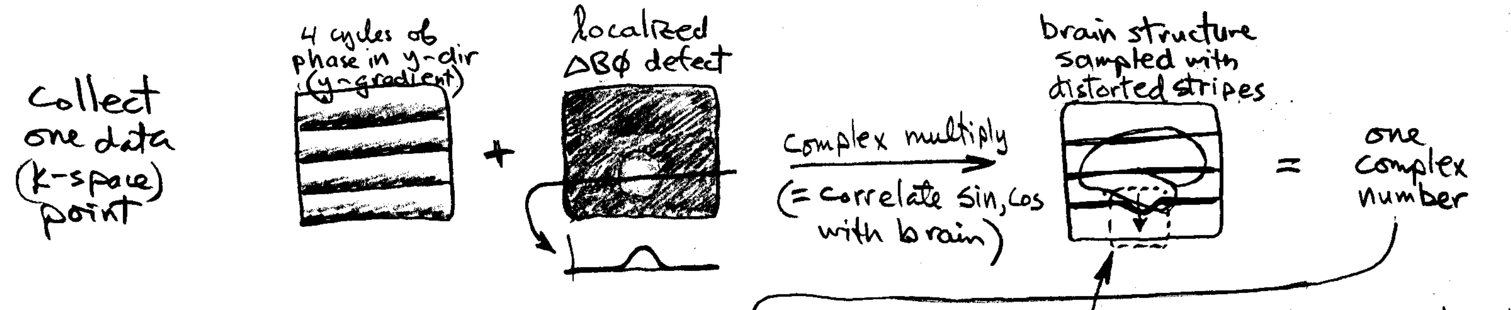
- thus, higher spatial frequencies have more error (= more shearing)
- for spurious x-direction gradient as above, there is a radial blurring, rather than a vertical shift because higher frequency phase stripes misaligned relative to low spatial freq

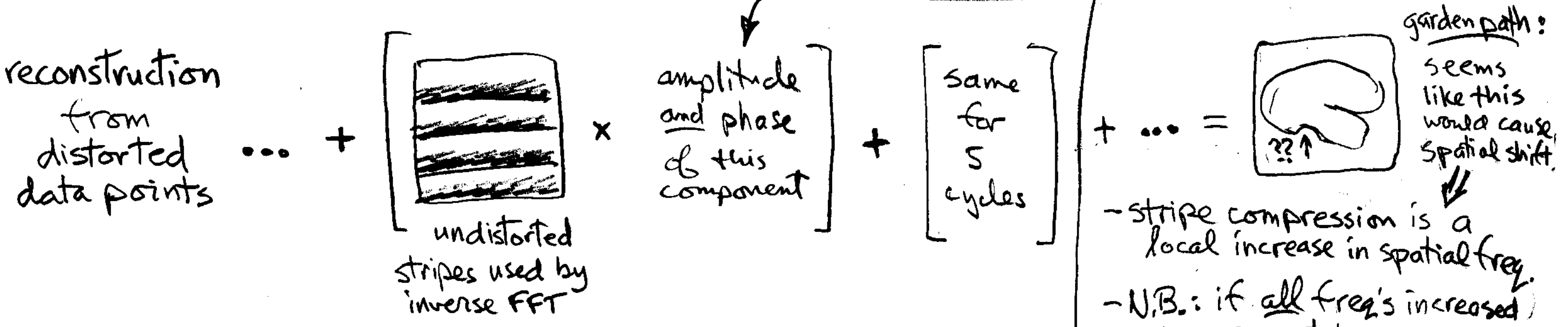


- For defects with more complex contours in the y-direction (than linear, as above) the vertical shifts (for EPI) will vary with y-position, and may result in signals from different y-positions being reconstructed on top of each other

IMAGE-SPACE VIEW OF LOCALIZED B0 DEFECT, EFFECT ON RECON

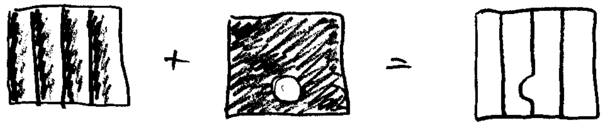
- localized B0 defects often arise from air pockets embedded in tissue
  - air in middle/outer ear → indentation in inferior temporal lobe
  - air under olfactory epithelium → orbitofrontal ctx, ant. thal. compression



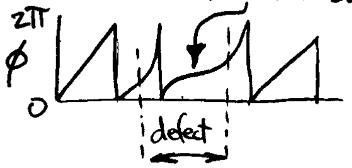


- stripe compression is a local increase in spatial freq.
- N.B.: if all freq's increased same amount, however, this is ex. of Fourier Freq. Shift Theorem → i.e. no amp. image shift! (only ϕ)
- But if phase error (stripe compression) incremented for successive freq's, shift occurs (see next page)

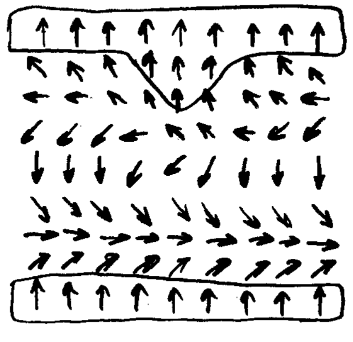
- same defect makes leftward dent in vertical phase stripes



- spatial information can be lost when continuous changes in phase are flattened by B0 defect



- shifts can pile multiple pixels on top of eachother into one bright pixel



Close-up of distorted phase stripes (one-cycle)

- local estimates of ΔB0 needed to correct images
  1) fieldmap method:
     - multiple TE's to est. local ΔB0 from ϕ/TE slope
     - shift each image pixel proportional to ΔB0 in phase-encode dir.
  2) point-spread-function:
     - extra phase encode to estimate P.S.F. (should be δ-function)
     - deconvolve distorted image in phase-encode direction

artifacts - 1c

LOCALIZED B0 DEFECT, EFFECT ON RECON (2)

- when local B0 defect disturbs image space phase stripes during signal acquisition, estimates of local spatial freq. are affected (compressed stripes = higher spat. freq.)

N.B. B0 affects image phase of all spatial frequencies. If we add, e.g., 90° phase, this means higher freq. image stripes move less since each cycle covers less space: ΔB0

- if each successive k_y line recorded w/ same echo time (e.g., w/ single line phase encoding) this will correspond to constant spat. freq. offset in k-space
- a k-space freq. offset only results in image space phase shift (Fourier freq. shift theorem), which is invisible in amplitude image (cf. echo cent. error)
- however, with w/EPI, static B0 defect causes more and more local displacement of image phase stripes for each additional k_y line
  - that is, later lines have greater spat. freq. offset
  - effectively stretches k-space in k_y direction
  - same num samples to higher spatial freq. shrinks FOV (squishes voxels — see FOV page)
- when image is reconstructed, region with local B0 defect shifted oppositely
- Thus, local shift effect due to combination of 3 things:
  1) static local ΔB0 defect
  2) successive increases in phase error for successive spat. freq. measurements during long EPI readout
  3) small size of k_y phase encode blips relative to B0 defect (much less of this effect in freq. encode direction)
- respiration (which affect B0) in 3D FLASH might cause similar effect within K_z partition (if successive spat. freqs.)

GRADIENT NON-LINEARITIES

< B_z
< B_x, B_y

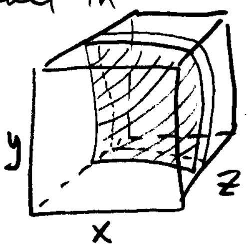
- ideally the G_x, G_y, and G_z gradient coils attempt to impress a linear variation onto to z-component of the B field — B_z — in the x, y, and z-directions
- in practice, gradient coils are non-linear (esp. printed-circuit-like)
- non-linearities are worse in smaller coils, but also in higher performance coils, designed for post-processing correction of distortions

these effects do not build up over time in phase-encode direction since they only occur when gradients are turned on.

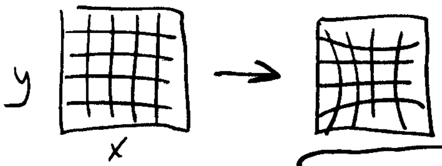
⇓

Fourier shift theorem

- non-linearities result in phase errors, which result in 3-D image distortion
  - a non-linear slice-select gradient will excite a curved slice



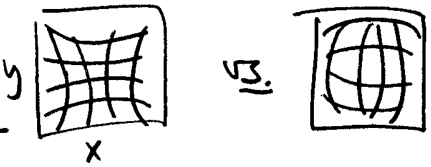
  - non linear phase and frequency encode gradients will distort in-plane features



- some scanners correct these differently for 3-D scans (all directions), 2-D scans (just in plane), and EPI scans (no corrections!)

these distortions are predictable and can be corrected

- this can result in errors approaching 1 cm in funct-struct overlays
- different coil designs have different directions of distortion (!)



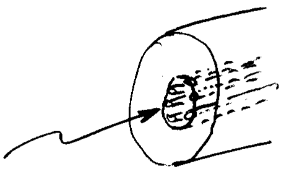
- the assumption of non-Maxwellian gradients results in additional phase errors

that is, the assumption that gradients cause no field in the B_x and B_y direction

- these can also be corrected since the B_x and B_y components are known artifacts - 3

SHIMMING AND B₀-MAPPING

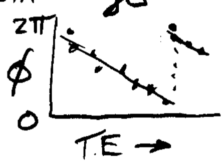
- passive iron shims inserted to flatten B₀ field



- additional coils (usu. in the gradient coils) can be statically energized in an attempt to flatten the B₀ field (a few ppm good)
- primary use is to compensate for defects in flatness present without a sample in the magnet (geometric imperfections, impurities in metal, etc) (= several hundred ppm)

[linear shim coils impose gradients in x, y, and z
higher order shims impose higher order spherical harmonic field components (e.g. z^2)

- secondary use is to compensate for inhomogeneities caused by introducing the sample into the B₀ field
- local resonance offsets caused by B₀ defects estimated from images
  - ↳ e.g., sample phase at multiple echo times



slope is resonance offset (after catching wraps)

⇓

because: longer time to echo means more time for phase to be retarded (or advanced) by B₀ defects

- fit defective field using combination of fields generated by shim coils, then add these corrections to base shim currents
  - ↳ this only corrects spatially gradual field defects
  - ↳ local defects due to air in sinuses much higher order than shims
- after shimming, field map measured again
- image voxel displacements calculated from resonance offset map are used to unwarp the reconstructed magnitude image
- for EPI images, assume displacements all in phase-encode direction (since freq encode gradient is strong relative to defects)

NAVIGATOR ECHOES

- 1D navigator

[daily B0 decay (e.g. 0.05 ppm/hr)
passive shim heating by gradients

B0 drift problem

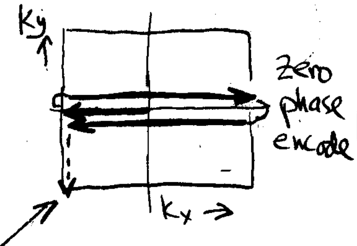
- slow up/down drifts in B0 continuously occur
- a pedestal in B0 is pedestal in phase (not gradient) which causes spatial shift (Fourier shift theorem)
- in EPI, mainly affects phase-encode dir b/c < small blips, long readout
- result is successive volumes drift in phase encode dir

Gradient balance problem

- unequal L/R readout gradients cause L/R shift in position of even/odd lines in k-space causing N/2 (Nyquist) ghosting → another phase error

fix both

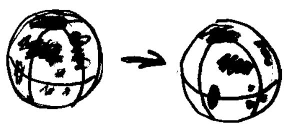
- per-slice navigators



begin std EPI

- measure phase errs
- add to k-space data before recon

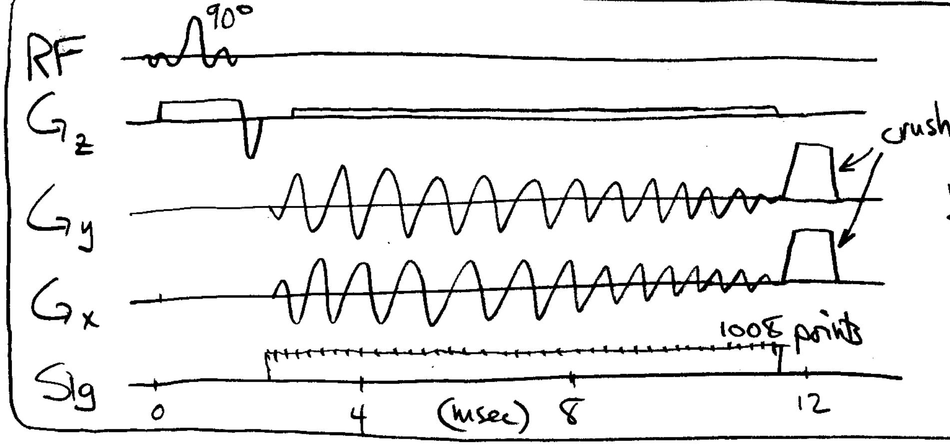
- 3D navigator: collect 3D sphere in k-space



[rotation of object → rotation of k-space amplitude pattern
translation of object → phase shift of k-space phase (Fourier shift)

- sample at sufficient k-space radius to pick up high spatial freq features
- N.B.: excite whole volume
- do N,S hemispheres separately (less T2*, cancel EPI-like error accumulation)
  ↳ equator → up, equator → down

Welch et al. (2002) MRM



$$z(n) = \frac{2n - N - 1}{N}$$

total pnts

$$y(n) = \cos\left(\sqrt{N\pi}\,\sin^{-1} z(n)\right)\sqrt{1 - z^2(n)}$$

$$x(n) = \sin\left(\sqrt{N\pi}\,\sin^{-1} z(n)\right)\sqrt{1 - z^2(n)}$$

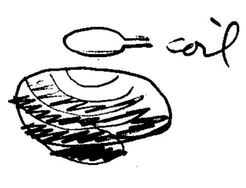
(skip poles — slew rate too high)

- can be used for prospective motion correction (rotate, translate w/ gradients)
- better estimate, because of speed, than full TR of EPI images (27 ms vs. 2-4 sec)
- may need to smooth rot, trans estimates across time (e.g. Kalman filter)

artifacts-5

RF FIELD INHOMOGENEITIES

B_1 inhomogeneities



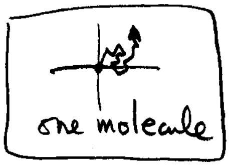
- receive coil inhomogeneities alter the amplitude of the received signal, altering the reconstructed proton density in a spatially varying way
  - ↳ variations can be used (cf. GRAPPA, SENSE) and/or corrected
- transmit coil inhomogeneities affect the flip angle in a spatially varying way (can affect contrast: FLASH)
  - ↳ potentially worse (why local transmit is still in progress)
  - ↳ usu. fixed by using a large transmit coil (e.g. body coil)
- RF penetration at higher fields (= higher RF frequencies) is less uniform:
  1) decreased RF wavelength (closer to size of head) at higher freq.
  2) increased permettivity (ε) and conductivity (σ) at higher field
- 2^{nd} advantage of the falloff in signal reorded with a small, receive-only RF coil is better signal-to-noise (less noise received from other parts of brain)
- different sensitivity functions from different coils can be used to scan less lines in k-space (GRAPPA/SENSE/SPACE-RIP)
  ↳ lin inv after 1D FT

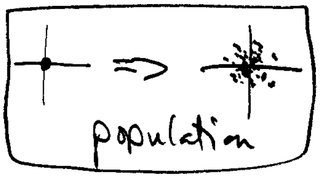
normalization ("pre-scan normalize")

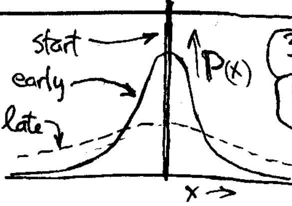
- record lo-res volume (b/c coil fall-off is smooth) through both body coil and small coil(s)
- divide $\frac{\text{small coil}}{\text{body coil}}$ at each voxel to determine receive field
- use receive field to normalize main image(s)

[see also: qT1, MP2RAGE, T1/T2]

DIFFUSION-WEIGHTED IMAGING







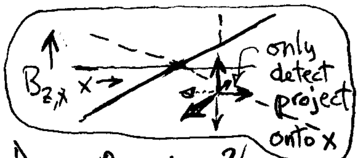
classical diffusion coefficient

time for diffusion

$3D: \sigma_x = \sqrt{6DT}$

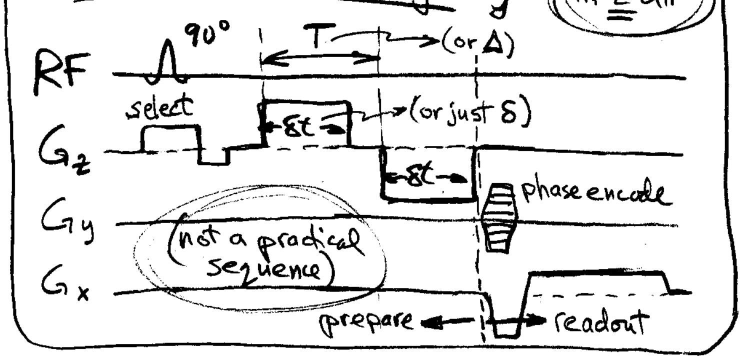
$1D: \sigma_x = \sqrt{2DT}$

↳ e.g. brain $D = 0.001\ mm^2/sec$



Simple diffusion weighting

(in z dir)



- assume Gaussian diffusion
- spins acquire phase during first δt
- if spins diffuse (= move) along gradient by time T, signal is lost because negative δt doesn't re-phase
- attenuation: $\boxed{A(D) = \dfrac{S_{\text{w/dw}}}{S_{0\ \text{no dw}}} = e^{-bD}}$

where $\boxed{b = \gamma^2 G^2 \delta t^2 \left(T - \delta t/3\right)}$

summarizes gradient amp, length, delay

- "apparent diffusion coefficient" → calculate from b=0 image and at least 6 b="large" (e.g. 1000) images
- after obtaining 3×3 tensor, calc eigenvectors/values to find orientation & shape of diffusion ellipsoid
  ↳ (see below)

- two useful scalar values from 3 eigenvalues:

apparent diffusion coefficient / mean diffusivity: $\boxed{ADC = MD = (\lambda_1 + \lambda_2 + \lambda_3)/3}$

fractional anisotropy: $\boxed{FA_{(0\text{-}1)} = \sqrt{\dfrac{(\lambda_1-\lambda_2)^2 + (\lambda_2-\lambda_3)^2 + (\lambda_1-\lambda_3)^2}{2(\lambda_1^2+\lambda_2^2+\lambda_3^2)}}}$

- to get large b, need large G, δt, T → long δt, T can give spurious T2-weighting
  ↳ can use stimulated echo to get long T w/ less T2-weighting

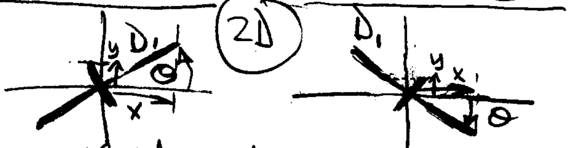
|← long T →|

$90°RF - \delta t_1 - 90°RF - 90°RF - \delta t_2$

$\Downarrow$ M_T (diffusion lobe 1) $\Downarrow$ park in M_L $\Downarrow$ back to M_T (diffusion lobe 2)

1) Anisotropic Diffusion (Gaussian)

- measure D along multiple axes
- have to measure tensor, not scalar →
  ↳ even for determining one primary direction



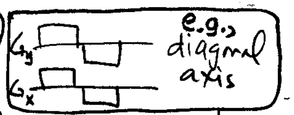
- without 3rd number (θ) x & y projections same

$$D = [u_x\ u_y\ u_z] \begin{bmatrix} D_{xx} & D_{xy} & D_{xz} \\ D_{yx} & D_{yy} & D_{yz} \\ D_{zx} & D_{zy} & D_{zz} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$$

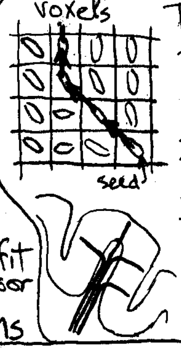
$\boxed{\mathbf{D}e_n = \lambda_n e_n}$ ← get eigenvectors e and eigenvalues, λ

$$\underbrace{D}_{\text{scalar diffusion}} = \mathbf{u}^T \cdot \underbrace{\mathbf{D}}_{\text{diffusion tensor}} \cdot \underbrace{\mathbf{u}}_{\text{measurement direction}}$$

- isotropic: [off diag = 0, diag equal]
- since **D** is symmetric, need minimum of 6 diff. measurement directions (more is better!)



