Foundations of Neuroimaging
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Hardware

**MAGNET HARDWARE**

1. $B_0$ field from
   - superconducting magnet

2. gradient coils

3. body RF transmit/receive

4. RF receive-only

5. shim coils
   - (in gradients)
   - $B_0 \rightarrow Z$
     - (longitudinal)
   - $B_1 \rightarrow x,y$
     - (transverse)

$max\ gradient: \ [80 \text{ mT/m, 200 T/m/sec}]$

(1) $B_0$ field

(2) body gradient coils

shim coils also embedded in here
   - (not shown)

(3) RF transmit body coil

(4) RF receive-only
   - head coils

$IT = 10,000 \text{ Gauss}$

Earth: 0.05 - 0.6 G

$25 - 65 \text{ MT}$

$\frac{Y}{2\pi} = 42 \text{ MHz/T}$

RF transmitter (30 kW)

RF receiver

$circularly\ polarized$

$B_1$ field rotating

$1.5\ million\ watt\ amplifiers$

to add ramps to $B_0$ field

$\leq$ non-superconducting water-cooled, external shield

(opposite direction $B_0$ shield coils outside these not shown)

superconducting coils in liquid helium

(no power required after current injected to bring up field using induction)
**Spin & Precession**

- Nuclei act like a spinning sphere of matter with an embedded equatorial charge (nuclei w/o odd atomic weight or odd proton numbers).

- Moving charge creates magnetic field

- Classical picture
  - Current loop from spinning charge (right-hand rule)
  - N.B.: classically this would cause EM radiation, spin-down

- Stern-Gerlach experiment
  - Pass silver atoms thru strong magnetic field → split into just 2 beams

---

**Microscopic picture**

- No strong magnetic field
  - \( B\phi = 0 \)

- Strong magnetic field, \( B\phi \)
  - \( \text{Precessing vectors are bunched at any one moment around circle} \)

**Macroscopic picture**

- All vectors some length, random directions
  - Slight excess of "up" (3 ppm)

- Bulk magnetization
  - \( M_z \)  

**Precession**

- Distinguish precession (slow) from spin (fast)
- Treat classically, like spinning top

\[ 2\pi f = \frac{\gamma B_0}{I} \]

- Larmor frequency (eg. 63 MHz)
- Gyromagnetic ratio (eg. 1.5T)

- Bulk equilibrium magnetization (parallel to Bo)

\[ M_z^0 = \left| \vec{M} \right| = \frac{\gamma^2 h^2 B_0 N_s}{4KTS} \]

- Where \( I = \pm \frac{1}{2} \)
The Bloch Equation describes the time-dependent behavior of the magnetization vector $\mathbf{M}$ in the presence of an applied magnetic field (excitation and relaxation). In the laboratory frame, the equation is:

$$\frac{d\mathbf{M}}{dt} = \mathbf{M} \times \mathbf{H} - \frac{M_z(t) - M^0_z}{T_1} - \frac{M_{x'y'}(t)}{T_2}$$

- **Precession**: $\mathbf{B} = B_0$
  - $\mathbf{B}$ is the equilibrium field in which the magnetization vector is almost parallel to the field, and $\mathbf{B} = B_0 + B_1$ for the rotating frame.

- **Transverse** and **Longitudinal** relaxations:
  - Transverse relaxation ($T_2$):
    - $M_{x'y'}(t) = -\frac{M_{x'y'}(0)}{T_2} e^{-t/T_2}$
  - Longitudinal relaxation ($T_1$):
    - $M_z(t) = \frac{M_z^0(1 - e^{-t/T_1}) + M_z'(0) e^{-t/T_1}}{1 + R_2 e^{-t/T_2}}$
    - $R_2 = M_z'(0) / T_2$

- Solution to equations above:
  - $M_z(t) = M_z(0) e^{-t/T_1} + M_z'(0) e^{-t/T_2}$
  - $M_{x'y'}(t) = M_{x'y'}(0) e^{-t/T_2}$

- In the Larmor-rotating coordinate system, a tilt is added to the phase shift in a standard $B_1$ excitation through rotation around the $x$-axis.
**Vector Add, Multiply**

- Adding vectors is easy
  \[ \mathbf{c} = \mathbf{a} + \mathbf{b} = [a_x + b_x, a_y + b_y] \]
  - Just add components (vector)
  - Applies to complex numbers

- Generalizes to any \( D \)

\[ \| \mathbf{c} \| = \sqrt{(a_x + b_x)^2 + (a_y + b_y)^2} \]

- Multiple ways to multiply vectors: here are 3

**Dot Product**

(= inner product)

(= "scaled projection onto")

\[ \mathbf{c} = \mathbf{a} \cdot \mathbf{b} = [b_x b_y b_z] \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = a_x b_x + a_y b_y + a_z b_z \]

- Generalizes to any \( D \)

\[ \mathbf{p} = \| \mathbf{a} \| \cos \Theta \]
\[ \mathbf{c} = \mathbf{p} \| \mathbf{b} \| \iff \| \mathbf{b} \| = 1 \]

**Cross Product**

(= outer product)

(see "geometric algebra")

\[ \mathbf{c} = \mathbf{a} \times \mathbf{b} = \begin{bmatrix} 0 & b_z & -b_y \\ -b_z & 0 & b_x \\ b_y & -b_x & 0 \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = [a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_x b_y - a_y b_x] \]

- Geometric algebra: bivector plane area

Righthand rule: curl fingers from \( \mathbf{a} \) to \( \mathbf{b} \): thumb is \( \mathbf{c} \)

- Unique orthogonal specific to 3D

\[ \| \mathbf{c} \| = \| \mathbf{a} \| \| \mathbf{b} \| \sin \Theta \]

\( \lor \) max if orthogonal

**Complex Multiply**

(see also quaternions, geometric algebra generalization)

\[ \mathbf{c} = \mathbf{a} \cdot \mathbf{b} = \begin{bmatrix} b_x & -b_y \\ b_y & b_x \end{bmatrix} \begin{bmatrix} a_x \\ a_y \end{bmatrix} = [a_x b_x - a_y b_y, a_x b_y + a_y b_x] \]

- Angles add
- Magnitudes multiply

\[ \| \mathbf{c} \| = \| \mathbf{a} \| \| \mathbf{b} \| \]

(like real nums)
**Effects of \( \vec{M}, \vec{B}, \) and \( \theta \) on Precession Freq.**

1. **Bloch 1st term**
   \[
   \frac{d\vec{M}}{dt} = \vec{M} \times \vec{B}
   \]

2. **Cross product properties review**
   \[
   \left\| \frac{d\vec{M}}{dt} \right\| = \left\| \vec{M} \right\| \left\| \vec{B} \right\| \sin \theta
   \]

3. **Starting condition**
   - Now see effects of changing \( \left\| \vec{M} \right\|, \left\| \vec{B} \right\|, \theta \)
   - Vectors define this

4. **Change \( \vec{M} \) length**
   - \( \frac{d\vec{M}}{dt} \) proportionally larger, so cancels effect of larger \( \vec{M} \)
   - Same precession freq. as starting cond.

5. **Change \( \theta \) between \( \vec{M} \) and \( \vec{B} \)**
   - \( \frac{d\vec{M}}{dt} \) goes up (then down) as \( \sin \theta \)
     - But circumference also goes up as \( \sin \theta \), cancelling again
   - Same precession freq.

6. **Change \( \vec{B} \) length**
   - \( \frac{d\vec{M}}{dt} \) goes up, proportional to \( \vec{B} \)
   - But circumference is same as starting cond.
   - Increased precession freq. \( (\omega = \gamma \vec{B}) \)
Simple Matrix Operations

Basic idea
- A matrix \( \begin{bmatrix} \text{rotates} \\ \text{scales} \end{bmatrix} \) a vector \( \vec{b} = M \vec{a} \)

3D example
- A 4D matrix \( \begin{bmatrix} \text{rotates/scales} \\ \text{then} \\ \text{translates} \end{bmatrix} \) a 3D vector
- \( b_x = M_{11}a_x + M_{12}a_y + M_{13}a_z \)
- \( b_y = M_{21}a_x + M_{22}a_y + M_{23}a_z \)
- \( b_z = M_{31}a_x + M_{32}a_y + M_{33}a_z \)

Add translate (after rotate/scale)
- Commonly used "hack" for aligning vols
- A 4D matrix \( \begin{bmatrix} \text{rotates/scales} \\ \text{then} \\ \text{translates} \end{bmatrix} \) (4th D = 1)