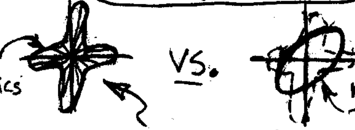
Tract Tracing



1) Markov process
2) crossing fibers
3) "freeway ramp" prob
4) sharp turns into gyri

Diffusion Surface (non-Gaussian)

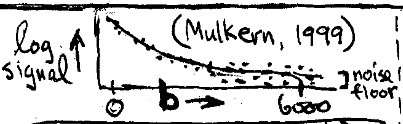
↳ fit w/ spherical harmonics



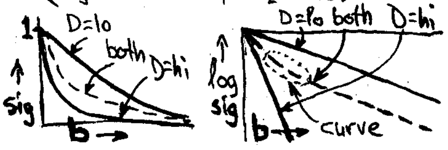
- need to measure diffusion in many directions (>6) to properly characterize even 2 main directions

2) Length Scale by multiple b-values

- fit line to semilog signal as funct of b
- if not straight line: multi-exponential, e.g. $\boxed{S = A_1 e^{-bD_1} + A_2 e^{-bD_2}}$
- hi ADC/fast/extra. vs. lo ADC/slow/intracellular

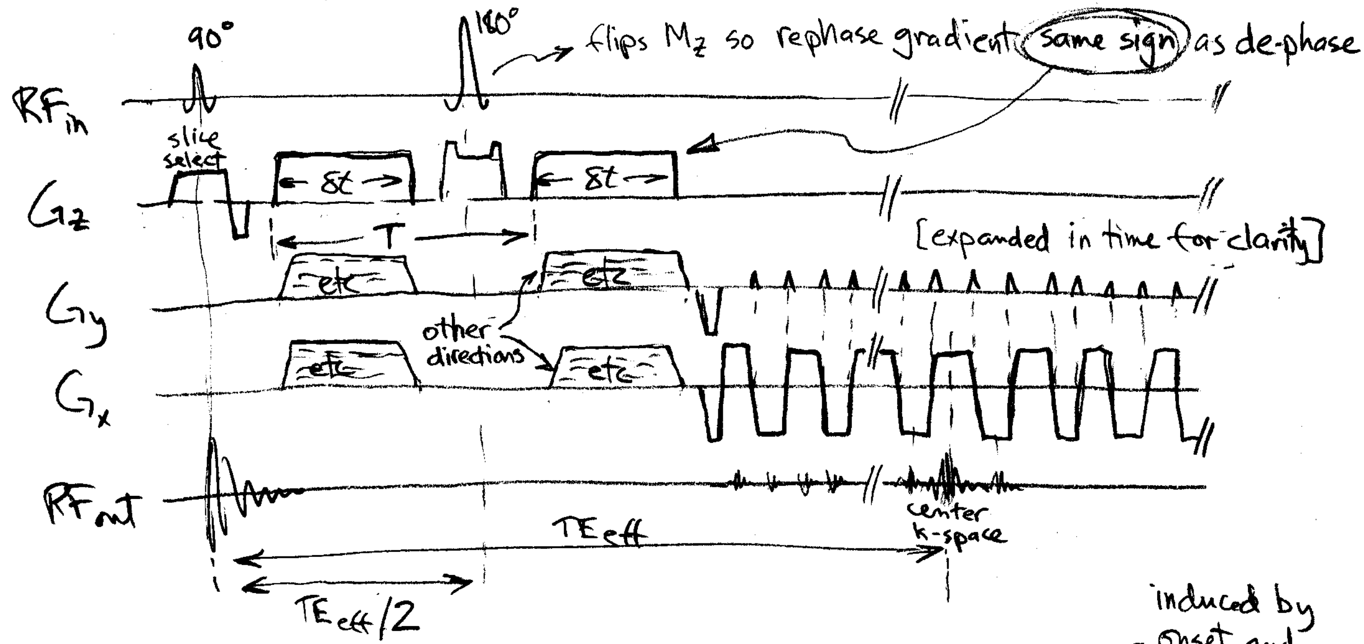


(e.g. assume $A_1 = A_2 = 0.5$)



PRACTICAL DIFFUSION-WEIGHTED PULSE SEQ

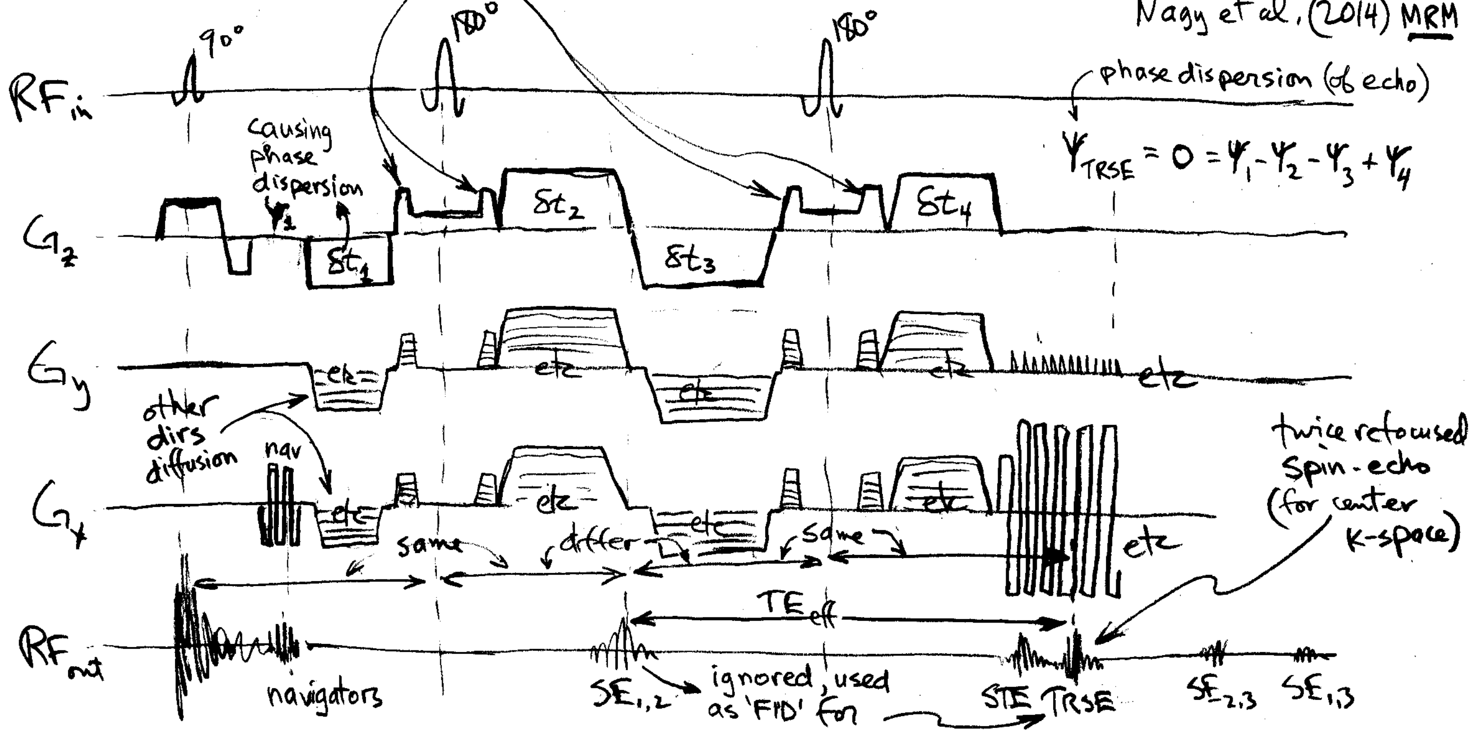
- spin-echo 'Stejskal-Tanner' EPI seq. (standard on scanners)
  ↳ allows longer TE



- eddy-currents are long time-constant currents in metal of scanner that distort B field → spatial image distortion
  induced by onset and offset of gradients

- "doubly-refocused" spin echo sequence (DSE) can cancel the effects of eddy current (w/partic. time constants)
  ↳ (also, keep crushers orthogonal to diffusion-encoding gradients)

Nagy et al. (2014) MRM



$$\Psi_{TRSE} = 0 = \Psi_1 - \Psi_2 - \Psi_3 + \Psi_4$$

PERFUSION – ARTERIAL SPIN LABEL

(like inversion recovery)

- basic idea:
  - tag blood below area of interest
  - collect control & tagged image
  - assume directional input flow!

- tag is <u>180° pulse</u>
- sign not problem when delay long enough (see below)

continuous ASL (CASL) — continuously tag a plane →

gradient on, blood gets adiabitically inverted as it passes through location w/ correct resonant freq.

pseudo-contin. ASL (pCASL) ↗ see next

pulsed ASL (PASL) — e.g., EPISTAR, FAIR, PICORE, QUIPPS II

tag block of tissue below slice(s) →

- small diffs between control and tag (~1%)
  - ↳ requires accurate balancing of control & tag images, control <u>mag. transfer</u>
- contrast problems:
  - transit delays → biggest confounding factor
  - relaxation rate diffs
  - venous clearance (vs. microspheres, which get stuck!)

↳ transit delays

TI →

500 — large vessels

1000 — frontal, temporal, parietal

1500 msec — occipital

<u>solutions for quantitative</u>
- insert delay so all spins arrive into low velocity capillaries
- kill end of tag to reduce spatial variation of tag

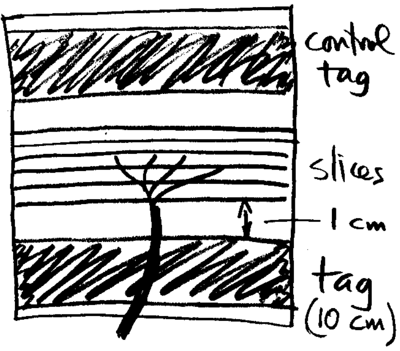
- QUIPSS II — quantitative perfusion

1) pre-saturate spins in target slices → adiabatic pulse (freq sweep)

2) tag – 180° pulse below slices
   control – 180° pulse above slices (to control off-resonance)

3) saturate tagged block to end tag (TI_1)
   - both tag <u>and</u> control
   - can use train of thin slices pulses at top of tag band

4) EPI or spiral images of target slices (TI_2)
   - ↳ image most distal slice last to cancel delays
   - ↳ fast between slice so imaging excitations don't get interpreted as flow



$$\boxed{\Delta M \cong \text{flow} \times \left[2M_0\, TI_1\, e^{-TI_2/T1A}\right]}$$

<u>TURBO ASL</u>
- use TI longer than TR
- omit QUIPPS tag ending

tag pulse
Control img.
control pulse
Tag image
tag pulse
Control img.

2X faster but limited slice #

---

can extract flow <u>and</u> BOLD adjacent subtractions minimize movement artifact

1) <u>alternate tag and control</u>, GRE TE=30ms
   ↳ <u>both</u> BOLD-weighted
   - control – tag → flow
   - control + tag → BOLD

   tag – control – tag – control ...

2) <u>dual echo spiral</u>
   - k=0 early ⇒ hi S/N flow
   - TE=30 ms ⇒ BOLD

PERFUSION — pCASL

- original CASL (continuous arterial spin labeling) requires RF on continuously to adiabatically invert blood flowing through one plane
  - → can only image one slice (b/c dephasing from gradient)
  - → hard to keep RF on continuously on modern scanner (esp. BC)
  - → can use special purpose RF transmit (separate xmit channel)

A) original CASL

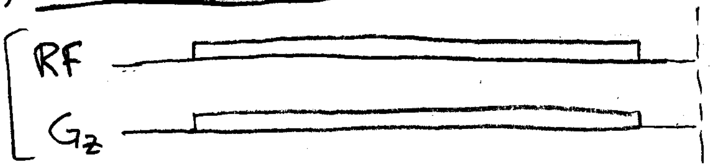
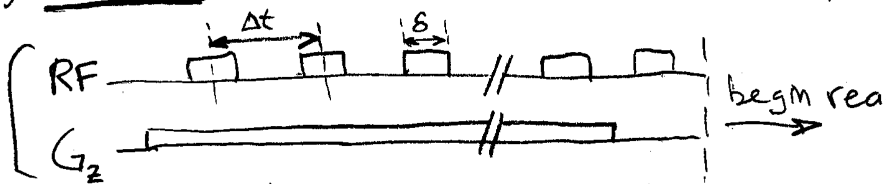


image formation module ("readout") → [multiple possibilities]

B) pCASL — pseudo (or pulsed!) continuous arterial spin labeling   Dai, Alsop (2008)



- problem: multiple pulses create aliased slice planes

$$RF(t) = \frac{1}{\Delta t}\,\text{comb}\left(\frac{t}{\Delta t}\right) \overset{\text{convolution}}{\circledast} \text{rect}\left(\frac{t}{\delta}\right) \longrightarrow$$

$$\boxed{\text{comb}(t) = \sum_n^{\infty} \delta(t-n)}$$ ← spike num

$$\boxed{\text{rect}(t) = \begin{cases} 1 & \text{if } |t| < \tfrac{1}{2} \\ 0 & \text{otherwise} \end{cases}}$$

[use: convolution of 2 funct equals multiplying their FT's]

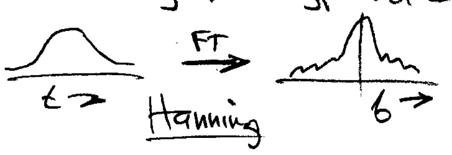
$$F[RF(t)] = \text{comb}(b\delta t) \overset{\text{multiply}}{\cdot} \delta\,\text{sinc}(\pi\delta b)$$

→ aliased labeling planes at: $b = n/\Delta t$ in frequency space, modulated by broad sinc()

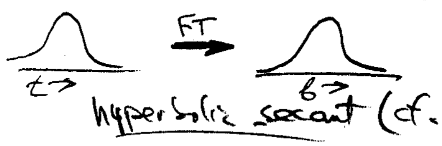
→ for const G_z: $\boxed{z = \frac{n}{\gamma G_z \Delta t}}$

→ i.e. spacing inversely proportional to Δt

- use Hanning or hyperbolic secant to reduce replicas



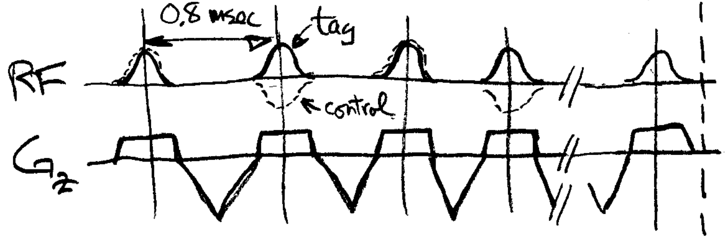
Hanning



hyperbolic secant (cf. Gaussian)

C) pCASL w/ shaped gradients

(control called a "transparent" pulse)



readout
- FLASH
- EPI
- SMS
- stack spirals 3D

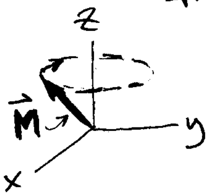
- tag pulses have phase offset respecting gradient
- control identical except every other has +π phase
  - → no net flip

perfusion-3

OFF RESONANCE EXCITATION

adiabatic RF pulse

- main idea: examine evolution of $\vec{M}$ vector in rotating coord syst set to "off-resonance" $\vec{B1}$ field freq (ω_{rf}), not Larmor freq of $\vec{M}$ (ω_0)
- normally, if rotating coord syst freq set to Larmor freq ($\omega_{rf} = \omega_0$), an actually precessing $\vec{M}$ will be stationary (ignoring decay) → implies effective $B_z = 0$ in rotating
- now, move $\vec{M}$ to rotating coord syst at $\vec{B1}$ freq lower than ω_0 (assume $\vec{B1} = 0 =$ off) : existing $\vec{M}$ will now appear to precess around z-axis:



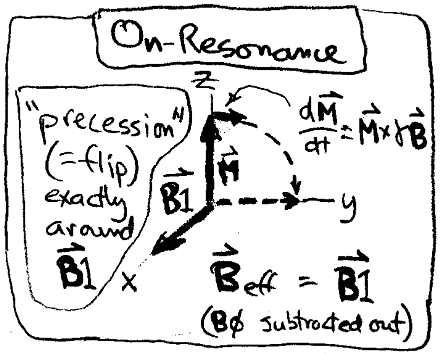
N.B.: this is precession in already rotating coordinate system! (slow relative to ω_0)

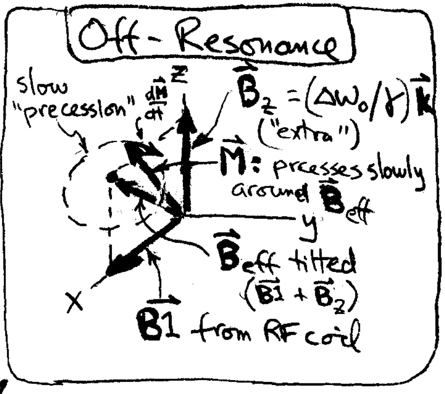
$$\underbrace{\Delta\omega_0}_{\text{freq of precession in rotating coordinate syst}} = \underbrace{\omega_0}_{\text{Larmor freq of }\vec{M}} - \underbrace{\omega_{rf}}_{\text{rotation freq }\vec{B1}\text{ (= "incorrectly set rotating coord syst frequency)}}$$

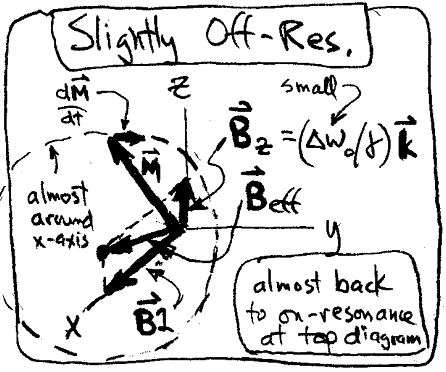
- thus, viewing $\vec{M}$ vector in off-resonance rotating coord syst makes it look like additional $\vec{B}_z$ field is causing "extra" precession
- "extra" $\vec{B}_z$ component is proportional to $\Delta\omega_0$ offset
  ⤷ can be pos or neg: rot coord syst freq: too low → pos $\vec{B}_z$; too high → neg $\vec{B}_z$
- extra $\vec{B}_z$ adds to $\vec{B1}$ resulting in slow precession around tipped axis: $\vec{B}_{eff}$ (effective)
- extra $\vec{B}_z$ from any gradient → same effect on $\Delta\omega_0$ (changes ω_0 instead of changing ω_{rf})

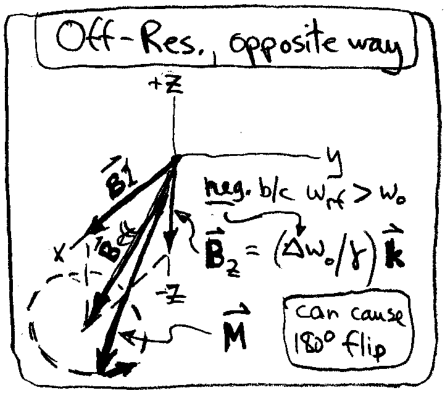
$$\underbrace{\vec{B}_{eff}}_{\text{effective }\vec{B}\text{ in rotating frame set to }\vec{B1}\text{ freq}} = \underbrace{\left(\frac{\Delta\omega_0}{\gamma}\right)\vec{k}}_{\text{apparent "extra" }\vec{B}_z\text{ from Larmor-B1 freq mismatch (pos or neg) (if on-res.}\to 0)} + \underbrace{B_{G_z}\vec{k}}_{\text{extra }\vec{B}_z\text{ from optional gradient (pos or neg) (additional source of }\vec{B}_{eff}\text{ tilt)}} + \underbrace{B1\,\vec{i}}_{\text{transverse RF stim (here, around x-axis)}}$$

← x-axis







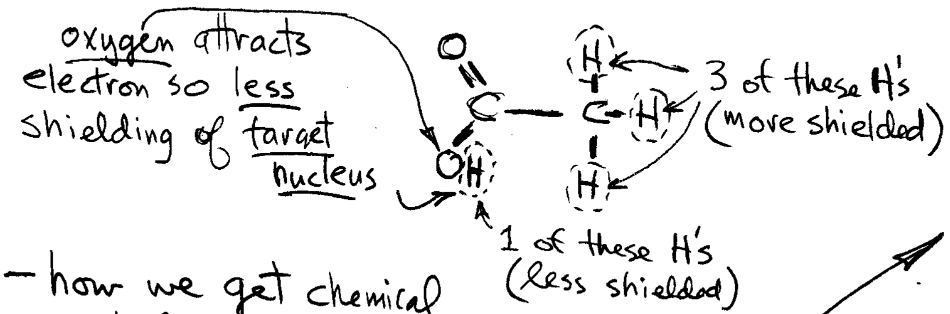


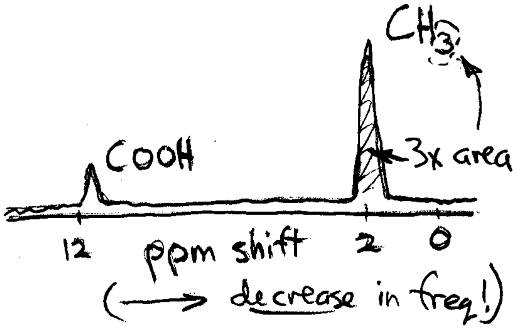
adiabatic RF pulse	≈	flow-driven CASL tag
RF: sweep freq.		RF: const freq
ω_0: ~constant (for given slice)		ω_0: sweeps because spins flow along gradient direction

spect-1

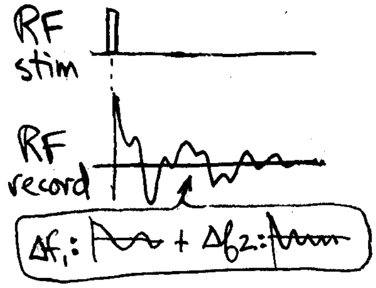
SPECTROSCOPY + IMAGE

- chemical shift: small displacement resonant freqs due to variable shielding of target nucleus (e.g. ^1H) by surrounding electron orbitals
- e.g., acetic acid:





- how we get chemical shift spectrum:

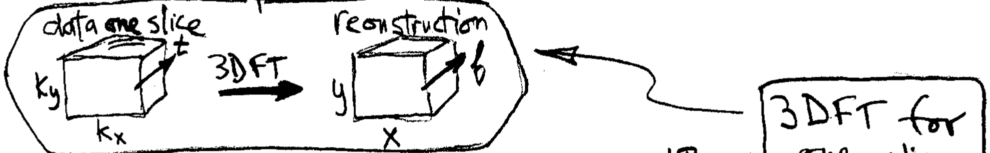


Larmor oscillations are multiplied (PSD) by center freq to obtain Δf (not MHz high freq)

- data before FT is a series of time-domain samples of the mix of shifted-freq offsets
- FT turns data into "shift spectrum"

- N.B.: opposite "direction" of FTs!

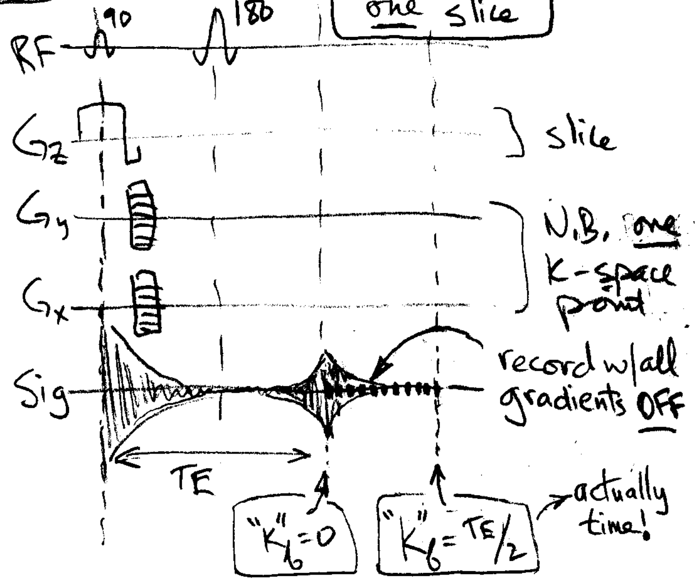
	signal	result
NMR	time domain oscillation samples (shift) → FT	shift freq. spectrum
MRI	spatial freq. samples → FT	spatial object (like time domain signal)



Pulse Sequence

- since we are already using phase (=freq) encoding for space, we need an "extra dimension" w/ all gradients OFF!
- use spin-echo to undo built-up chemical shifts, then record chemical-NMR-like signal

↳ and FT-it like chemists do!