N.B.: Have to keep track of order!!
- rotate/scale then translate ≠ translate then rotate/scale
- Change rot component: untranslate, rot, retranslate

3 special cases (3D): rotate around each major axis without changing length (Scale = 1.0)

- Rotate around x-axis: \( R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{bmatrix} \)
e.g., 90° flip

- Rotate around y-axis: \( R_y(\alpha) = \begin{bmatrix} \cos \alpha & 0 & \sin \alpha \\ 0 & 1 & 0 \\ -\sin \alpha & 0 & \cos \alpha \end{bmatrix} \)
e.g., 180° flip to avoid add 180° phase after 90° flip on x'

- Rotate around z-axis: \( R_z(\alpha) = \begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \)
e.g., precession with 3φ along z'

General case
- Rotate around general z'-axis:

\[ R_{\alpha'}(\gamma) = R_z(-\Theta) R_y(-\Phi) R_z(\Alpha) R_y(\Phi) R_z(\Theta) \rightarrow \text{(quaternions are more efficient)} \]
SOLUTIONS TO SIMPLE DIFFERENTIAL EQ.

**diff. eq.**

\[ dM_{x'y'}(t) = -\frac{M_{x'y'}(t)}{T_2} \]

**Solution:**

\[ M_{x'y'}(t) = M_{x'y'}(0) \cdot e^{-t/T_2} \]

How this works, and where \( M_{x'y'}(0) \) comes from...

Goal:

1) Find eq. whose derivative satisfies diff. eq.
2) Also find soln (one of many) that passes thru init condition

Since our diff. eq. is:

- derivative of funct. = const. same funct.

Try exponential, since derivative \( e^x \)

\[ e^x \rightarrow (e^x)' (\text{slope}) \]

\[ \begin{array}{c}
\text{deriv.} \\
M'(t) = \frac{-1}{T_2} \cdot M(t)
\end{array} \]

Diff eq.

One soln

\[ \begin{array}{c}
\text{take deriv. to check} \\
M(t) = e^{-t/T_2} = e^{1-1/T_2}
\end{array} \]

\[ \begin{array}{c}
\text{family of solutions} \\
\text{N.B., this function is the} \\
\text{"unknown" like the X} \\
\text{in } X + 1 = 3
\end{array} \]

Diff eq.

Another soln

\[ \begin{array}{c}
\text{take deriv. to check} \\
M(t) = \text{const.} \cdot e^{-t/T_2}
\end{array} \]

\[ \begin{array}{c}
\text{const is "initial condition"} \\
\text{information added to soln} \\
(\text{not from diff. eq.)}
\end{array} \]

\[ \begin{array}{c}
\text{磁化} \\
\text{immed. after RF (BI) ends}
\end{array} \]

\[ \begin{array}{c}
\text{const} = M_{x'y'}(0) \\
M'(t) = M_{x'y'}(0) \cdot e^{-t/T_2}
\end{array} \]
VERIFY SOLUTION TO T1 REGROWTH

- Slightly more complex T1 sol'n compared to T2 sol'n

T2 sol'n verify (comp prev)

T1 solution verify

\[
\frac{dM}{dt} = \frac{M_{xy}}{T2}
\]

Original diff eq.

\[
M'(t) = \frac{-1}{T2} \cdot M(t)
\]

Make unknown funct M(t) more visible

\[
M(t) = M_{xy}(0) e^{-t/T2}
\]

Proposed solution

\[
M'(t) = \frac{-1}{T2} \cdot M_{xy}(0) e^{-t/T2}
\]

Test by take deriv.

\[
M(t) = M_{2}^0 \left( 1 - e^{-t/T2} \right) + M_{2}(0) e^{-t/T2}
\]

Init cond.

\[
M'(t) = \frac{-1}{T2} \left( M(t) - M_{2}^0 \right)
\]

\[
M(t) = \frac{M_{2}^0}{T1} \left( 1 - e^{-t/T1} \right) + M_{2}(0) e^{-t/T1}
\]

\[
M'(t) = 0 + \frac{1}{T1} M_{2}^0 e^{-t/T1} - \frac{1}{T1} M_{2}(0) e^{-t/T1}
\]

\[
M'(t) = \frac{-1}{T1} \left( M_{2}^0 e^{-t/T1} + M_{2}(0) e^{-t/T1} \right)
\]