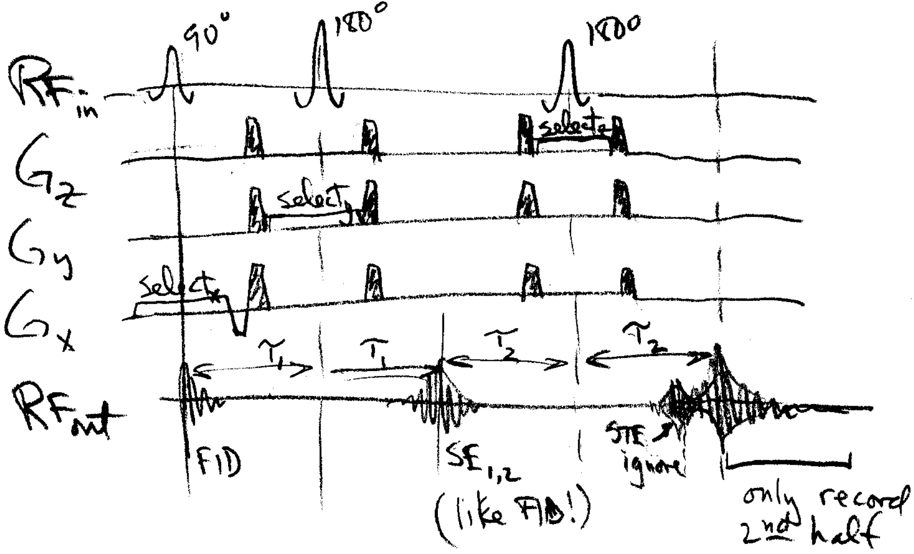


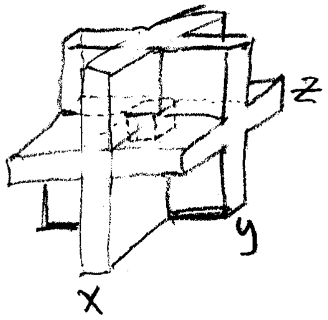
Spect-2

PRESS, MEGA-PRESS

- usu. single voxel by using 3 orthog. slice selects (tho can add PE gradients & more excitations to get multiple vox)

- **PRESS** — 3 orthog slice select

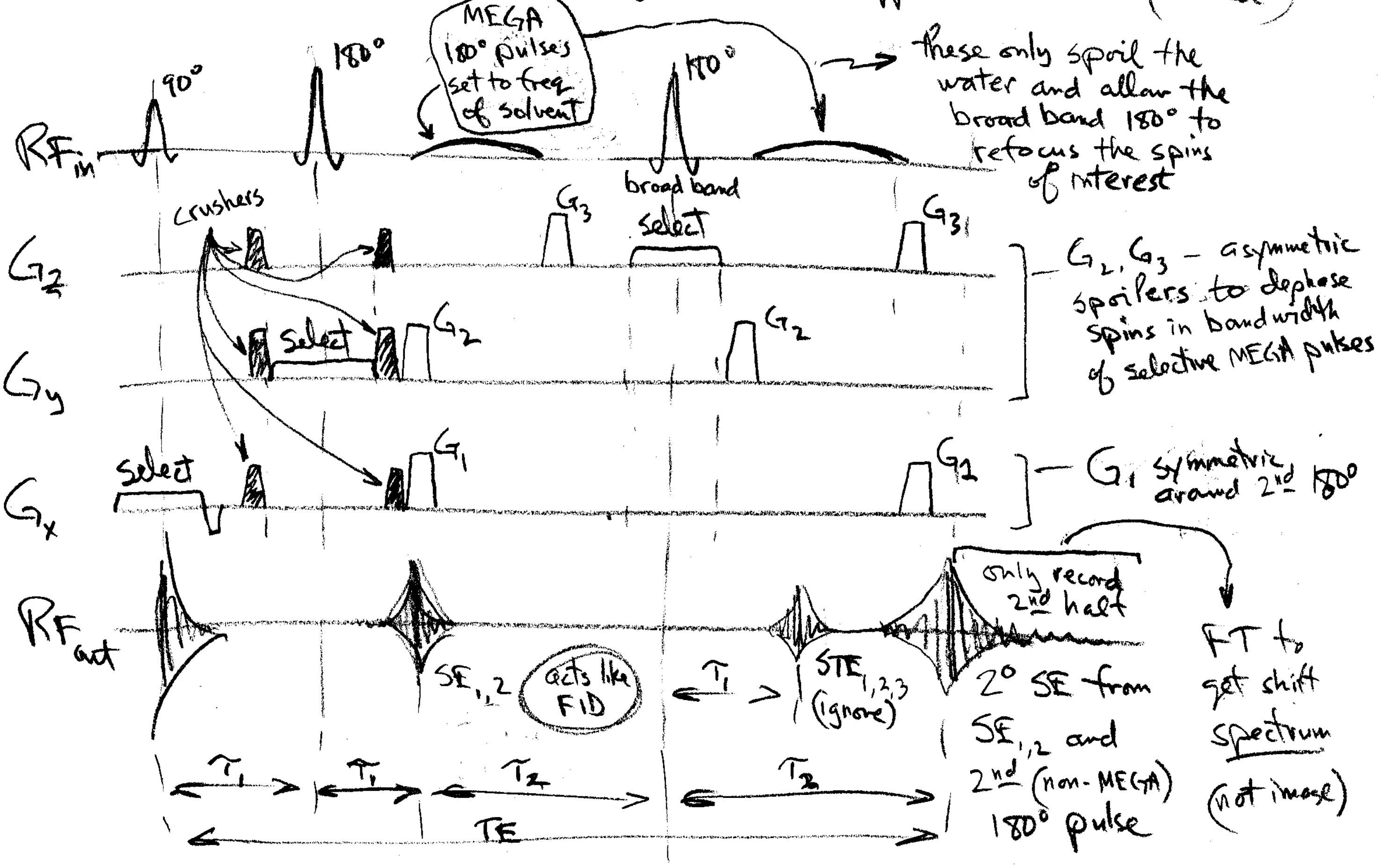




It's OK, now to think of multiple freqs here!

FT to get shift spectrum

MEGA-PRESS — add "editing" RF's to suppress solvent (water)



these only spoil the water and allow the broad band 180° to refocus the spins of interest

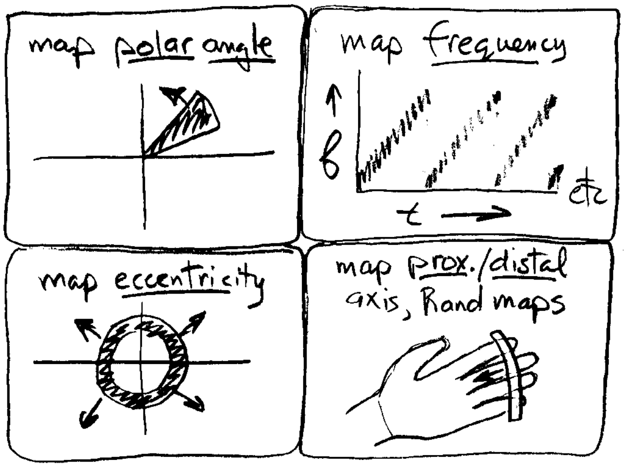
G_2, G_3 – asymmetric spoilers to dephase spins in bandwidth of selective MEGA pulses

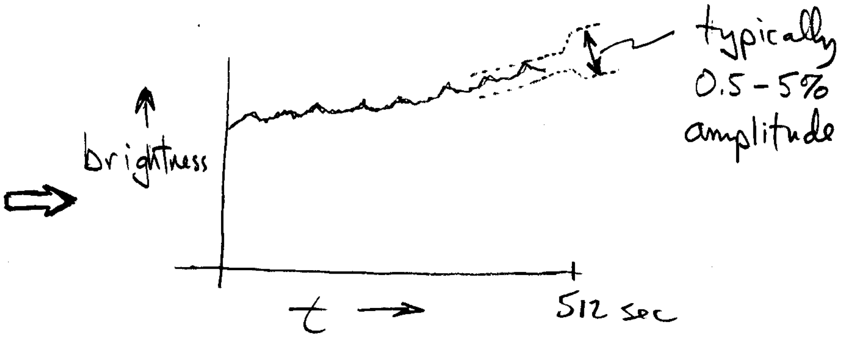
G_1 symmetric around 2nd 180°

2° SE from SE_1,2 and 2nd (non-MEGA) 180° pulse

FT to get shift spectrum (not image)

PHASE-ENCODED STIMULUS & ANALYSIS

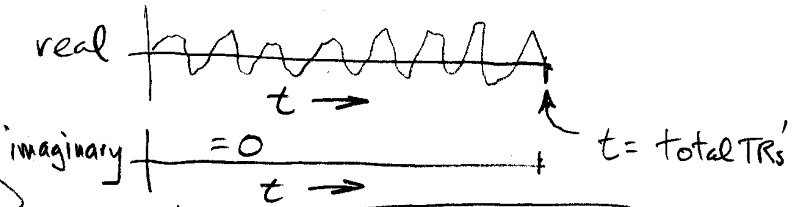




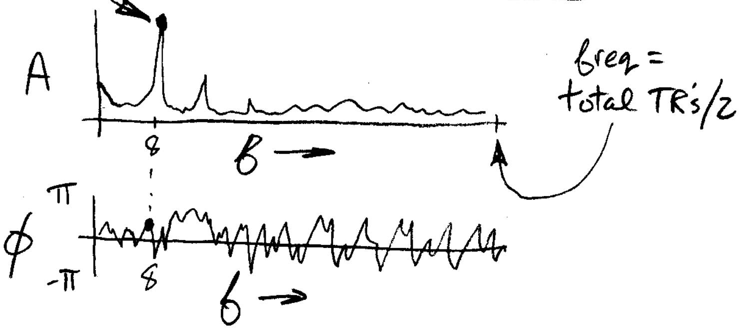
periodic stimuli (phase-encoded) – e.g., 8 cycles at 64 sec/cycle

strongly periodically activated single voxel time course

remove constant (avg) and linear trend



FFT, convert to A, φ



calculate significance

- ratio between amplitude at stimulus frequency (=signal) and average of amplitudes at other frequencies (=noise)
- ignore harmonics, lo freq (=movement)

smooth

- vector average of complex significance (A, φ) with that at nearest neighbor surface points

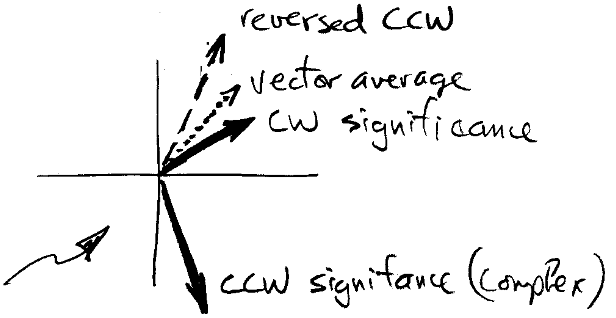
display

- plot phase using hue and saturation to indicate significance

delay correction

- record responses to opposite directions of stimulus (ccw/cw, in/out, up/down)
- vector average after reversing angle of one

↳ penalizes inconsistent more than just avg of angles



CONVOLUTION

for one x

$$h(x) = f(x) \circledast g(x) = \int_{z=-\infty}^{z=+\infty} f(z) \cdot g(x-z)\, dz$$

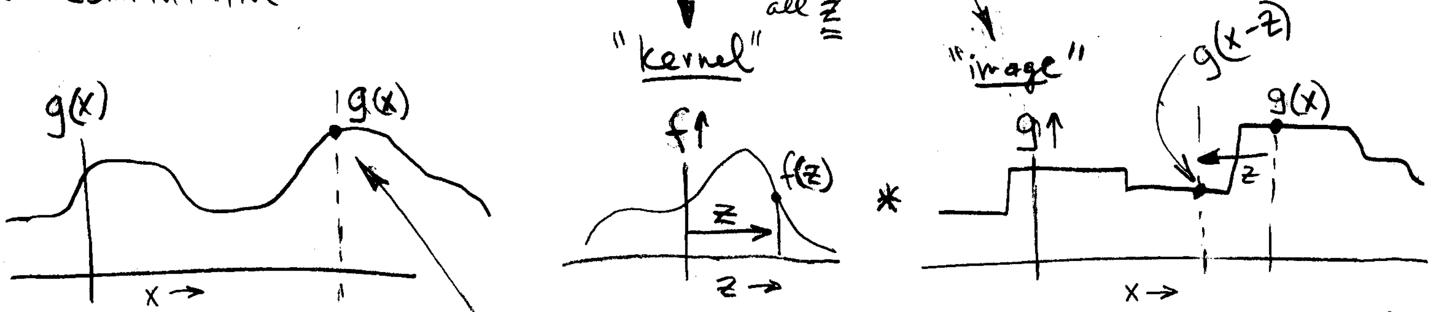
or $(f*g)(x)$

- definition of convolution
- commutative

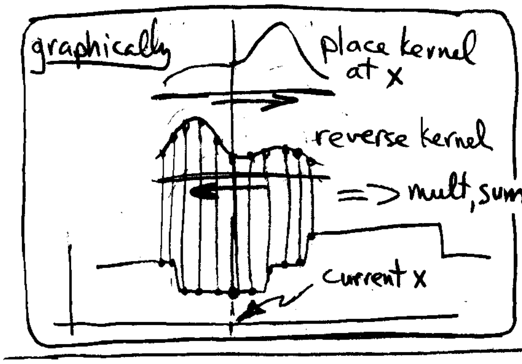
sum this product across all $\underline{\underline{z}}$

linear time/space invariant system

"kernel" "image"

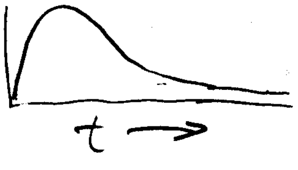


- how to calculate one term
- sum across all $\underline{z}$ to get the value of the convolution at at point $\underline{x}$
- move kernel to calculate next $\underline{x}$

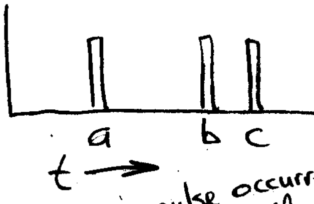


Why reverse makes sense

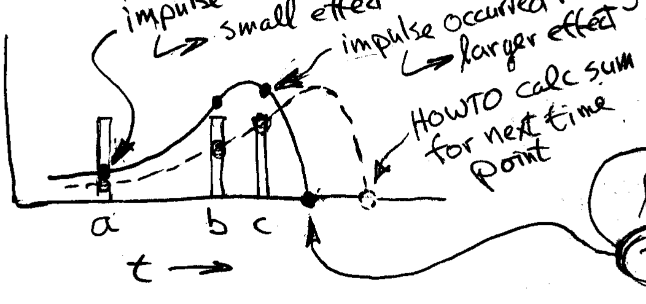
b/c commutative, can think like: $h(x) = \int \underbrace{g(z)}_{\text{image}} \cdot \underbrace{f}_{\text{kernel}}(x-z)\, dz$

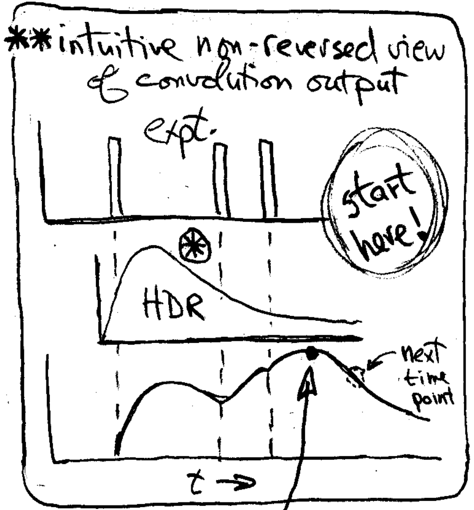


impulse response function (HDR)



impulses (expt. design)





N.B. cross-corr same as convolution except no reversal $g(x+z)$ instead of $g(x-z)$

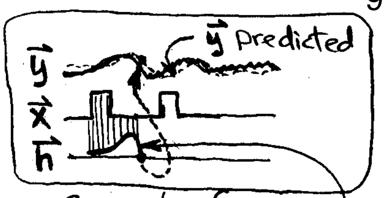
N.B. auto-corr same, except no-reversal and use same funct for both f, g

how to calculate convolution output for this time point (only 3 terms in sum, all other zero)

GENERAL LINEAR MODEL

"block design"
↳ better to detect resp.

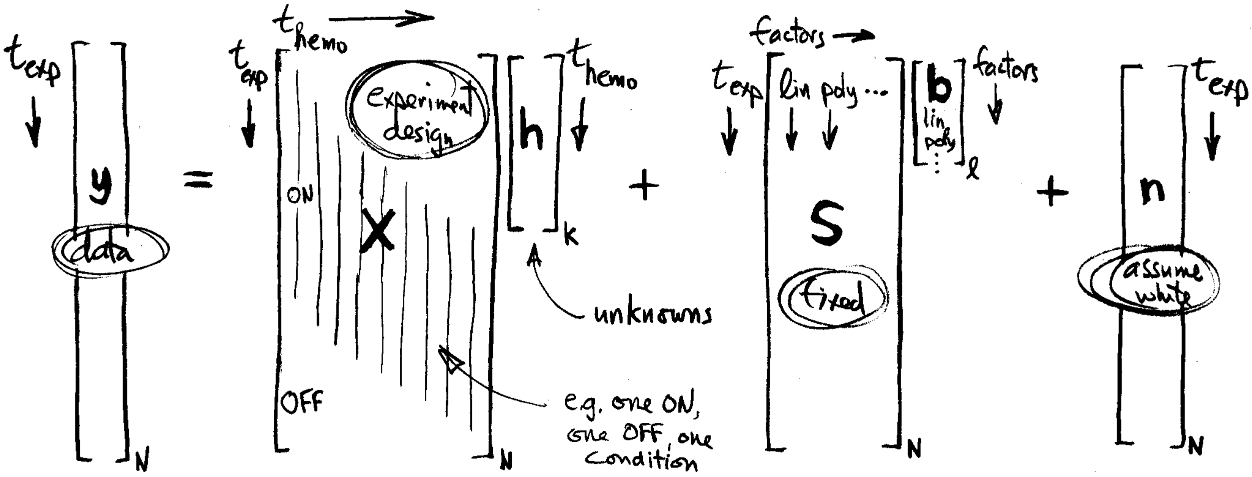
"event-related design"
↳ better to measure HDR shape



— goal: solve for the hemodynamic response functions, $\vec{h}$

$$\vec{y} = \overset{\square}{X}\vec{h} + \overset{\square}{S}\vec{b} + \vec{n}$$

data = design · HDR + drifts · weights + noise



can concat X,S

$$y = \left[X \vdots S\right]\begin{bmatrix} h \\ b \end{bmatrix} + \left[n\right]$$

simpler: preconvolve

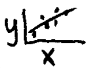
1) conv. $\overset{\square}{X}$ w/ fixed $\vec{h}$

2) solve for β

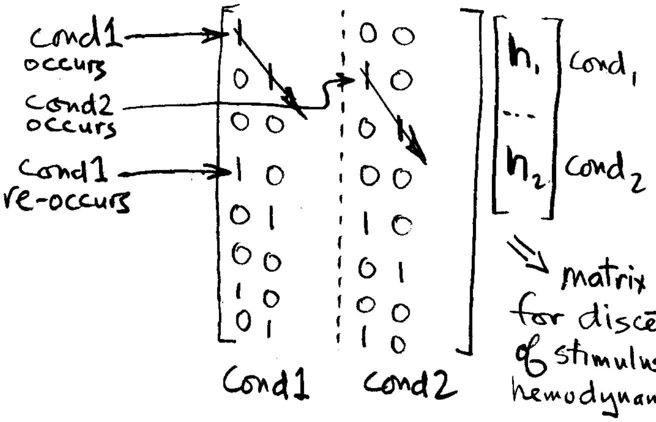
$$\vec{y} = \vec{x}\beta + \vec{n}$$

vect

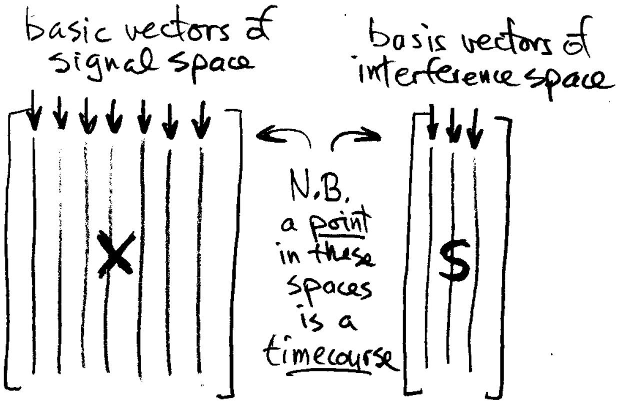
scalar



⇓ multiple conditions



⇒ matrix notation for discrete convolution of stimulus pattern with hemodynamic resp. functs.



[orthogonal col's most efficient
minimize trace$\left[(X_\perp^T X_\perp)^{-1}\right]$ to get

maximum likelihood estimate (Liu et al. 2001 Neuroimage)

1) assume white noise, solve for $\vec{h}$

2) $\hat{h} = (X^T P_S^\perp X)^{-1} X^T P_S^\perp y$ where $P_S^\perp = I - S(S^T S)^{-1} S^T$

↳ projection matrix that removes part of vector that lies in S space

or

$= (X_\perp^T X_\perp)^{-1} X_\perp^T y$ where $X_\perp = P_S^\perp X$

↳ design matrix w/nuisance effects removed from col's

3) significance (how to construct F-ratio)

$$F = \frac{N-k-\ell}{k}\left[\frac{y^T(P_{XS} - P_S)y}{y^T(I - P_{XS})y}\right]$$

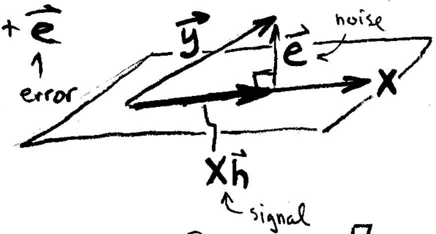
P_{XS} - projects data on expt + nuisance subspace → see diagram next page for geometric interp

P_S - projects data onto nuisance subsp.

GEOMETRIC INTERPRETATION OF GENERAL LINEAR MODEL

- with no nuisance functions (S), we could look at orthogonal projection of data onto experimental design and compare that to error to determine significance

$$\underset{\text{data}}{\vec{y}} = \underset{\text{expt}}{\overset{\square}{X}}\,\underset{\text{HDR}}{\hat{\vec{h}}} + \underset{\text{error}}{\vec{e}}$$

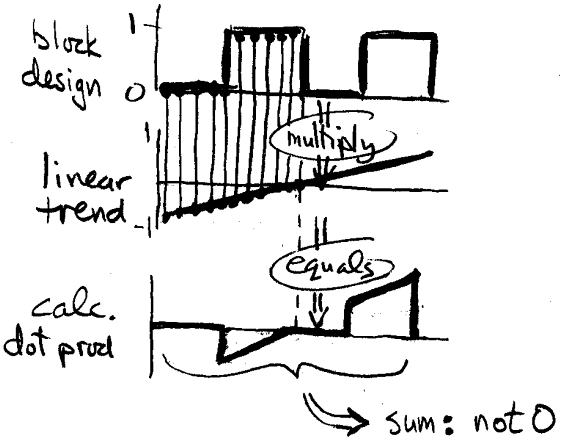


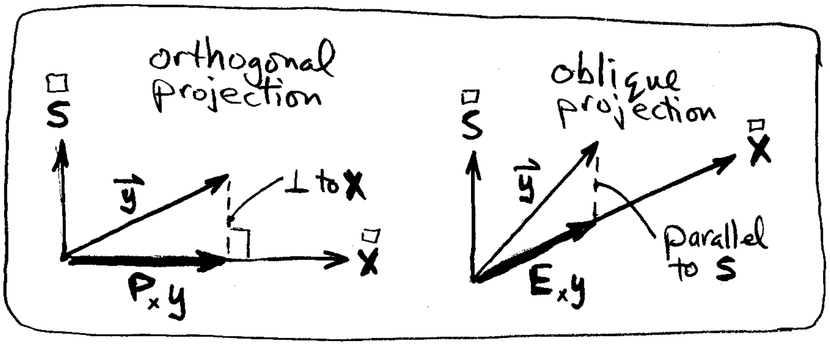
$$\overset{\square}{X}\vec{h} = \overset{\square}{P}_x\vec{y}$$

Projection matrix, P_x, operates on $\vec{y}$ to give projection of data into experiment space, X

- when nuisance functions, $\overset{\square}{S}$, are considered, problem: $\overset{\square}{S}$ may not be orthogonal to $\overset{\square}{X}$
  - for example: linear trend not orthogonal to std. block design

[remember: "orthogonal" means [dot prod. = 0
corr = 0]]

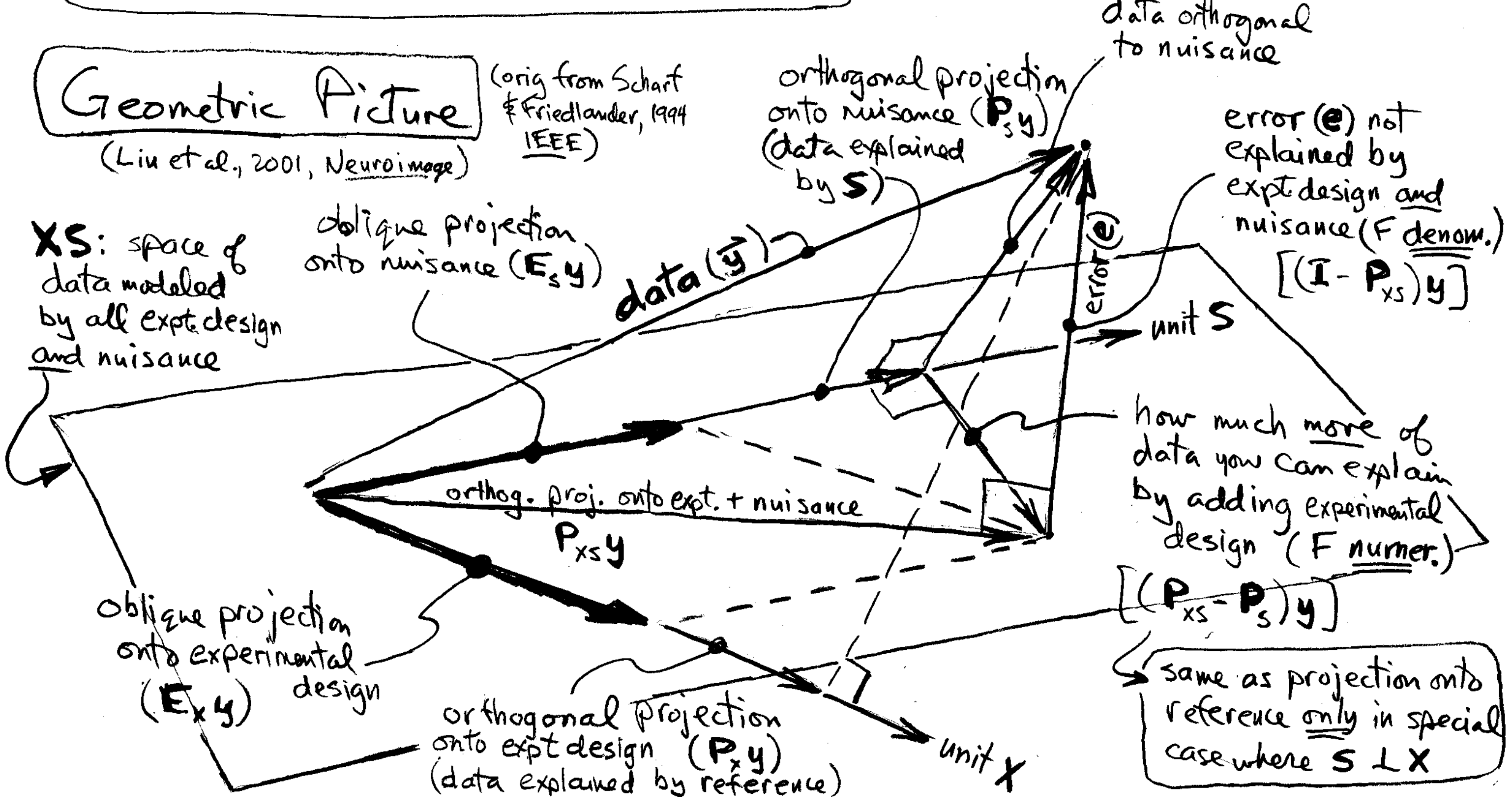




Geometric Picture

(Liu et al., 2001, Neuroimage)

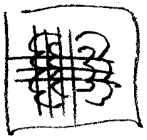
(orig from Scharf & Friedlander, 1994 IEEE)



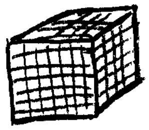
surf-0

WHY USE SURFACES?

- raw MRI data is a 2D flat slice <u>or</u> a 3D volume
  ↳ $I(x,y)$ or $I(x,y,z)$



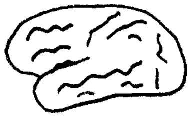
or



but...

1) the neocortex (and cerebellar cortex) are thin, folded <u>2D sheets</u>

   cortex starts as smooth "balloon" → 
   major sulci, temporal lobe form → 
   great size increase, "crinkles" form →



2) neocortex contains many topological maps along its surface

   retinotopy
   tonotopy
   somatotopy
   musculotopy

   plus higher level maps → ~2/3 of its area

3) surface displays allow seeing (almost) all of data at once

only 30% exposed

→

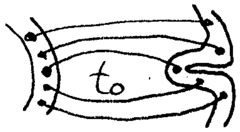
everything visible

→ [flattened]

4) differences in major sulci make 3D-based aligment difficult

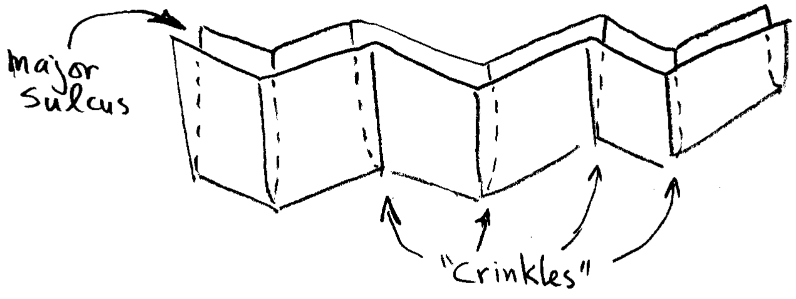
e.g. STS, monkey-like IPS vs. postcentral plus extra

requires mapping



↳ extremely anisotropic def.

5) idiosyncratic sulcal crinkles



- these introduce additional noise into alignment in 3D
- exact position of crinkles unlikely to have functional implications (tho 3D align might respect them)

SEGMENTATION & SURFACE RECON

Talairach, Normalize, StripSkull

1) <u>MNI auto-Talai</u>rach → generates 4×4 matrix

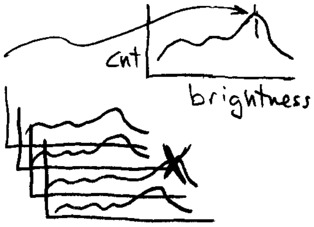
$$\begin{bmatrix} \begin{bmatrix} 3\times 3 \\ \text{(rotate} \\ \text{\& scale)} \end{bmatrix} & \begin{matrix} x \\ y \\ z \end{matrix} \\ 0\ \ 0\ \ 0 & 1 \end{bmatrix} \quad \text{translate}$$

- make average brain target (blurry)
- blur target (further), blur single brain (a lot), gradient descent on xcorr
- repeat w/ less bluring of avg target and current brain
- problems: variable neck cutoff
  - ↳ but *much* better than standard! ← fit to bounding box ← only 2 points near *center* of brain!

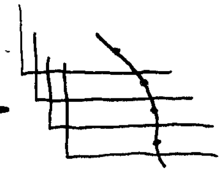
2) <u>Intensity Normalization</u> (output: "T1")

- histogram of pixel values in 10 mm thick HOR slices

- smooth histogram
- peak find to get initial estimate of white matter



- discard outlier peaks across slices →
- fit splines to peaks across slices
  - ↳ *interpolates* scaling factor ⊥ to HOR



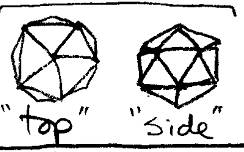
- scale each pixel so WM peak is 110
- refine estimate to interpolate in 3D

5-10 times:
- find points in 5×5×5 within 10% of WM, get new scale for them
- build Voronoi to interpolate scales *unset* above
- soap-bubble-smooth Voronoi boundaries (3 iterations)
- re-scale each voxel

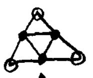
3) <u>Skull Stripping</u> (output: "brain")

- "shrink-wrap" algorithm
- start with ellipsoidal template → subtesselated icosahedron
- minimize *brain penetration* and *curvature*

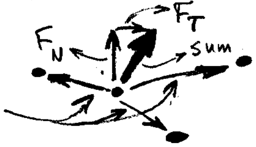
12 vertices
20 tri faces
→ even distribution of points on a *sphere*



- to go finer, subdivide each triangle & promote new vertices to sphere



- *curvature*: spring force (from center-to-neighbor vect sum)



- decompose into ⊥ and tangential (local normal from summed normed cross products)

- *brain penetration*
  - apply force along surface normal that prevents surface from entering gray matter

SEGMENTATION & SURFACE RECON

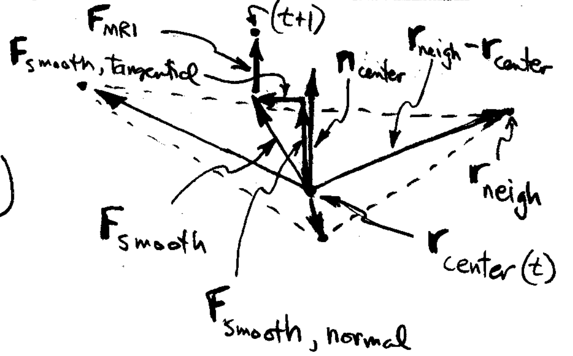
spring force in detail

- implementing a "force" is like directly constructing the operator that minimizes something (without first defining the "something")
- more formally, we would define cost function, <u>then</u> take its derivative (gradient) to minimize it

<u>Shrinkwrap update eq.</u> (skull strip, orignal Dale & Sereno surface refinement)

$$\underbrace{\mathbf{r}_{center}(t+1)}_{\text{for one vertex}} = \underbrace{\mathbf{r}_{center}(t)}_{\text{previous}} + \underbrace{\mathbf{F}_{smooth}(t) + \mathbf{F}_{MRI}(t)}_{\text{local quantities}}$$

rule for each vertex, $\vec{\mathbf{r}}_{center}$



$$\mathbf{F}_{smooth} = \underbrace{\lambda_{tang.}}_{\substack{\text{stronger}\\ \text{than normal}\\ (0.5)}} \sum_{neigh} \underbrace{(\overset{\text{identity }3\times3}{\mathbf{I}} - \mathbf{n}_{center}\mathbf{n}^T_{center})}_{\substack{\text{expand by}\\ \text{distribution to:}\\ \text{neighbor vector}\\ \underline{\text{minus}}\text{ projection}\\ \text{of neighbor onto}\\ \text{normal} \rightarrow \underline{\text{tangential}}!}} \underbrace{\cdot}_{\substack{\text{project}\\ \text{second}\\ \text{onto}\\ \text{first}}} \underbrace{(\mathbf{r}_{neigh} - \mathbf{r}_{center})}_{\substack{\text{vector to}\\ \text{neighbor vertex}}}$$

HOWTO

$\mathbf{n}^T\mathbf{n} = x^2+y^2+z^2$

$\mathbf{n}\mathbf{n}^T = \begin{bmatrix} x^2 & xy & xz \\ yx & y^2 & yz \\ zx & zy & z^2 \end{bmatrix}$

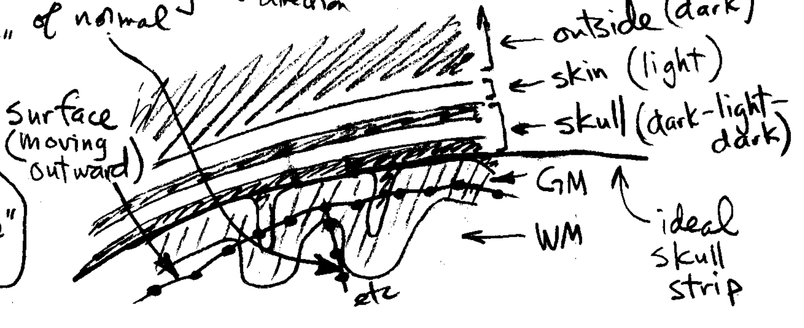
$(\mathbf{n}\mathbf{n}^T)\mathbf{v} = \mathbf{n}(\underbrace{\mathbf{n}^T\mathbf{v}}_{\text{dot prod.}}) =$

$d\mathbf{n}$ = "project $\mathbf{v}$ onto $\mathbf{n}$" (if $\|\mathbf{n}\|=1$)

$$+ \underbrace{\lambda_{normal}}_{\substack{\text{weaker}\\ \text{than tangential}\\ (0.1)}} \left[\sum_{neigh} \underbrace{(\mathbf{n}_{center}\mathbf{n}^T_{center})\cdot(\mathbf{r}_{neigh} - \mathbf{r}_{center})}_{\substack{\text{projection of }\underline{\text{a}}\text{ neighbor vertex}\\ \text{vector onto normal in the}\\ \text{direction of the normal (}\mathbf{n}\text{ is}\\ \text{squared (as above) so we get}\\ \text{a vector out (not a scalar)}}} \underbrace{-}_{\substack{\text{vector}\\ \text{subtract}\\ \text{off}}} \underbrace{\frac{1}{\#\text{vertices}} \sum_{v} \sum_{neigh} (\mathbf{n}_v\mathbf{n}_v^T)\cdot(\mathbf{r}_{neigh} - \mathbf{r}_v)}_{\text{average normal component}} \right]$$

$$\mathbf{F}_{MRI} = \underbrace{\lambda_{MRI}}_{\substack{\text{strong}\\ (1.0)}} \mathbf{n}_{center} \underbrace{\prod_{d}^{30}}_{\substack{\text{multiply all}\\ \text{terms} \rightarrow\\ F_{MRI} = 0 \text{ if}\\ \underline{\text{any}}\text{ are zero}}} \max\Big[\underbrace{0,}_{\substack{\text{don't}\\ \text{allow}\\ \text{neg}}} \underbrace{\tanh}_{\substack{\text{max force}\\ \text{saturates}\\ \text{at 1.0}\\ (\max\text{ "}\Pi\text{"}\\ \text{product}\\ =1.0)}} \big[\overset{\text{intensity MRI data}}{I} \underbrace{(\mathbf{r}_{center} - d\,\mathbf{n}_{center})}_{\substack{d\text{ sample points }\underline{\text{into}}\\ \text{brain along the direction}\\ \text{of normal}}} - \underbrace{I_{thresh}}_{\text{low (40)}}\big]\Big]$$

Snapshot of surface and "core sample" from one vertex



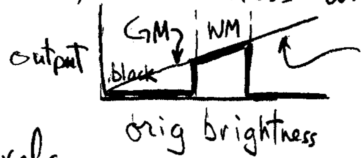
surf-3

SEGMENTATION & SURFACE RECON

filter, cut, tessellate

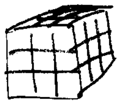
4) Non-isotropic filtering (output: "wm") — "floss" and "spackle"

- preliminary hard threshold:



blood vessels (bright) also truncated to black

- find ambiguous/boundary voxels
  - ↳ 20% or more of 26 immediate neighbors different



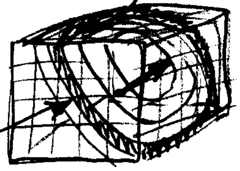
↳ to avoid expensive calc below...

- find plane of least variance

for each ambiguous voxel

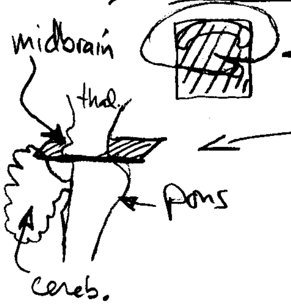
[for each direction (from icosahedral supertessellation)
consider 5×5×5 volume around 1 voxel
find plane of least variance in this hemisphere
median filter w/ hysteresis

(1 vox thick)



↳ if 60% of within-slab differ, reverse classification
↳ "flosses" sulci without blurring

5) Find cutting planes



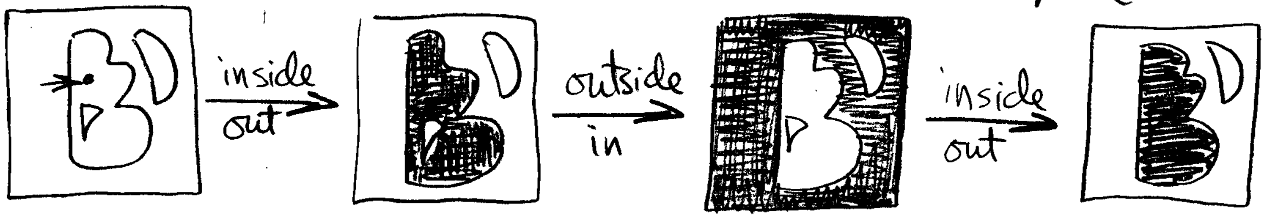
{ callosum, to separate hemispheres (SAG)
midbrain, to avoid fill into cerebellum (HOR) }

→ Talairach to start; fill WM in SAG or HOR till min area

6) Region-growing to define connected parts (output: "filled")

- inside-out, outside-in, inside-out — for each hemisphere

- up/down cycles within each plane
- plane-by-plane

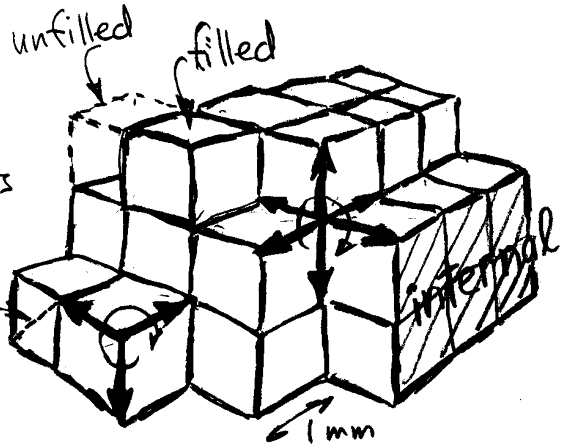


- "wormhole filter" (3×3×3 = center + 26)
  - ↳ fill (unfilled) voxel if 66% neighbors differ → eliminates structures w/ thin, 1-D structure

surf-4

SEGMENTATION & SURFACE RECON 3D → 2D

7) Surface Tessellation (output: rh.orig, lh.orig)



- variable num neighbors possible!