Derivative in original T1 eq. says: M(t) minus M_{2}^0

\[
M'(t) = \frac{-1}{T1} \left( M(t) - M_{2}^0 \right)
\]

Solution \[
M_{2}^0 - M_{2}^0 e^{-t/T1} + M_{2}(0) e^{-t/T1}
\]

Which equals our re-calculated derivative:

\[
M'(t) = \frac{-1}{T1} \left( -M_{2}^0 e^{-t/T1} + M_{2}(0) e^{-t/T1} \right)
\]
Bloch Eq. - Matrix Version

Differential Eq.:
\[
\frac{d\vec{M}}{dt} = \vec{M} \times \vec{\gamma} \vec{B}_0
\]

\[\vec{a} \times \vec{b} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix}
\]

\[
d\vec{M} = \begin{bmatrix} \frac{dM_x}{dt} \\ \frac{dM_y}{dt} \\ \frac{dM_z}{dt} \end{bmatrix}
= \begin{bmatrix} 0 & \gamma B_0 & 0 \\ -\gamma B_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix}
\]

Solution:
\[
\vec{M}(t) = \begin{bmatrix} M_x(t) \\ M_y(t) \\ M_z(t) \end{bmatrix}
= \begin{bmatrix} \cos \omega t & \sin \omega t & 0 \\ -\sin \omega t & \cos \omega t & 0 \\ 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} M_x(0_+) \\ M_y(0_+) \\ M_z(0_+) \end{bmatrix}
\]

Initial condition:
\[
\vec{R}_z(\omega t) \vec{M}(0_+)
\]

Include Relaxation:
\[
\frac{d\vec{M}}{dt} = \vec{M} \times \vec{\gamma} \vec{B}_0 - \frac{M_x i + M_y j}{T_2} - \frac{(M_z - M_z^0) k}{T_1}
\]

Differential Eq.:
\[
\frac{d\vec{M}}{dt} = \begin{bmatrix} -\frac{1}{T_2} \gamma B_0 & 0 \\ 0 & -\frac{1}{T_2} \gamma B_0 \\ 0 & 0 & -\frac{1}{T_2} \end{bmatrix}
\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix}
+ \begin{bmatrix} 0 \\ 0 \\ M_z^0 / T_1 \end{bmatrix}
\]

Solution:
\[
\vec{M}(t) = \begin{bmatrix} e^{-\frac{t}{T_2}} & 0 & 0 \\ 0 & e^{-\frac{t}{T_2}} & 0 \\ 0 & 0 & e^{-\frac{t}{T_1}} \end{bmatrix}
\begin{bmatrix} M_x(0_+) \\ M_y(0_+) \\ M_z(0_+) \end{bmatrix}
+ \begin{bmatrix} 0 \\ 0 \\ M_z(0_+) (-e^{-\frac{t}{T_1}}) \end{bmatrix}
\]
EXCITATION IN THE ROTATING FRAME

- Original Bloch Eq. in laboratory frame
  \[ \frac{d \vec{M}}{dt} = \vec{M} \times \vec{B} \]

- Add on-resonance B1 to refer to 3D system
  \[ \vec{B} = B_1(t) (\cos \omega t \hat{i} - \sin \omega t \hat{j}) + B_0 \hat{k} \]

- Lab frame < no gradient

- Basic excite

- Matrix version
  \[ \frac{d \vec{M}}{dt} = \begin{bmatrix} \frac{dM_x}{dt} \\ \frac{dM_y}{dt} \\ \frac{dM_z}{dt} \end{bmatrix} = \begin{bmatrix} 0 & \omega_0 & \omega(t) \\ -\omega_0 & 0 & 0 \\ \omega(t) & 0 & 0 \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} \]

- Substitution to convert to the rotating frame

- After substitution any off-resonance appears as residual B0 (B2)
  (See off-res notes next page)

- Rotating frame < on-resonance
  * Basic excite, B1x-only no gradient

- Removes ω0, cos/sin

- Rotating frame < off-resonance
  * General, B1x-only incl gradients

- Gradient: ω(t) = kGx \( \hat{z} \)
  * Off-res: appears as residual B0, lifting B1 vect.
  * Out of x-y plane

- This means \( \vec{M} \) vect. update will contain component that rotates \( \vec{M} \) around \( \hat{z} \)-axis (in rotating coords = phase)

- Rotating frame < on-resonance
  * Ind Gradient

- Small tip approx.
  * Small tip \( \frac{dM_z}{dt} \approx 0 \)

- Small tip \Rightarrow easier to solve!
Block 1g

BLAICH EQ. SUMMARY

\[
\frac{d\hat{M}}{dt} = \hat{M} \times \hat{B} - \frac{M_x \hat{i} + M_y \hat{j}}{T_2} - \frac{(M_z - M_z^0) \hat{k}}{T_1}
\]

(lab-frame)

(vector lengths not to scale!)