- quads to triangles

3D data set → $I(x,y,z)$

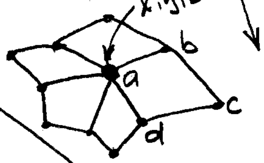
⇓

2D surface → vertex list (coords)

$x_1 \quad y_1 \quad z_1$ → vtx nums implicit
$x_2 \quad y_2 \quad z_2$
⋮

2D surface → face list (vertex nums)

$a_1 \; b_1 \; c_1 \; d_1$ → face nums implicit
$a_2 \; b_2 \; c_2 \; d_2$
⋮



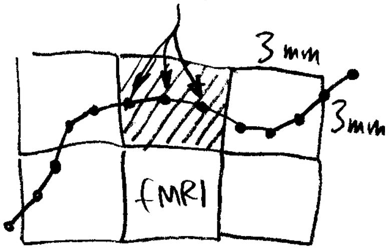
- find filled voxels bordering unfilled
- make ordered list of neighboring vertices
  ↳ so cross-products oriented properly

- long list of values associated with each <u>numbered</u> <u>vertex</u>

  e.g.
  position (orig, morphed)
  area (orig, morphed)
  curvature (intrinsic, Gaussian)
  "sulcus-ness" (summed $\perp$ movement during unfolding)
  cortical thickness
  fMRI data $\Leftarrow$
  EEG/MEG dipole strength

- separate fMRI data set must be aligned, sampled
  - fMRI voxels larger
  - sample at each <u>surface vertex</u>
  - nearest-neighbor "soap bubble" smoothing to interpolate data onto hi-res mesh

fMRI → surface initial sample same value here



⇓

interpolate <u>on surface</u>

- some quantities only well-defined on surface
  ↳ gradient of magnitude of cortical map measure (e.g., eccentricity)

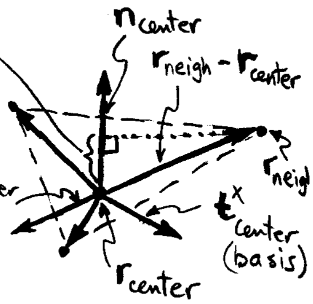
SEGMENTATION & SURFACE RECON

smooth, inflate, final surfaces

– smoothing/inflation/WM, pial done as <u>derivative</u> of <u>energy functional</u>

$$\underbrace{J}_{\substack{\text{total}\\\text{scalar}\\\text{error to}\\\text{minimize}}} = \underbrace{J_{\text{tangential}}}_{\substack{\text{scalar tangential}\\\text{error (fixed by}\\\text{redistributing vertices)}}} + \underbrace{\lambda_{\text{normal}}}_{\substack{\text{small}\\(0.25)}} \underbrace{J_{\text{normal}}}_{\substack{\text{scalar curvature}\\\text{error (fixed by}\\\text{reducing curvature)}}} + \underbrace{\lambda_{\text{image}}}_{\substack{\text{smaller}\\(0.075)}} \underbrace{J_{\text{image}}}_{\substack{\text{scalar image error}\\\text{(fixed by moving toward}\\\text{target image value)}}}$$

$$\underbrace{J_{\text{normal}}}_{\substack{\text{across all}\\\text{vertices,}\\\text{curvature}\\\text{error}}} = \underbrace{\frac{1}{2\,\#\text{vert}}}_{\substack{\text{1/2 so no}\\\text{coefficient}\\\text{on derivative}}} \underbrace{\sum_{\text{centers}}}_{\substack{\text{across}\\\text{all vertices}}} \underbrace{\sum_{\text{neighbors}}}_{\substack{\text{neighbors}\\\text{of one}\\\text{vertex}}} \Big[\underbrace{\mathbf{n}_{\text{center}}}_{\substack{\text{vertex}\\\text{unit}\\\text{normal}}} \underset{\substack{\text{project}\\\text{onto}\\\text{normal}}}{\cdot} \underbrace{(\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}})}_{\substack{\text{vector from}\\\text{current center}\\\text{to one neighbor}\\\text{(position vector diff.)}}}\Big]^2$$



$$\underbrace{J_{\text{tangential}}}_{\substack{\text{"squishing"}\\\text{of mesh}}} = \frac{1}{2\,\#\text{vert}} \sum_{\text{centers}} \sum_{\text{neighbors}} \Big[\underbrace{\mathbf{t}^x_{\text{center}}}_{\substack{\text{x-direction}\\\text{in tangent}\\\text{plane}}} \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}})\Big]^2 + \Big[\underbrace{\mathbf{t}^y_{\text{center}}}_{\substack{\text{y-direction}\\\text{in tangent}\\\text{plane}}} \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}})\Big]^2$$

project vector to neighbor onto x & y

$\mathbf{t}^x, \mathbf{t}^y$ are first 2 eigenvect. of neighbor vector cloud ($\mathbf{n}$ is third)

$$\underbrace{J_{\text{image}}}_{\text{image error}} = \frac{1}{2\,\#\text{vert}} \sum_{\text{centers}} \Big[\underbrace{I^{\text{targ}}_{\text{center}}}_{\substack{\text{target}\\\text{brightness}}} - \underbrace{I(\mathbf{r}_{\text{center}})}_{\substack{\text{brightness at}\\\text{current location}}}\Big]^2$$

I^{targ} for WM: mean of voxels labeled WM in 5 mm neighborhood

I^{targ} for pia: global – small num for C.S.F.-like

[– take directional derivative of energy functional (to find steepest uphill)
[– move each vertex in the opposite (negative) direction w/ self-intersect test

N.B.: eq. 9 in Dale, Fischl & Seren different – and <u>in</u>correct!

$$\underbrace{-}_{\substack{\text{go the}\\\text{opposite}\\\text{direction}}} \underbrace{\frac{\partial J}{\partial \mathbf{r}_{\text{center}}}}_{\substack{\text{direction}\\\text{(vector) of}\\\text{largest scalar}\\\text{error for}\\\text{one vertex}\\\text{movement}}} = \lambda_{\text{image}} \underbrace{\Big[I^{\text{targ}}_{\text{center}} - I(\mathbf{r}_{\text{center}})\Big]}_{\text{scalar}} \underbrace{(-\nabla I(\mathbf{r}_{\text{center}}))}_{\substack{\text{vector – calculate}\\\text{gradient on image}\\\text{(first blur w/ Gaussian)}}}$$

N.B.: cancels neg on LHS so ∂J says which way to go

Σ_{centers} gone b/c const

$$+ \sum_{\text{neighbors}} \lambda_{\text{normal}} \underbrace{\Big[\mathbf{n}_{\text{center}} \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}})\Big]}_{} \underbrace{(-\mathbf{n}_{\text{center}})}_{\substack{\text{unit normal}\\\text{vector}}}$$

scaled by

HOWTO derivative:

constants, var

$$\frac{\partial J}{\partial \mathbf{r}} = \partial(C_1 \cdot (C_2 - V))$$
$$= \partial(C_1 C_2 - C_1 V)$$
$$= -C_1$$

$$+ \sum_{\text{neighbors}} \overbrace{\Big[\mathbf{t}^x_{\text{center}} \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}})\Big](-\mathbf{t}^x_{\text{center}})}^{\substack{\text{x-component}\\\text{of tangential}}} + \overbrace{\Big[\mathbf{t}^y_{\text{center}} \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}})\Big](-\mathbf{t}^y_{\text{center}})}^{\text{y-component}}$$

SULCUS-BASED CROSS-SUB. ALIGN

- use summed perpendic. vertex move during inflation as vtx measure of "sulcus-ness"
- add term to error function, J: "sulcus-ness" error

$$J_{sulc} = \frac{1}{2\#vert}\sum_{centers}\left[S^{sub_1}_{cent} - S^{targ}(\mathbf{r}_{center})\right]^2$$

find neg of steepest uphill direction of change in sulcus-ness of target

take deriv.

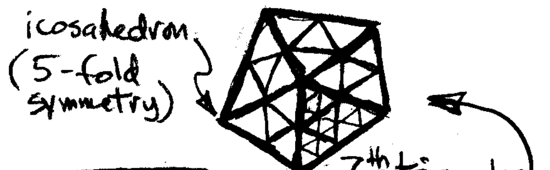
$$-\frac{\partial J_{sulc}}{\partial \mathbf{r}_{cent}} = \lambda_{sulc}\left[S^{subj}_{cent} - S^{targ}(\mathbf{r}_{center})\right]\left(-\nabla S^{targ}(\mathbf{r}_{center})\right)$$

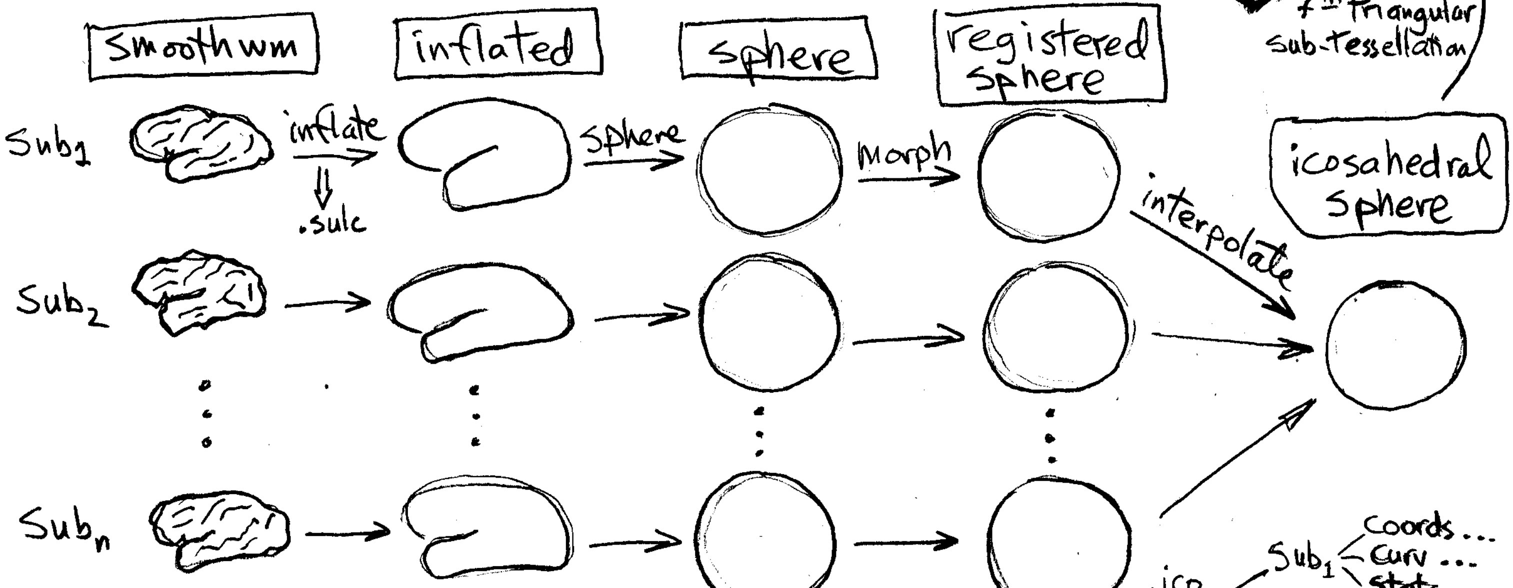
grad descent

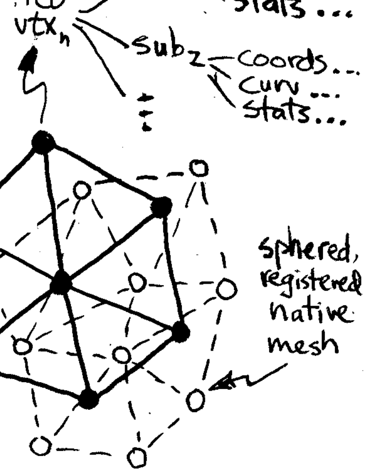
2D

sulcus-ness of moveable subj vtx

- bootstrap
  - morph to one brain
  - make avg targ
  - re morph to avg targ





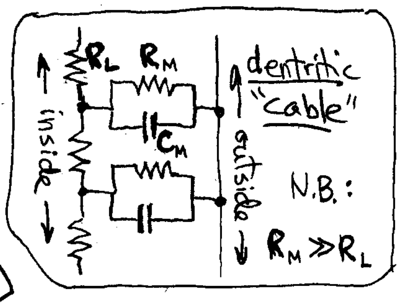


- each sub's native surf has diff # vertices
- interpolate values (coords, surface measures, stats) to each icosahedral vertex from neighboring vertices of native mesh (dashed lines)
- average surface made from folded/inflated avg coords
  - folded: loses area from sulcal crinkles (fsaverage "inflated")
  - inflated: retains orig area, correct sulc/gyrus ratio ("inflated_avg")
- can use sampled-to-icos individual subj coords to draw icosahedral surface in shape of an indiv. brain

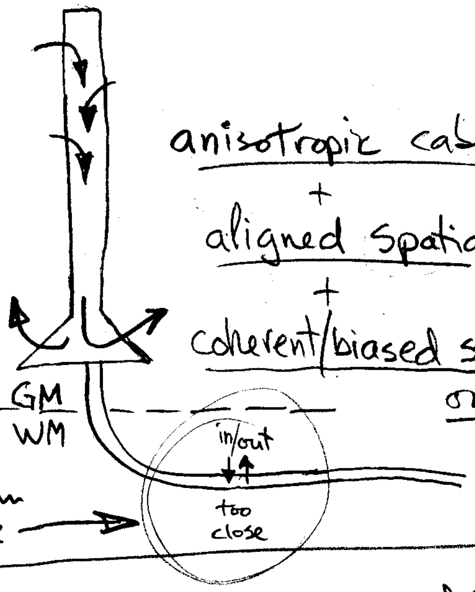
↳ N.B.: morph will have changed local vertex density compared to more uniform native mesh (use native for sing. sub.)

source-1

SOURCE OF EEG/MEG

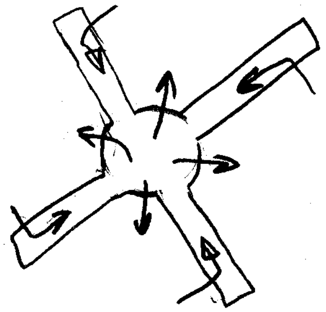


PSPs



anisotropic cables
+
aligned spatially
+
coherent/biased stim one end

- no distant signal from axon spike



isotropic

"closed field" (invisible at distance)

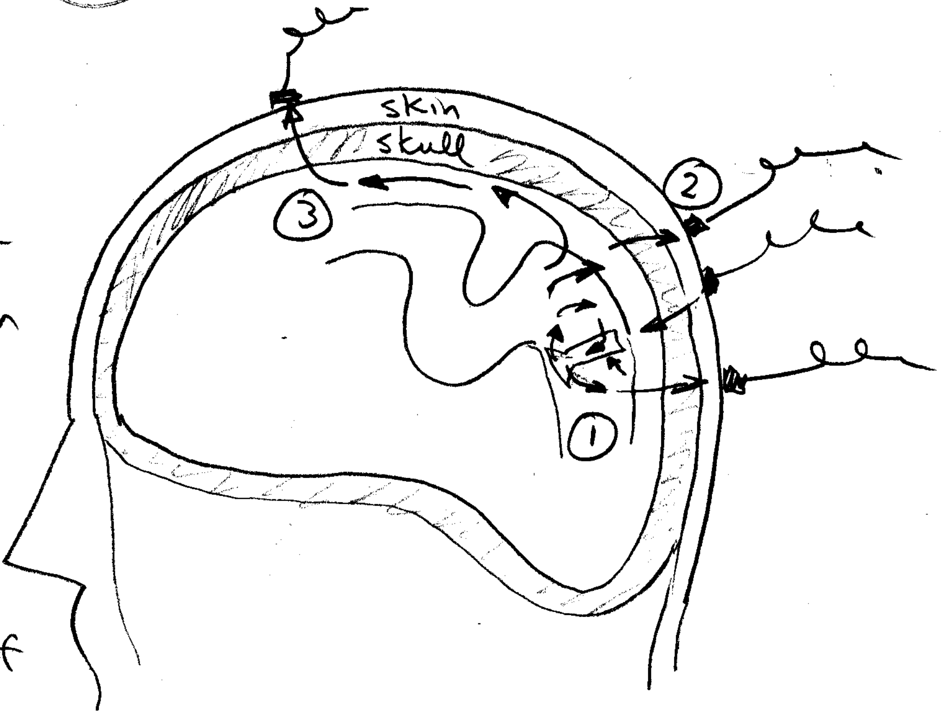
N.B.: spikes only detected by 15 μm microelectrode in gray matter!

Head

1) - local dipole

2) - EEG through skull, skin

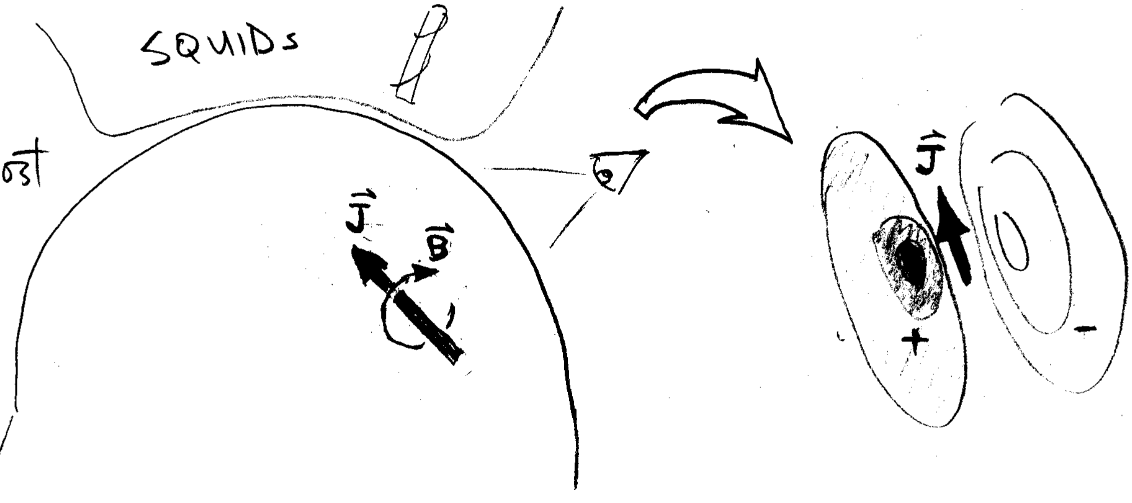
3) - smearing because skull 1/80 conductivity of brain



the effect on current flow is like a higher order (larger) "cable"

MEG

- radial dipoles lost
- tangential dipole generates Gabor-like scalp distrib. of $\vec{\mathbf{B}}$ field

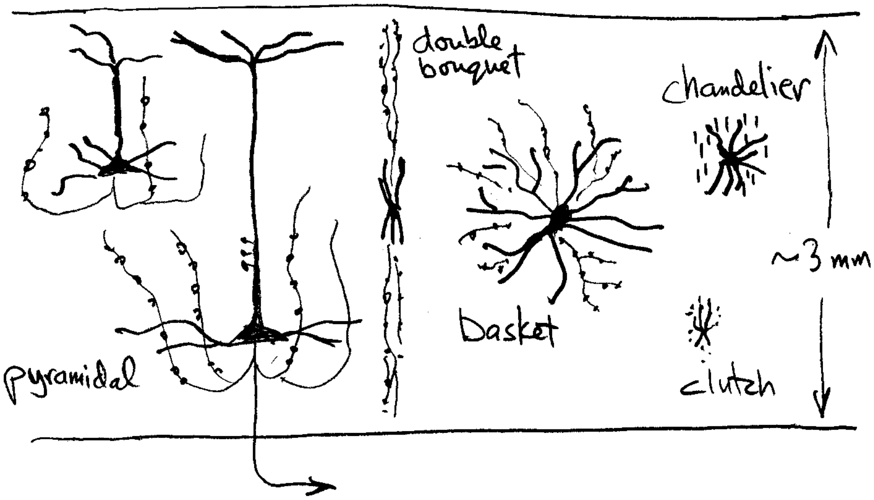


Source-2

INTRACORTICAL CIRCUITS & ORIGIN OF EEG

Cell types

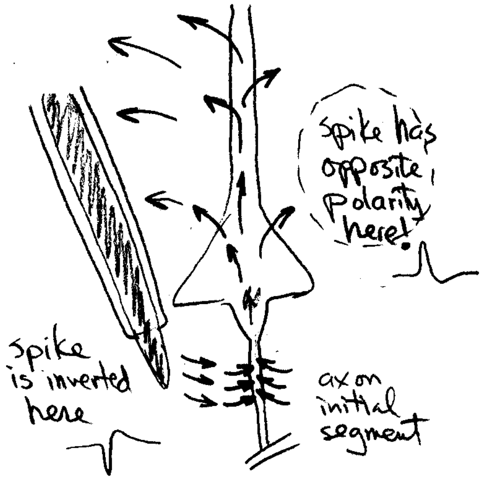
- excitatory (spiny)
  - pyramidal
  - spiny stellate (e.g. V1 layer 4C)
- inhibitory (smooth)
  - basket
  - double bouquet
  - chandelier
  - clutch



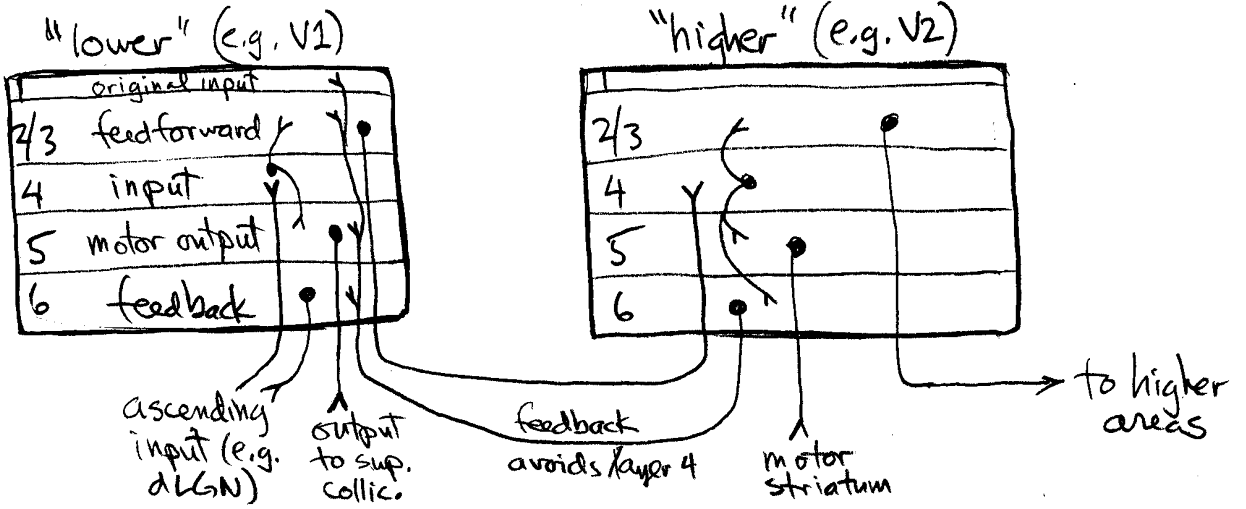
Circuits

- huge complexity
- first principal components: input → layer 4 → layer 2/3 → feedforward; layer 4 → layer 5/6 → output feedback; layer 2/3 ↑↓ layer 5/6
- microelectrode recording (e.g. 10 μm tip)
  - high pass → spikes
  - low pass → local field potentials

- spikes only recordable in gray matter
- white matter spikes only recordable with pipette w/ very fine tip b/c inward & outward currents so spatially close in axon/spike (> 1 μm)



intra/inter cortical connections cartoon

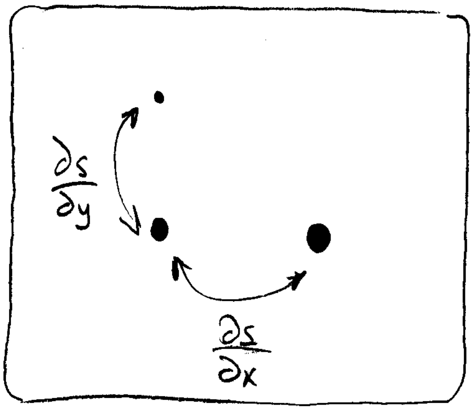


graddivcurl-1

GRADIENT, DIVERGENCE, CURL

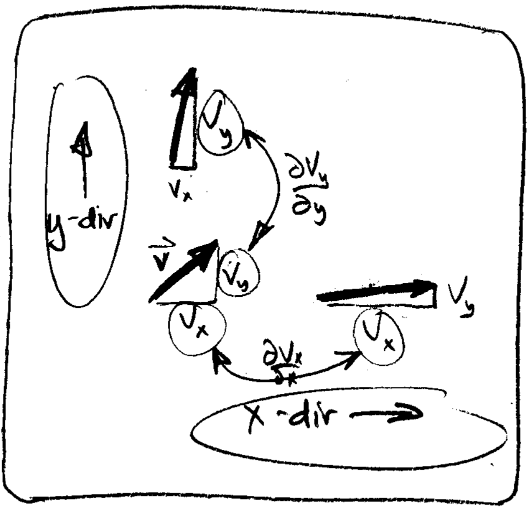
gradient (∇) (generalized deriv.)

$$\underbrace{\nabla}_{\text{turns scalar field into vector field}} \underbrace{s(\overset{x,y,z}{\vec{r}})}_{\text{scalar funct defined at each } x,y,z \text{ point}, \mathbf{r}} = \underbrace{\frac{\partial s(\vec{r})}{\partial x}}_{\text{change of } s \text{ in } x \text{ direction at point } \vec{r}} \underbrace{\vec{i}}_{\text{unit vector in } x \text{ dir}} + \frac{\partial s(\vec{r})}{\partial y}\vec{j} + \frac{\partial s(\vec{r})}{\partial z}\vec{k}$$



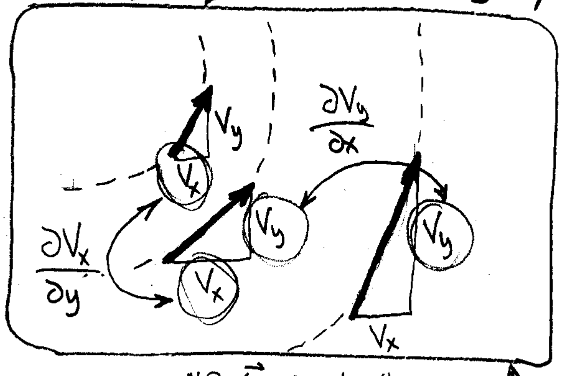
divergence ($\nabla\cdot$) (deriv. "dot prod")

$$\underbrace{\nabla\cdot}_{\text{turns vector field into scalar field}} \underbrace{\vec{v}(\vec{r})}_{\text{vector funct defined at each } x,y,z \text{ point}, \mathbf{r}} = \underbrace{\frac{\partial v_x(\vec{r})}{\partial x}}_{\text{change of just } x\text{-component of } \mathbf{v} \text{ in } x\text{-direction at point } \mathbf{r}} + \frac{\partial v_y(\vec{r})}{\partial y} + \frac{\partial v_z(\vec{r})}{dy}$$



curl ($\nabla\times$) (deriv. "cross product")

$$\underbrace{\nabla\times}_{\text{turns vector field into another vector field}} \underbrace{\vec{v}(\vec{r})}_{\text{vector funct defined at each } x,y,z \text{ point}, \mathbf{r}} = \left(\underbrace{\frac{\partial v_z(\vec{r})}{\partial y}}_{\text{change of just } z \text{ component of } \vec{v} \text{ in } y\text{-direction at point } \vec{r}} - \frac{\partial v_y(\vec{r})}{\partial z}\right)\vec{i} + \left(\frac{\partial v_x(\vec{r})}{\partial z} - \frac{\partial v_z(\vec{r})}{\partial x}\right)\vec{j} + \left(\frac{\partial v_y(\vec{r})}{\partial x} - \frac{\partial v_x(\vec{r})}{\partial y}\right)\vec{k}$$



N.B. $\vec{k}$ ⊥ to this plane

Vector identities

$\nabla\times\nabla s = 0$ curl of the gradient of any scalar field is zero

$\nabla\cdot(\nabla\times\vec{A}) = 0$ divergence of the curl of any vector field is zero

$\vec{A}\cdot(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{B})\cdot\vec{C}$

csd-1

POTENTIAL (Φ), ELECTRIC FIELD ($\nabla\Phi$) & C.S.D. ($\nabla\cdot(-\nabla\Phi) \equiv \nabla^2\Phi$)

low-frequency field approximation

- electric fields uncoupled from magnetic (vs. electromagnetic radiation)
  - ↳ pre-Maxwellian approx. (EEG freq.'s ≪ 1 MHz)
- calculate electric fields as if magnetic fields don't exist
- calculate magnetic fields strictly from distribution of currents
- ignore capacitative effects, too

scalar potential, Φ (what we measure with electrode)

ignore coupling: $\vec{B} \equiv \nabla\times\vec{A}$ ← vector potential ⟹ because EEG ≪ 1 MHz

scalar potential at one position

$$\underbrace{\vec{E}}_{\text{electric field vector}} = -\underbrace{\nabla}_{\text{turns scalar field }(\Phi)\text{ into vector field }(\vec{E})}\Phi \boxed{-\frac{\partial\vec{A}}{\partial t}} \cong -\left[\frac{\partial\Phi}{\partial x}\vec{i}+\frac{\partial\Phi}{\partial y}\vec{j}+\frac{\partial\Phi}{\partial z}\vec{k}\right]+0$$

gradient of scalar field

(1) $\vec{E}$ defined as force (vector) acting on unit charge at a given point in space (as result of arbitrary distribution of other charges)

(2) Current density, $\vec{J}$ (not curr. source dens.!) is proportional to $\vec{E}$ ⟶ still a vector!

$$\vec{J} = \sigma\vec{E} \quad \text{(Ohm's law)} \quad \sigma \text{ is conductivity}$$

(3) Two def's of $\vec{A}$:

$$\vec{B} = \nabla\times\vec{A}$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{vol}\frac{\vec{J}}{\|\vec{r}\|}\,d\,vol \quad \boxed{\text{dist-weighted sum of directional currents}}$$

CSD is Laplacian of Φ (= div $\vec{E}$)

$$\overset{\text{divergence}}{\underbrace{\nabla\cdot}_{\substack{\text{measures rate of change}\\ \text{of vector field across space}}}}(-\nabla\Phi) = \text{scalar field} = -\left[\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}\right] \equiv -\nabla^2\Phi$$

makes scalar!

$\nabla\cdot$ turns vector field into scalar field

meaning of $\nabla\cdot$ ↑,↓

find radiating and contracting regions of $\vec{E}$ vector field ⟹

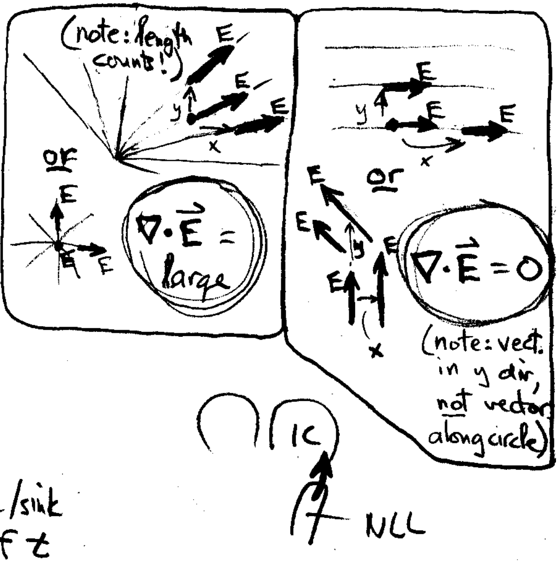
sum of:

change in x component in x direction

+

change in y comp. in y dir

⋮



3D CSD gold standard (rat BAER paper)

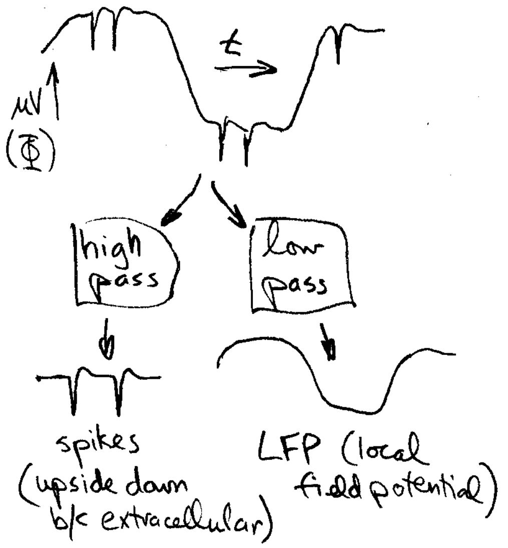
$$\underset{(360\text{ points})}{\Phi\text{ data}} \xrightarrow{\nabla} \underset{\text{electric vector field}}{\nabla\Phi} \longrightarrow \underset{\substack{\text{scalar field source/sink}\\ \text{movie as function of } t}}{\nabla\cdot(-\nabla\Phi)}$$

csd-2

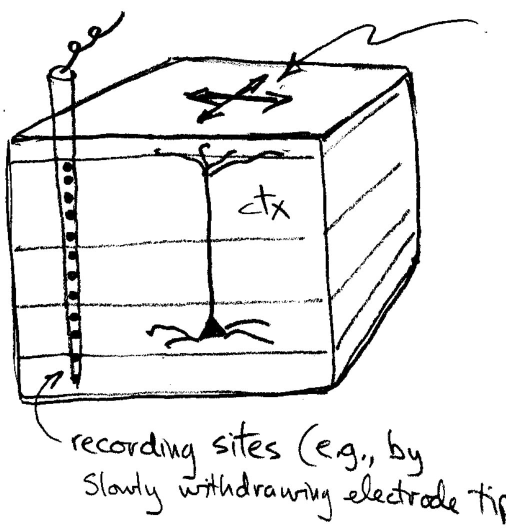
1D AND 2D CURRENT SOURCE DENSITY EXPTS.

1D CSD

- raw, event-related signal relative to ground, ⏚ (eg. skull)



both types of data can be recorded from same electrode

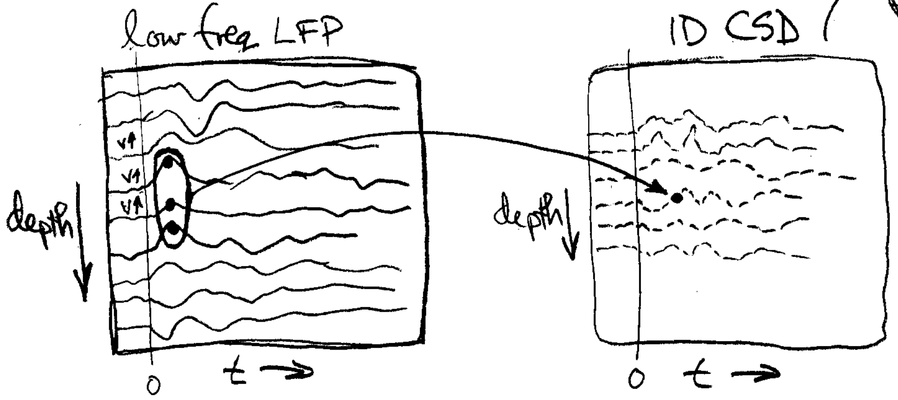


ignore these dimension parallel to cortical sheet

recording sites (e.g., by slowly withdrawing electrode tip)

rationale: CSD changes much more slowly parallel to cortex than perpendicular to cortical sheet

↳ assume approx constant (≈0) parallel to ctx



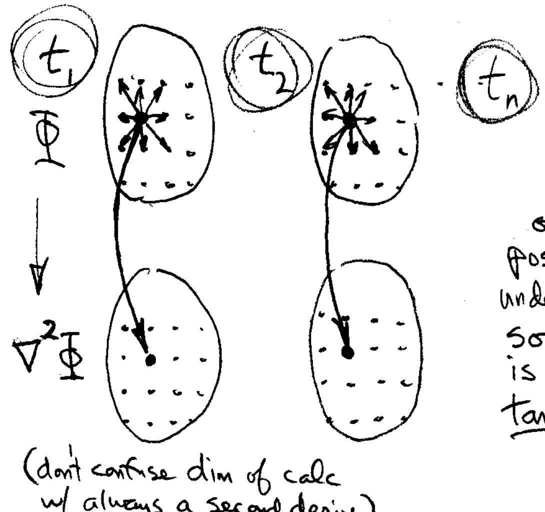
e.g. shows prominent early sinks in layer IV consistent w/dLGN inputs in V1

- ∇^2 means find spatial (i.e., 1D depth) curvature of potential
- discrete approx: center $-$ $\frac{\text{above}+\text{below}}{2}$
- N.B. in example above, even though all 3 potentials are positive, smaller value of center point implies sink!

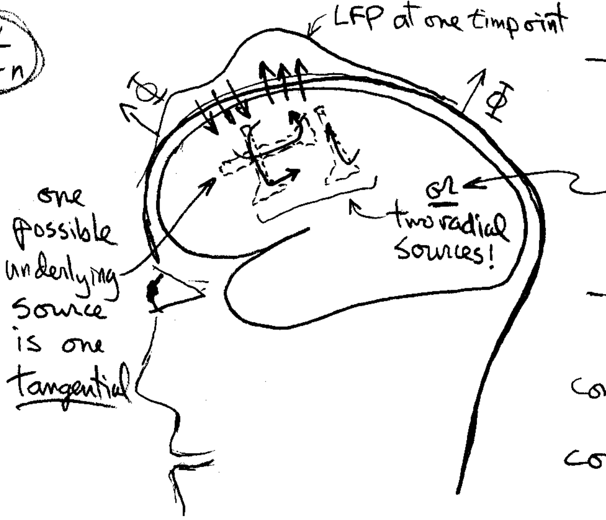
2D CSD

- 2D array of electrodes on pial surface or on scalp

rationale: all electrodes record along same surface so assume depth profiles are constant



(don't confuse dim of calc w/ always a second deriv.)



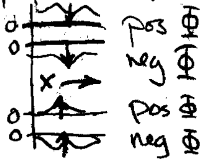
one possible underlying source is one tangential

or two radial sources!

- for scalp recordings, sources and sinks are at the scalp (not a depth loc method unless done in 3D)
- can't tell curvature from sign of potential!

concave → sink

convex → source

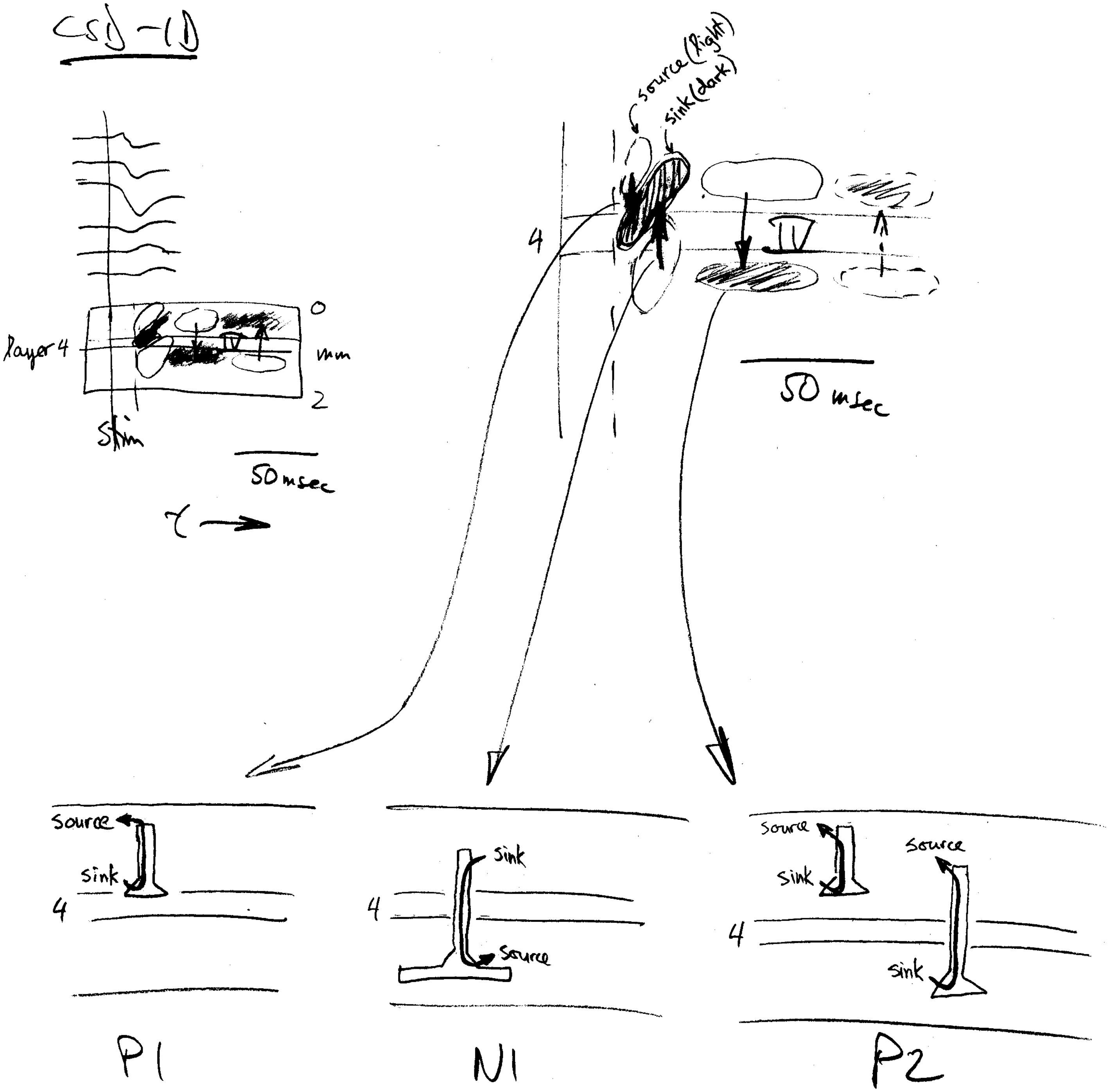


csd-3

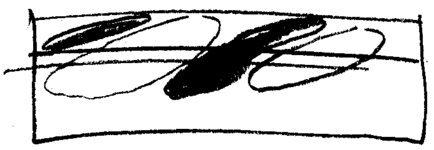
INTRACORTICAL C.S.D.

- e.g. click evoked rat A-I
  (Sukov & Barth, 1998)

CSD-1D



- phase-locked CSD ρ
  gamma shifts w/each cycle →



surface electrode is *temporal* ref

forward-1

MAXWELL EQUATIONS

Electrostatics, Magnetostatics
low freq limit

Ohm's law: $\sigma \vec{E} = \vec{J}$

$$\underbrace{\nabla\cdot}_{\text{divergence}} \Big(\underbrace{\sigma}_{\substack{\text{conductivity}\\ \text{constant (or}\\ \text{tensor constant}\\ \text{if inhomogeneous}\\ \text{in different directions)}}} \underbrace{\nabla \Phi}_{\substack{\text{gradient}\\ \text{of scalar}\\ \text{potential}\\ \text{(what we}\\ \text{measure}\\ \text{invasively)}\\ \text{at each point}}}\Big) = \underbrace{\nabla\cdot}_{\text{divergence}} \underbrace{\vec{J}^i}_{\text{impressed currents}}$$

impressed currents

currents due to ionic flow that "appear out of nowhere" (Nernst batteries)

like CSD but times σ, conductivity (cf. $gV = I$, $V = IR$)

N.B. these are all defined at a (every) point in space

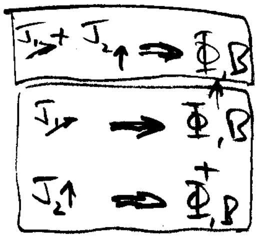
$$\underbrace{\nabla\cdot}_{\text{divergence}} \underbrace{\vec{B}}_{\substack{\text{magnetic}\\ \text{field vector}\\ \text{at each point}}} = 0$$

[incidentally, this Maxell equation violated by a linear gradient in B_z in x, y or z dir]

$$\underbrace{\nabla \times}_{\text{curl}} \underbrace{\vec{B}}_{\substack{\text{magnetic}\\ \text{field}}} = \underbrace{\mu_0}_{\text{permeability}} \Big(\underbrace{\vec{J}^i}_{\substack{\text{impressed}\\ \text{currents}}} - \underbrace{\sigma}_{\text{conductivity}} \underbrace{\nabla \Phi}_{\substack{\text{gradient of}\\ \text{salar potential}\\ (= \vec{E})}}\Big)$$

- propagation of potentials, magnetic fields instantaneous (no capacitance)
- simultaneous eq's to solve: $\vec{J}$ are <u>sources</u>, Φ, $\vec{B}$ are <u>data</u>
- <u>linear</u>

potential (Φ) and magnetic fields ($\mathbf{B}$) produced by a weighted sum of two current source distributions are <u>equal</u> to weighted sum of fields produced by each current source distribution by itself



forward-2

WHY WE CAN IGNORE MAGNETIC INDUCTION

(from Nunez, 1981)

$$\underbrace{\vec{E}}_{\text{electric field}} = \underbrace{-\nabla\Phi}_{\text{field component due to charge distribution}} - \underbrace{\frac{\partial \vec{A}}{\partial t}}_{\text{field component due to coupling between electric \& magnetic fields}}$$

$$\vec{B} \equiv \nabla\times\underbrace{\vec{A}}_{\text{"vector potential"}}$$

$$\nabla\times\underbrace{\vec{H}}_{\text{magnetic field (in given medium) induced}} = \underbrace{\vec{J}}_{\text{currents (in given medium)}} + \underbrace{\frac{\partial \vec{D}}{\partial t}}_{\text{time varying electric fields (in given medium)}}$$

$\vec{H} = 1/\mu\,\vec{B}$ permeability μ

$\vec{J} = \sigma\,\vec{E}$ conductivity σ

$\vec{D} = \varepsilon\,\vec{E}$ permettivity ε

characterize substance linear in all three

this induced magnetic field induces an electric field in turn

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

take $\nabla\times$ of both sides

use $\vec{B} = \mu\vec{H}$

substitute this into this

$$\nabla\times\nabla\times\vec{E} = -\mu\frac{\partial}{\partial t}\left(\vec{J} + \frac{\partial\vec{D}}{\partial t}\right)$$

if linear in conductivity and dielectric, too, and fields periodic w/ f

$$\nabla\times\nabla\times\vec{E} = -2\pi f i\mu(\sigma + 2\pi f i\varepsilon)\vec{E}$$

to neglect: $$\frac{2\pi f\mu(\sigma + 2\pi f\varepsilon)|\vec{E}|}{|\nabla\times\nabla\times\vec{E}|} \ll 1$$

1) $|\nabla\times\nabla\times\vec{E}| \approx |\vec{E}|/L^2$ where L is dist over which $\vec{E}$ varies significantly