- Full lab-frame picture is complex:
  - 3 component of \( \frac{d\hat{M}}{dt} \) update vector
  - Larmor freq. component 7-9 orders magnitude larger than \( T_2 \), \( T_1 \) decay
  - \( \hat{B}_1 \) is also rapidly wiggling

- Conceptual simplification in 4 stages:

1) **lab frame**
   - Just precession

2) **rotating frame**
   - \( \hat{M} \) stopped
   - That is, \( B_0 = 0 \)

3) **add \( \hat{B}_1 \)**
   - \( \hat{B}_1 \) also stopped!
   - But \( \hat{M} \times \hat{B} \) still works!!
   - "Precess" around \( \hat{B}_1 \) axis

4) **off-resonance**
   - Slow precess, now around tilted \( \hat{B}_{eff} \)

Slow precess. now around tilted \( \hat{B}_{eff} \)

Tilted plane

Apparent \( B_2 \) comp. from residual precess. around \( z \) from off-resonance
RF FIELD POLARIZATION

- polarization (change of direction)

- linearly polarized field
  \[ \vec{B}_1(t) = B_1 \cdot \cos \omega t \hat{x} \]
  magn. strength: \{1, 1, 1\}

- N.B.: \( B_1 \) adds to much larger \( B_0 \)

- circularly polarized field (quadrature)
  \[ \vec{B}_{1\text{circ}}(t) = B_1 \left( \cos \omega t \hat{x} - \sin \omega t \hat{y} \right) \]
  \[ = B_1 \cdot e^{-i\omega t} \]

- in the rotating coordinate system, flipping around x-axis vs. y-axis is just difference in phase of RF field

B1 generated/recorded by RF coil

\( \phi \)

\( \omega t \)

\( e^{i\omega t} \)

180° flip

90° flip

(around x-axis)

(around opposite y-axis)

(same coords as above, z at top)
**SIGNAL EQUATION**

\[ \Phi(t) = \int \mathbf{B}(\mathbf{r}) \cdot \mathbf{M}(\mathbf{r}, t) \, d\mathbf{r} \]

- Magnetic flux through coil \( \rightarrow \) scalar (integral of mag. field perpendicular to area)
- Magnetic field detected/generated by coil geometry at each point in object
- Local magnetization of object (time-dependent)
- Position: \( \mathbf{r} \rightarrow x, y, z \)
- Time derivative inside
  - Use Bloch equations
  - \( \mathbf{M} \rightarrow \mathbf{M} + \frac{\partial \mathbf{M}}{\partial t} \) for Bloch equation

\[ V(t) = -\frac{\partial \Phi(t)}{\partial t} = -\frac{\partial}{\partial t} \int \mathbf{B}(\mathbf{r}) \cdot \mathbf{M}(\mathbf{r}, t) \, d\mathbf{r} \]

Farnady Law & Induction

- Ignore change in z-comp. \( \mathbf{M} \) because so slow \( \rightarrow \) i.e., we only see \( \mathbf{M}_{xy} \), not \( \mathbf{M}_z \)
- Substitute \( \mathbf{M}(t) \) with lab frame \( \mathbf{M}_{xy}(t) = \mathbf{M}_{xy}(0) e^{-i\omega_0 t} \)
- Simplify:
  1. Ignore decay (assume this \( t=0 \))
  2. Assume phase-sensitive detection \( \mathbf{S}_w \) (difference from \( \mathbf{S}_0 \) \( \rightarrow \) rotating frame)

Omit receive and limit excite flip phase offsets

\[ \mathbf{S}(t) = \int \mathbf{S}(\mathbf{r}) \, d\mathbf{r} \]