2) μ of tissue similar to empty space

3) assume conservative (large) σ, dielectric const., and EEG freq

↳ number is about 10^{-6} → small

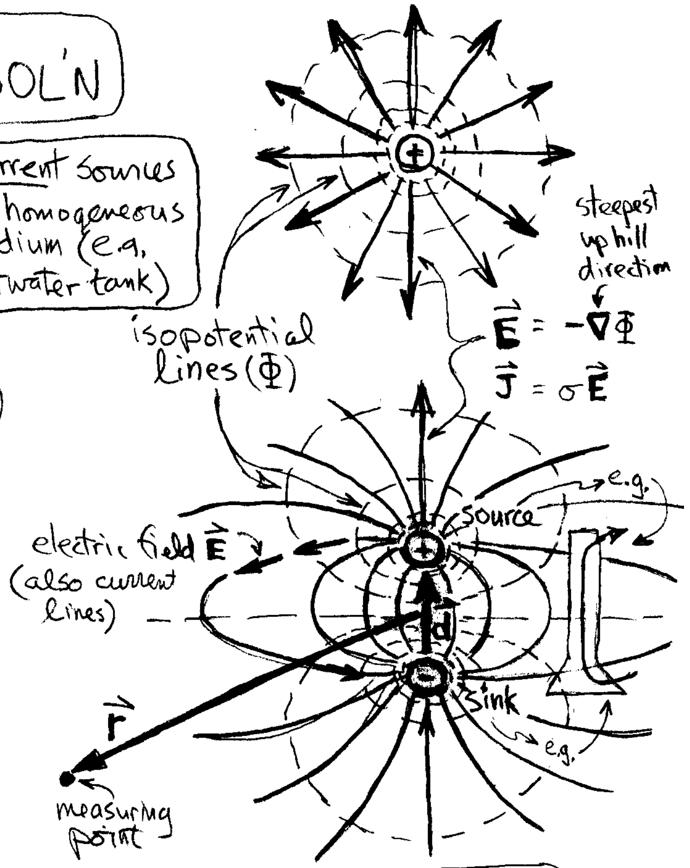
forward-3

MONOPOLE, DIPOLE FORWARD SOL'N

$$\underbrace{\Phi_1}_{\substack{\text{potential}\\ \text{recorded}\\ \text{for source monopole}}} = \frac{\overset{\text{strength of source}}{s}}{4\pi\sigma\,\underbrace{r}_{\substack{\text{distance, source to}\\ \text{measuring point } (=\|\vec{r}\|)}}}$$

Current sources in homogeneous medium (e.g. saltwater tank)

$$\underbrace{\Phi_2}_{\substack{\text{potential}\\ \text{recorded for}\\ \text{source-sink}\\ \text{pair}\\ (\text{"near field"})}} = \frac{s}{4\pi\sigma}\left(\frac{1}{\underbrace{r_1}_{\substack{\text{dipole}\\ \text{source to}\\ \text{measuring}\\ \text{distance}}}} - \frac{1}{\underbrace{r_2}_{\substack{\text{dipole}\\ \text{sink to}\\ \text{measuring}\\ \text{distance}}}}\right)$$



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scalar $$\Phi_2 \approx \left(\frac{1}{4\pi\sigma}\right)\frac{\overset{\vec{J}}{\underbrace{s}_{\substack{\text{scalar}\\ \text{strength}}}\,\underbrace{\vec{d}}_{\substack{\text{dipole}\\ \text{vector,}\\ \text{sink to}\\ \text{source}}}}\cdot\underbrace{\vec{r}}_{\substack{\text{measuring}\\ \text{point vector}\\ \text{(center of dipole}\\ \text{to point)}}}}{\underbrace{r^3}_{\substack{\text{distance}\\ \text{center of}\\ \text{dipole to}\\ \text{measuring}\\ \text{point}}}}, \quad r \overset{3\text{-}4\text{x}}{\gg} d$$

now: also explicitly include dipole orientation and measuring point in equations

approximations for "far enough away" measurements (subtracting two 1/r's gives inverse square)

N.B.: reverse of current flow + → −

** since $\vec{r}$ also in numerator, this now inverse square

vector $$\vec{B}_2 \approx \left(\frac{\mu_0}{4\pi}\right)\frac{\overset{\vec{J}}{s\,\vec{d}}\times\vec{r}}{r^3}, \quad r \underset{3\text{-}4\text{x}}{\gg} d$$

N.B.: both assume inside infinite isotropic conductor

$$\underbrace{\Phi_i}_{\text{electrode}}(t) = \underbrace{e_i}_{\text{gain}}\,\underbrace{s(t)}_{\text{one source strength}}$$

$$\vec{b}_i(t) = \vec{m}_i\, s(t)$$

$$b_i(t) = m_i\, s(t) \longrightarrow \text{SQUIDs measure component of } \vec{B}$$

$$\Phi_i(t) = \sum_j e_{ij}\,\underbrace{s_j(t)}_{\text{all sources}}$$

$$\underbrace{b_i(t)}_{3 \text{ for } \vec{B}} = \sum_j m_{ij}\, s_j(t)$$

linear superposition with fixed electrodes and sensors

$$\underbrace{\vec{x}(t)}_{\text{elec \& mag. measures}} = \sum_j \underbrace{\vec{g}_j}_{\text{gain vect}}\,\underbrace{s_j(t)}_{\text{strength}} \longrightarrow \vec{x}(t) = \overset{\square}{G}\,\vec{s}(t)$$

forward-4

Forward Solution (1)

- well-posed (one answer)
- linear: $b(A) + b(B) = b(A+B)$
- approximations due to unknown electrical properties of head

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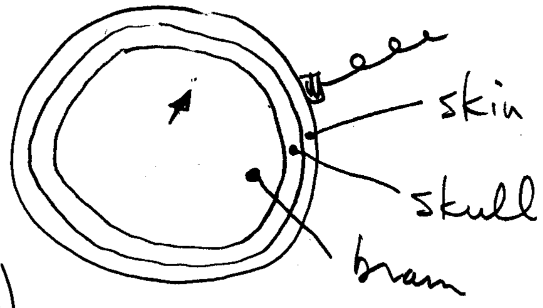
- 3-shell spherical analytic

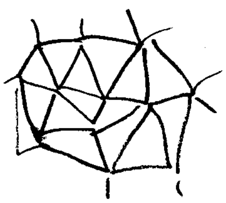
- skull 1/80 conductivity of brain
- "smearing" (cf. cable theory)



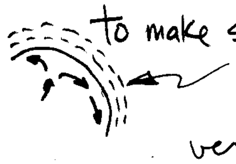
- remember, we only need to be able to calc. weight for each dipole/electrode pair independently

- 3-shell boundary element

[arbitrary shape
homogeneous conductivity



- for magnetic, only need one shell b/c currents thru skin/skull too small to make sig. $\vec{B}$

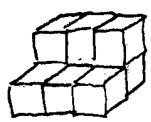


solution = infinite homogeneous + matrix of correction factors

vertex (columns) / vertex (rows)

calc for one head

- finite element



[most general
computational intensive w/ small grid
many unknown parameters to estimate

forward-S

FORWARD SOL'N (2)

recording

sources

forward

source strength at:

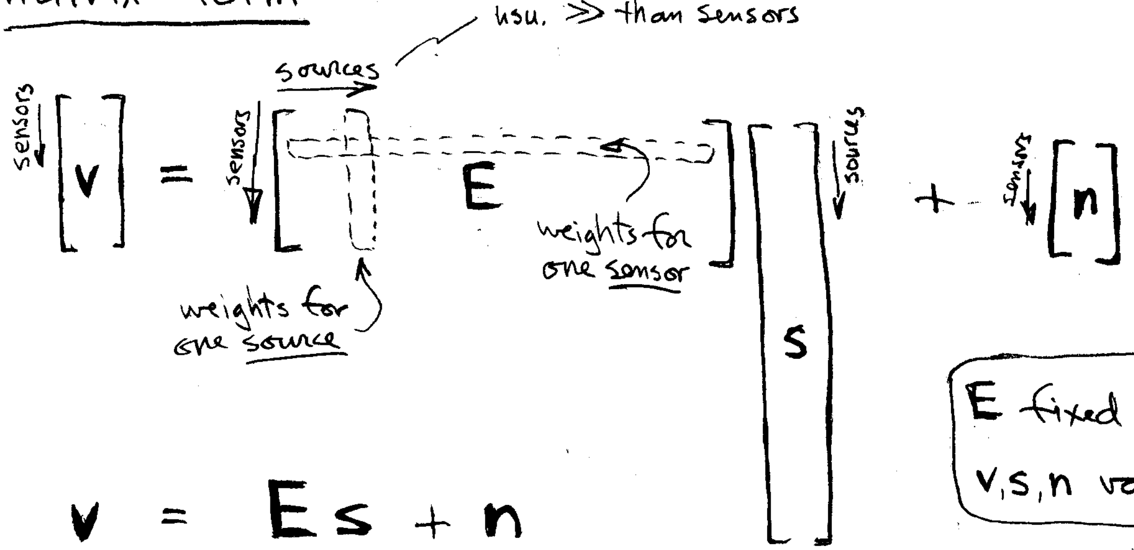
position (x, y, z)
angle (θ, φ)
in brain

$$V_i = \sum_j e_{ij} s_j + n_i$$

sensor

noise

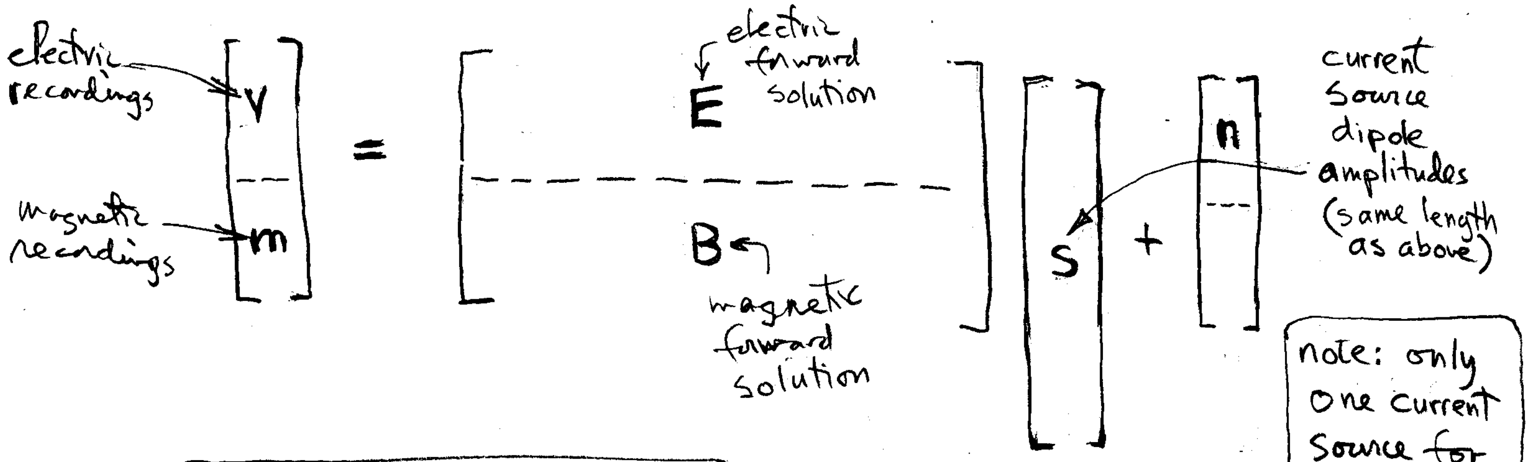
matrix form



E fixed across time
v, **s**, **n** vary

$$\mathbf{v} = \mathbf{E}\mathbf{s} + \mathbf{n}$$

lower case bold → vector
upper case bold → matrix



$$\boxed{\vec{\mathbf{x}} = \overset{\square}{\mathbf{A}}\vec{\mathbf{s}} + \vec{\mathbf{n}}}$$

note: only one current source for each column in the **E+B** matrix!

WHY LOCALIZE?

- most of ERP literature based (instead) on temporal "components"

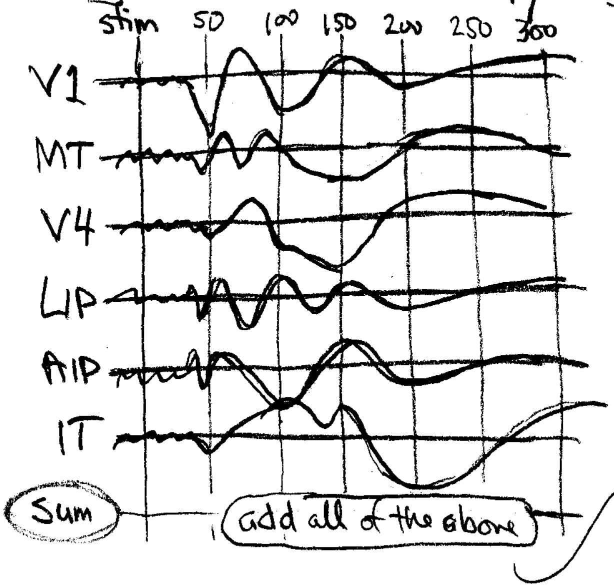
but: 1) underlying local cortical generators (from microelectrode LFP, CSD)
- extended in time (400 msec), visible from every scalp electrode
- multiphasic in every cortical area
- temporally non-static depending on stimulus
  - e.g. simple contrast, brightness diffs can modulate retinal delay by 50 msec!

2) thus, any "component" consists of sum of activity from multiple cortical areas at different hierarchical levels

3) stimulus manipulations will change temporal overlap
- may cause "component" peak to disappear without changing cortical areas being activated

4) verified by intracortical LFP/CSD (Schroeder et al, 1998)

macaque monkey intracortical data



→ these areas span the visual system from bottom to top, accounting for roughly 50% of the entire macaque monkey ctx

LFP's from approx. layer 4 in cortex ("input layer")

psychologists now identify a few temporal "component" peaks...
- but each peak comes from every one of these cortical areas!!

- by contrast, the spatial signature of the signal from one cortical area is static — a better area-based "component"

- temporal "components" should be retired!!
- their original reason for being no longer relevant!!
  - easier to record more temporal points (EEG started w/ few electrodes, many time points)
  - easier to "paste" high level psychological functions onto a few waveform deflections

inverse-1

Derivation of Ill-posed Inverse

(from Dale & Sereno, 1993)

$$\mathbf{x} = \mathbf{A}\mathbf{s} + \mathbf{n}$$

$\mathbf{x}$ = sensor data vector

$\mathbf{A}$ = forward soln matrix $(\mathbf{E}+\mathbf{B})$

$\mathbf{s}$ = source vector

$\mathbf{n}$ = sensor noise vector

$$\mathrm{Err}_{\mathbf{w}} = \left\langle \| \overset{\text{solve for inverse operator}}{\mathbf{W}} \mathbf{x} - \mathbf{s} \|^2 \right\rangle \longleftarrow \text{expectation:} \left[= \sum_k \overset{\text{probability that rand. var. value is k}}{p_k} \overset{\text{rand var. value}}{k} \right]$$

assume $\mathbf{n}, \mathbf{s}$ normal, zero-mean w/ corresponding covar. matrices $\mathbf{C}, \mathbf{R}$

$$\begin{aligned}
\mathrm{Err}_{\mathbf{w}} &= \left\langle \| \mathbf{W}(\mathbf{A}\mathbf{s}+\mathbf{n}) - \mathbf{s} \|^2 \right\rangle \\
&= \left\langle \| (\mathbf{W}\mathbf{A}-\mathbf{I})\mathbf{s} + \mathbf{W}\mathbf{n} \|^2 \right\rangle \\
&= \left\langle \| \mathbf{M}\mathbf{s} + \mathbf{W}\mathbf{n} \|^2 \right\rangle \quad \text{where } \mathbf{M} = \mathbf{W}\mathbf{A}-\mathbf{I} \\
&= \left\langle \| \mathbf{M}\mathbf{s} \|^2 \right\rangle + \left\langle \| \mathbf{W}\mathbf{n} \|^2 \right\rangle \\
&= \mathrm{tr}(\mathbf{M}\mathbf{R}\mathbf{M}^T) + \mathrm{tr}(\mathbf{W}\overset{\text{diag is noise variance (already squared)}}{\mathbf{C}}\mathbf{W}^T) \quad \text{trace is sum of diag elements}
\end{aligned}$$

[re-expand] $= \mathrm{tr}(\mathbf{W}\mathbf{A}\mathbf{R}\mathbf{A}^T\mathbf{W}^T - \mathbf{R}\mathbf{A}^T\mathbf{W}^T - \mathbf{W}\mathbf{A}\mathbf{R} + \mathbf{R}) + \mathrm{tr}(\mathbf{W}\mathbf{C}\mathbf{W}^T)$

Explicitly minimize by taking derivative w.r.t. W, set to zero, solve for W

$$0 = 2\mathbf{W}\mathbf{A}\mathbf{R}\mathbf{A}^T - 2\mathbf{R}\mathbf{A}^T + 2\mathbf{W}\mathbf{C}$$

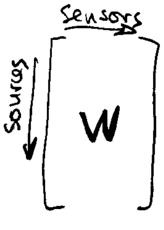
$$\mathbf{W}\mathbf{A}\mathbf{R}\mathbf{A}^T + \mathbf{W}\mathbf{C} = \mathbf{R}\mathbf{A}^T$$

$$\mathbf{W}(\mathbf{A}\mathbf{R}\mathbf{A}^T + \mathbf{C}) = \mathbf{R}\mathbf{A}^T$$

$$\boxed{\mathbf{W} = \mathbf{R}\mathbf{A}^T(\mathbf{A}\mathbf{R}\mathbf{A}^T + \mathbf{C})^{-1}} \longrightarrow$$

equivalent to minimum norm and Tikhonov regularized inverse if **C**, **R** are proportional to identity matrix (i.e., sensor noise & sources independent and equal variance)

W is inverse solution operator :



inverse-2

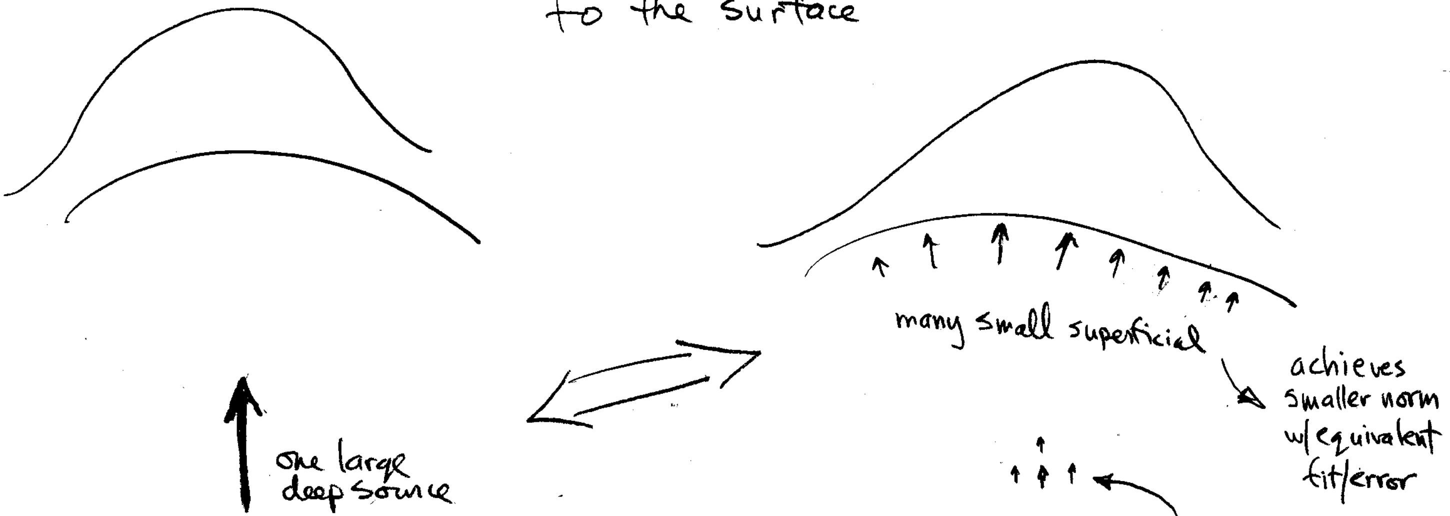
INVERSE SOL'N (2)

w/ liberal priors [sources uncorrelated (R)
noise uncorrelated (C)

$$W = RA^T(ARA^T + C)^{-1}$$

↳ "minimum norm" solution
(find $\vec{s}$ w/ smallest norm $= \|\vec{s}\|$)

- the minimum norm solution appropriately downplays deeper (= weaker scalp signal) sources since these are more likely to fall into the noise floor
- "problems" of minimum norm:
  ↳ deeper sources get displaced to the surface

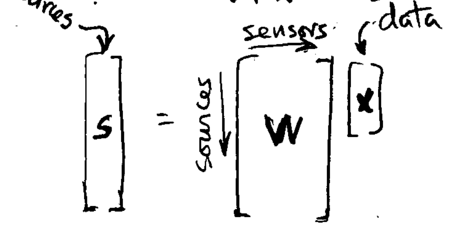


- small superficial sources "win" because of approx. inverse square form of fwd solution
  ↳ smaller norm of distributed superficial sol'n
- can't fix by increasing priors of deep sources!!
  ↳ that will give deep sources given noise as input!!

INVERSE SOLUTIONS TO ILL-POSED COMPARED

over-determined ↪ least squares
under-determined ↪ minimum norm

$$\mathbf{s} = \mathbf{W}\mathbf{x}$$



- how to use the inverse solution, **W**
- same **W** for all timepoints

"minimum norm" solution

i.e., norm $\|\vec{\mathbf{s}}\|$ of solution is smallest of infinitely many alternate solutions

---

linear inverse operator; spatial covariance sources; forward soln; spatial covariance sensor noise

$$\mathbf{W} = \mathbf{R}\mathbf{A}^T(\mathbf{A}\mathbf{R}\mathbf{A}^T + \mathbf{C})^{-1}$$

→ from error minimization derivation

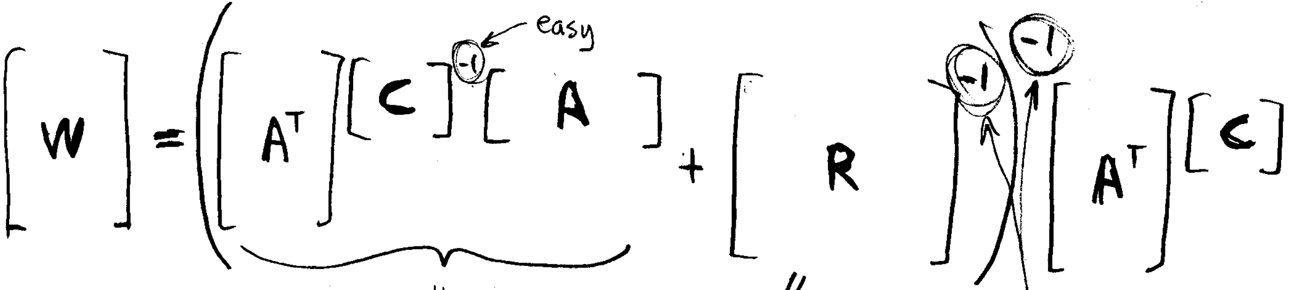
$[ARA^T] \Rightarrow$ square in # of sensors (small)

---

linear inverse operator

$$\hookrightarrow \mathbf{W} = (\mathbf{A}^T\mathbf{C}^{-1}\mathbf{A} + \mathbf{R}^{-1})^{-1}\mathbf{A}^T\mathbf{C}$$

→ alternate, algebraically equivalent Bayesian derivation (w/ bigger inverses!)



$[A^T C^{-1} A] \Rightarrow$ both square in # of sources (large)

hard inverses

use inverse-1

PROBLEMS W/ SURFACE NORMAL

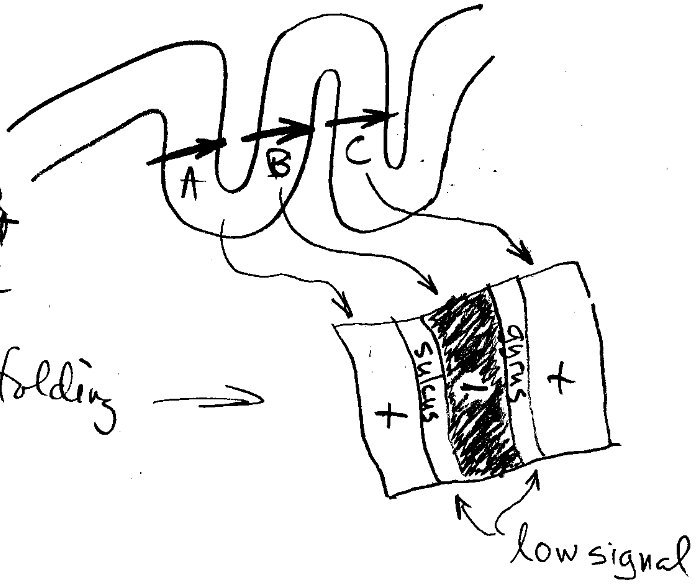
- since nearby points on surface often have different orientation, surface normal constraint can help (since fwd soln A,B very different)

- but, since point spread function: typically extends across sulci, artifactual sign reversals occur

point $\xrightarrow{fwd}$ data

spread $\xleftarrow{invers}$

positive current at A indistinguishable from negative current at B, or pos at C



↳ after unfolding →

- Solutions

  1) ignore sign → saves useful orientation info!

  2) solve onto 3 orthogonal dipoles at each cortical point instead of a single oriented dipole

     ↳ more appropriate when averaging across subjects, since detailed orientations vary a lot

     → also, fills in bottom of sulcus (else unsigned stripes)

useinverse-2

fMRI Constrained Inverse

[could be combined w/ MUSIC!]

- insert fMRI values for R_{ii}'s
- but still allow other sites to have non-zero R_{ii}'s
- pathologies occur if solution restricted completely to fMRI points by setting non fMRI R_{ii}'s to zero ↗ set to small number instead!
- this allows extracting time course from sources visible in EEG/MEG and fMRI
- N.B.: sources that are only visible in EEG/MEG will be dispersed to small distributed values at a large number of vertices

visible in both EEG/MEG and fMRI

visible only in EEG/MEG and not fMRI → distributed at small amplitude across many vertices

useinverse-3

NOISE SENSITIVITY NORMALIZATION (1)

(Liu, Dale, and Belliveau, 2002)

forward: $x = As$ well-posed

inverse: $s = Wx$ ill posed

Solve: $x = As + n$ for $\underline{s}$

$$W = RA^T(ARA^T + C)^{-1}$$

variance of dipole strength estimate due to additive sensor noise

$$Var(s_i) = \langle (W_i\, n(t))^2 \rangle = W_i C W_i^T$$

already squared

that is, sensor noise run through inverse

– multiply inverse operator by noise sensitivity matrix, D (diagonal)

$$\underset{\text{sources}}{\breve{D}_{ii}} = \frac{1}{\text{diag}_i \sqrt{WCW^T}}$$

$$W^{ns\_norm} = DW$$

normalize each $\underline{\text{row}}$ of inverse operator by noise sensitivity at that location

$$\begin{array}{cc} & \text{sensors} \\ \text{sources} & \begin{bmatrix} W \end{bmatrix} \end{array}$$

$$s_i^{ns\_norm} = (W^{ns\_norm}x)_i = (DWx)_i = \frac{W_i x}{\sqrt{(WCW^T)_i}} = \sqrt{\frac{(Wxx^TW^T)_i}{(WCW^T)_i}}$$

$$\|W_i\| = \sqrt{(WCW^T)_i}$$

$$\underbrace{\begin{bmatrix} W \end{bmatrix} \overset{I}{\begin{bmatrix} \cancel{C} \end{bmatrix}}}\begin{bmatrix} W^T \end{bmatrix}$$

$\to WW^T \Rightarrow$ each entry is sum of squares

if assume Gaussian white noise, noise covariance, C, is multiple of $\mathbf{I}$, so

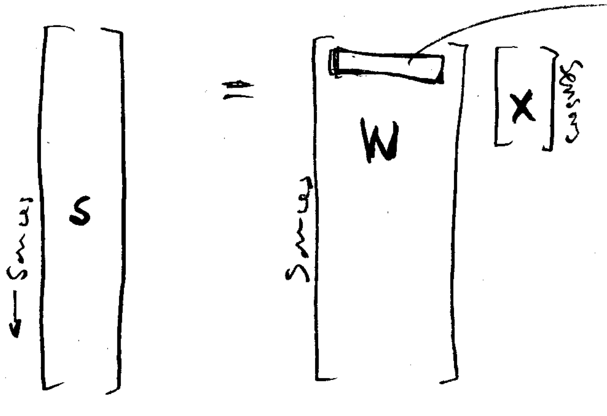
$$W_i^{ns\_norm} = \frac{W_i^{orig}}{\|W_i^{orig}\|}$$

i.e., scale each $\underline{\text{row}}$ of W by single value — the $\underline{\text{norm}}$ of that row

$$\underset{\text{one source}}{s_i} = \frac{\overset{\text{one row}}{W_i^{orig}} \cdot \overset{\text{sensors}}{\vec{x}}}{\|W_i^{orig}\|}$$

row of $\mathbf{W}$ is:

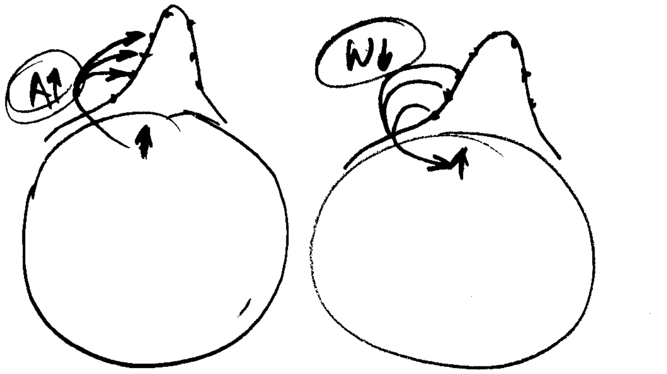
inverse solution coefficients for $\underline{\text{one source}}$ $\Rightarrow$ scale (divide) by $\underline{\text{norm}}$ of this row



that is, if inverse sol'n for deep source is $\underline{\text{reduced}}$ by interaction of inverse square nature of fwd $\underline{\text{and}}$ min norm, dividing by norm of row of inverse (1 source) will $\underline{\text{increase}}$ / rescue deep source use inverse-4

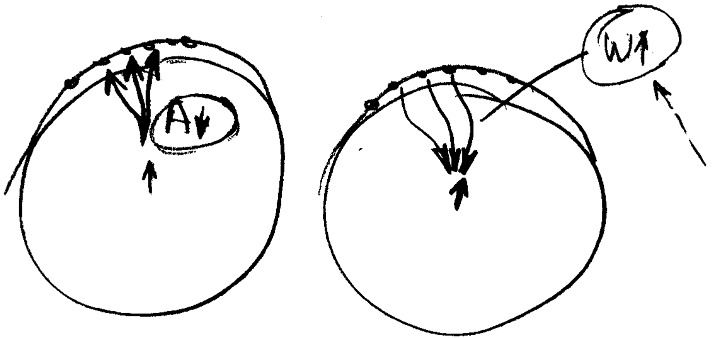
NOISE SENSITIVITY NORMALIZATION (2)

shallow source (unit strength)



fwd big inv <u>small</u>

deep source (unit strength)



fwd small

inv <u>reduced</u> because of minimum norm (N.B. W should be bigger than for superficial source <u>but</u>, min norm reduces b/c of inverse square)

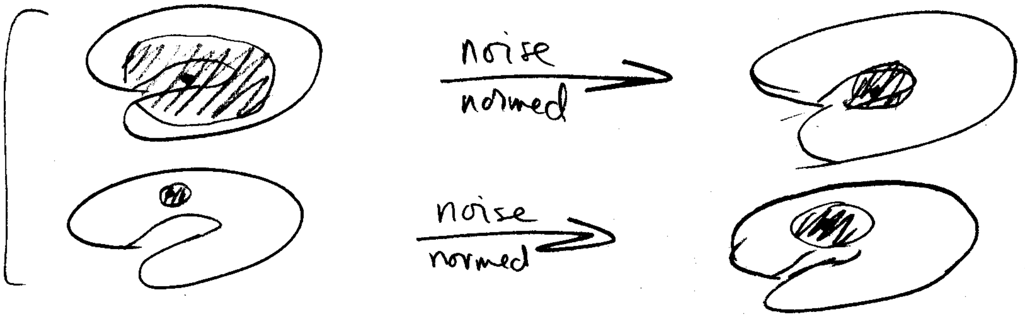
$$S_i = \frac{\vec{W}_i^{orig} \cdot \vec{x}}{\| \vec{W}_i^{orig} \|}$$

→ shallow: small

→ deep: <u>even smaller</u>! → therefore, fixed estimate <u>increased</u> relative to shallow

- effect on inverse solution → more like <u>significance</u> <u>vs</u> <u>actual power</u>
- effect on point-spread function is to equalize shallow & deep
  - shallow spread out more than min norm
  - deep shrunk to same as shallow

point spread functions



<u>point-spread</u>

point $\xrightarrow{fwd}$ data

spread $\xleftarrow{in}$

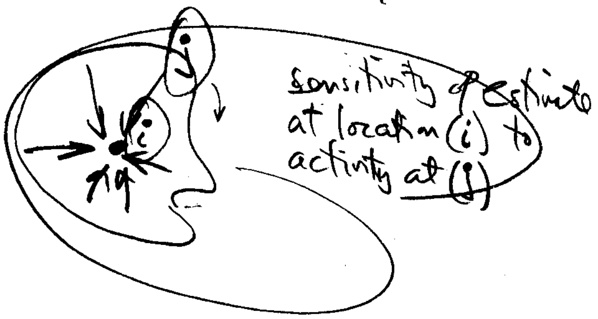
useinverse-5

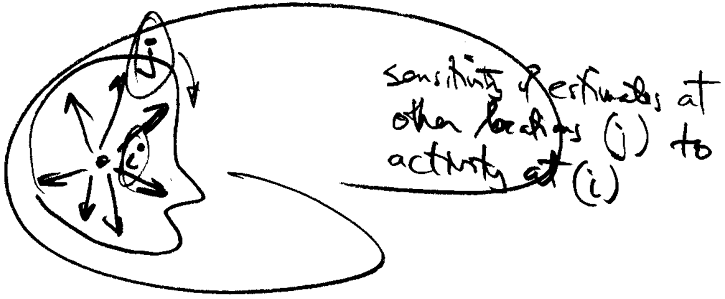
POINT-SPREAD / CROSS TALK

point $\xrightarrow[\text{fwd}]{A}$ $\xrightarrow[\text{inv}]{W}$ blur

thus, **WA** measures blur!

just "take inverse of fwd"





Crosstalk (rows of resolution matrix, WA)

$$\xi_{ij}^2 = \frac{|(W\overset{\text{true fwd}}{\tilde{A}})_{ij}|^2}{|(W\tilde{A})_{ii}|^2} = \frac{|w_i \tilde{a}_j|^2}{|w_i \tilde{a}_i|^2}$$

Point spread (columns of resolution matrix, WA)

$$\rho_{ij}^2 = \frac{|(W\tilde{A})_{ji}|^2}{|(W\tilde{A})_{ii}|^2} = \frac{|w_j \tilde{a}_i|^2}{|w_i \tilde{a}_i|^2}$$

avg crosstalk

$$ACM_i = \frac{\sum_j \xi_{ij}^2}{j}$$

avg point spread

$$APSF_i = \frac{\sum_j \rho_{ij}^2}{j}$$

- PSF & crosstalk maps identical for standard inverse (WA, the "resolution matrix" is symmetric)
- PSF affected by noise-normalized; crosstalk same ((DW)A not symmetric)

Conclusions

- more EEG or more MEG better
- EEG better than MEG (cf. radial) (EEG fwd_{real} currently less accurate)
- biggest gain from adding small # EEG (or MEG) (e.g. 30) to many MEG (or EEG) (e.g. 150)
- easier to add many MEG, so: optimal < 30 EEG, 300 MEG
- EEG/MEG forward-solution-scaling-factor error causes → more cross talk

music-1

Music (1)

(from Dale & Sereno, 1993) (cf. Mosher & Leahy)

– using <u>sensor</u> covariance

$$\mathbf{D} = \overbrace{\langle \mathbf{x}\mathbf{x}^T \rangle}^{\text{average value of covariance}} = \overbrace{\sigma^2 \mathbf{I}}^{\text{sensor noise}} + \sum_i \sum_j \overbrace{\sigma_i \sigma_j \,\mathrm{Corr}(i,j)}^{\substack{\text{source correlation} \\ \mathbf{R}_{ij}}} \overbrace{\mathbf{A}_i \mathbf{A}_i^T}^{\text{times fwd soln}}$$

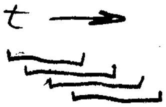
recording vector at one time point

$$\approx \frac{[\mathbf{x}_1 \ldots \mathbf{x}_n][\mathbf{x}_1 \ldots \mathbf{x}_n]^T}{n}$$

n — recording time points

sensors
$\Leftarrow$ [ ] sensors

can take this estimate over short time epochs



$$\mathbf{D} = \mathbf{U}\Lambda\mathbf{U}^T = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} & & \end{bmatrix}^T$$

↑ eigenvector — corresponding eigenvalue

** columns of **U** matrix are orthogonal basis vectors of "spatial pattern" space (one pnt is spatial patt across sensors)

= find, order most significant <u>spatial</u> patterns in <u>sensors</u>, over time

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<u>Project</u> forward sol'ns onto these spatial patterns for each point in brain
("project" = dot prod = similarity)
(fwd defines spatial sensor pattern for unit source)

$$\underset{\substack{\text{one num for} \\ \text{each source}}}{\xi_i} = \mathbf{A}_i^T \underbrace{\mathbf{U}}_{\substack{\text{all eigenvectors} \\ \text{of sensor} \\ \text{spatial patterns}}} \overbrace{\Lambda}^{\text{eigenval's}} \mathbf{U}^T \underbrace{\mathbf{A}_i}_{\substack{\text{fwd for} \\ \text{one source} \\ \text{("gain vector")}}} \Rightarrow$$

→ 1 <u>column</u> of **A** matrix

big single number if forward sol'n looks like **U**'s music-2

MUSIC (2)

how to weight the minimum norm inverse

$$R_{ii} \approx \frac{A_i^T A_i}{A_i^T U \Lambda^{-1} U^T A_i}$$

cf.

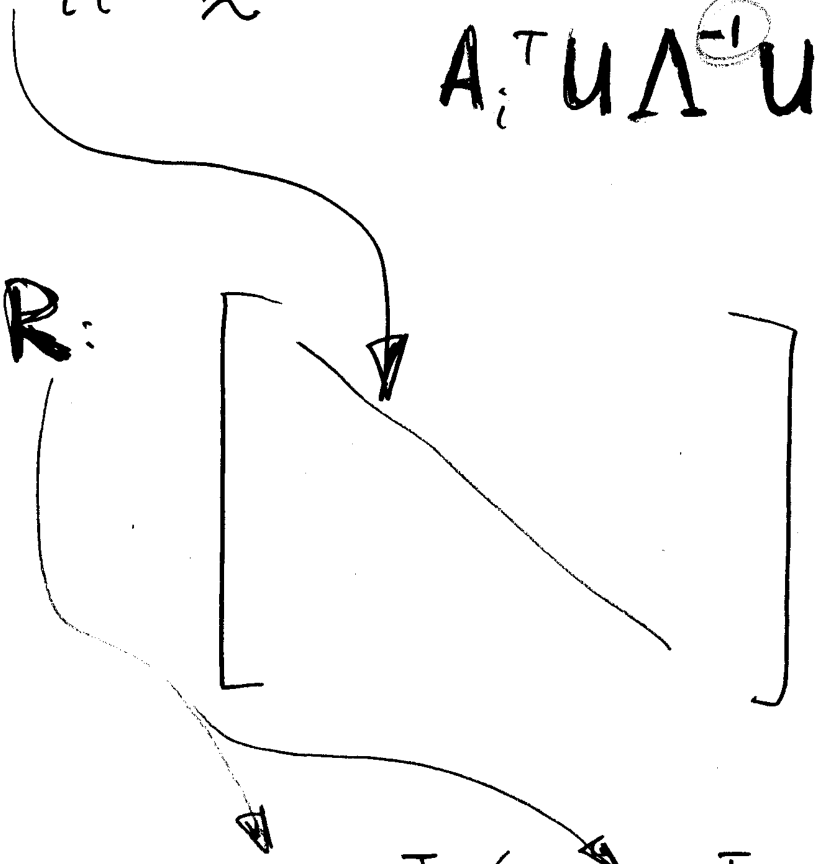
- like parallel resistance

$$R_{parallel} = \frac{1}{1/R_1 + 1/R_2 + \dots}$$

→ any low resistance (ξ_i) decreases overall resistance (overall R_{ii})

→ i.e., if forward sol'n has appearance like any low eigenvalue spatial pattern, it gets devalued

total $\mathbf{R}$:

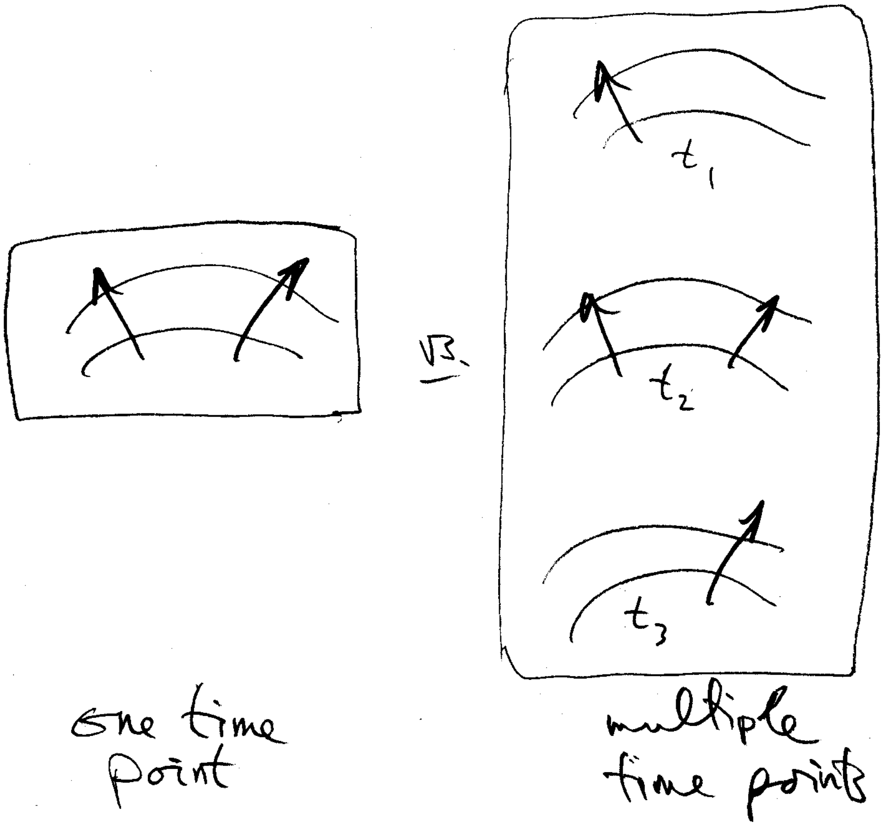


$$W = RA^T(ARA^T + C)^{-1}$$

music3

MUSIC (3)

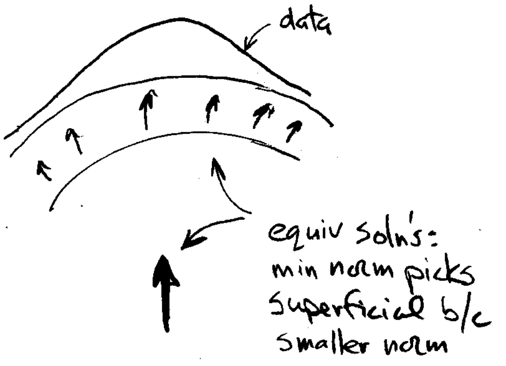
- how it works: take advantage of spatial information that changes over time



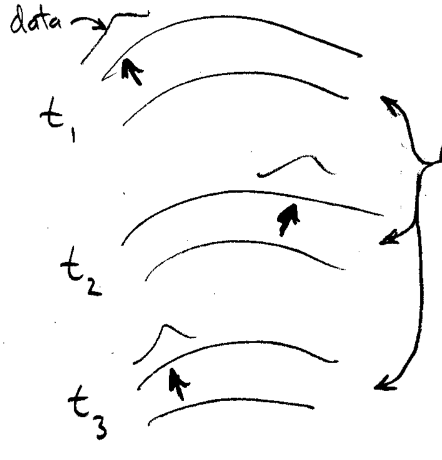
N.B.: Problem if assumption about lack of perfect correlation is violated

e.g., if two widely separated sources (i.e. diff fwd sol'ns) are highly correlated, MUSIC will eliminate <u>both</u> since no single fwd sol'n will look like that "2-separated dipole" pattern (e.g., L/R A-I)
↳ "dual MUSIC" hack to fix...

- how it fixes min norm problem



equiv soln's: min norm picks superficial b/c smaller norm



if sources really were superficial, these spatial patterns would appear in D

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if not, superficial sources would be eliminated, leaving deep