Laboratory frame Bloch solutions:
- \( \mathbf{M}_L \rightarrow \mathbf{M}_T \)
- \( \mathbf{M}_T = \mathbf{M}_0(0) e^{-i\omega_0 t} \)
- Demodulate to spatially dependent resonant freq in rotating frame \( \rightarrow \) standard signal expression

Phase angle in rotating frame

\[ \omega t = \text{radians/second} = \frac{\text{radians}}{\text{sec}} = \frac{\text{radians}}{\text{sec}} \]

[Getting difference converts lab \( \rightarrow \) rotating]
**Phase-Sensitive Detection**

- Method for moving very high frequency Larmor oscillations down to tractable frequency range.

**Diagram:**
- **V(t) ~ 123 MHz**
- **Reference Signal (123 MHz)**
- **Low-Pass Filter**
- **S(t) ~ 50 kHz**
- **Demodulated Signal**
  - RF coil signal \( \times \) reference (transmitter)
  - \( \propto \sin[(\omega_0 + \omega) t] \times \sin[\omega_0 t] \)
  - \( \propto \frac{1}{2} \left[ \cos \omega t - \cos (2\omega_0 + \omega) t \right] \)
- **Chirp - Time Domain**
  - Chirp input
  - Center
  - Demodulated
  - Low freq signal
  - Filter
  - Phaseshift
- **Quadrature Coils**
  - Two signals are made from a single receiving RF coil.
  - A quadrature coil can be treated the same way (OK to combine after adding \( \frac{1}{2} \) phase, then PSD).
  - Quadrature coil has better S/N since noise in each part is uncorrelated (\( \frac{1}{2} \) better).

**Equations:**
- Triangular identity:
  - \( \sin a \sin b = \frac{1}{2} [\cos (a-b) - \cos (a+b)] \)
  - \( \sin a \cos b = \frac{1}{2} [\sin (a+b) + \sin (a-b)] \)

**Notes:**
- This signal is digitized.
- Filter this one out with low pass filter.
**FID - FREE INDUCTION DECAY, T2**

T2 - unrecoverable (= rapid)
T2* - add recoverable (= rapid + static)

- Signal (FID) resulting from RF pulse w/ angle α

\[ S(t) = \sin \alpha \int_{w=-\infty}^{w=\infty} \rho(w) \cdot e^{-t/T_2(w)} \cdot e^{-i\omega t} \cdot dw \]

ignoring space in obj

An example spectral density ("Lorentzian inhomogeneity")

\[ \rho(w) = \frac{M_0^2}{\Delta w} \cdot \left( \frac{\Delta w^2}{(\Delta w)^2 + (w-w_0)^2} \right) \]

w = \Delta w (Bloch)

\[ \Delta w = \Delta \phi \]

[w = \Delta \phi (fixed height) under curve]

[subst \( \rho(w) \), rearrange to extract \( w_0 \), take integral]

\[ \tilde{S}(t) = \frac{\pi \cdot M_0^2 \cdot \Delta \phi \cdot \sin \alpha}{c^2 + w^2} \cdot e^{-\frac{\Delta \phi}{c} \cdot t} \cdot e^{-t/T_2} \cdot e^{-i\omega t} \]

[combine T2 + static terms]

\[ \tilde{S}(t) = \frac{\pi \cdot M_0^2 \cdot \Delta \phi \cdot \sin \alpha}{c^2 + w^2} \cdot e^{-t/T_2} \cdot e^{-i\omega t} \]

\[ \frac{1}{T_2^*} = \frac{1}{T_2} + \frac{1}{T_2'} \]

overall decay rate including inhomogeneous Bφ

unrecoverable + recoverable "intrinsic" + "static"

suggestive, since 10^8 million cycles per second

N.B. complex \( \rho(w) \)

only approximated by \( e^{-t/T_2^*} \)
**Echoes — Spin Echo**

- Just after 90° x' pulse, \( f_{x0} + f_{x1} \) have same phase.
- Relaxation + phase dispersion of \( f_{x0} + f_{x1} \) (both from \( B > B_0 \)).

- Remember brief RF just tips vectors while retaining length.
- Relaxation includes tips and shrinks (\( M_r \)) and grows (\( M_z \), echo).

- 180° x' pulse works too, but echo will have \(+\pi\) phase (left side in figs above).
- Echo generated even if second pulse not 180° (see next).

**Rotating Coords**

- Echo caused by re-phasing of \( f_{x0} + f_{x1} \) (w/ decay due to \( T_2 \)).

- FID decay (and echo growth/decay) described by \( T_2^* \), from inhomogeneity.

- Reduction in height of echo compared to initial described by \( T_2 \), echo fixes the star.

**Diagram**

- RF transmit and receive signal ampl.
- 90°, 180°, TE, \( T \), \( T_1 \), \( T_2 \).
- FID (just suggested, really about 1,000,000 cycle here).
- Emitted dephased transverse mag, here.
- Echo (would be complete disk above).

N.B.: Black spins in voxel.
**ECHOES — Spin echo**

\[ \alpha_1 - T - \alpha_2 - T \]  
(both pulses along \( y' \) for simplicity)

**Effect of \( \alpha_y \) pulse**

\[
\begin{align*}
M_x' &\rightarrow M_x, \cos \alpha - M_z', \sin \alpha \\
M_y' &\rightarrow M_y \\
M_z' &\rightarrow M_x, \sin \alpha + M_z', \cos \alpha
\end{align*}
\]

(etc for \( \alpha_{x_1}, \alpha_{x_2} \))

**Effect of \( T \) delay**

\[
\begin{align*}
M_x' &\rightarrow (M_x, \cos \omega T + M_y, \sin \omega T) e^{-T/2} \\
M_y' &\rightarrow (-M_x, \sin \omega T + M_y, \cos \omega T) e^{-T/2} \\
M_z' &\rightarrow M_z^0 (1 - e^{-T/2}) + M_z' e^{-T/2}
\end{align*}
\]

Immediately after \( \alpha_1 \) pulse

\[
\begin{align*}
M_x(w, 0) &= -M_z^0(w) \sin \alpha, \\
M_y(w, 0) &= 0, \\
M_z(w, 0) &= M_z^0(w) \cos \alpha
\end{align*}
\]

**For one isochromat of freq. \( w \)**

Immediately after \( \alpha_2 \) pulse (no effect on \( M_y \); rewrite \( y' \) in terms of \( x' \) and \( y \) e.g.)

\[
\begin{align*}
M_{x'}(w, T) &= -M_z^0(w) \sin \alpha, \cos \omega T e^{-T/2} \\
M_{y'}(w, T) &= M_z^0(w) \sin \alpha, \sin \omega T e^{-T/2} \\
M_{z'}(w, T) &= M_z^0(w) \left[ 1 - (1 - \cos \alpha_2) e^{-T/2} \right]
\end{align*}
\]

**Time dependent free precession around \( z' \)**

\[
\begin{align*}
M_{x'}(w, t) &= M_{x'}(w, T) e^{-(t-T)/T_e} e^{-i\omega(t-T)} \\
&\quad = M_z^0(w) \sin \alpha, \sin^2 \alpha_2 e^{-t/T_e} e^{-i\omega(t-2T)} \\
&\quad \quad - M_z^0(w) \sin \alpha, \cos^2 \alpha_2 e^{-t/T_e} e^{-i\omega t} \\
&\quad \quad - M_z^0(w) \left[ 1 - (1 - \cos \alpha_2) e^{-T/2} \right] \sin \alpha_2 e^{-(t-T)/T_e} e^{-i\omega(t-T)}
\end{align*}
\]

**For a large num of freq's:**

\[
\begin{align*}
\text{terms (1) nephasing} &\quad \rightarrow \text{FID of echo} \\
\text{terms (2) & (3) are dephasing} &\quad \rightarrow \text{nephase at } t = 2T_e
\end{align*}
\]

**Echo Signal**

\[
S(t) = \sin \alpha, \sin^2 \alpha_2 \int_{-\infty}^{\infty} \rho(w) e^{-t/T_e} e^{-i\omega(t-T_e)} \, dw
\]

**Peak amplitude**

\[
A_E = \sin \alpha, \sin^2 \alpha_2 \int_{-\infty}^{\infty} \rho(w) e^{-TE/T_e} \, dw = M_z^0 \sin \alpha, \sin^2 \alpha_2 e^{-TE/T_e}
\]

**Special Case**

\[
90^\circ_y - 90^\circ_y: \quad S_1(t) = \frac{1}{2} \int_{-\infty}^{\infty} \rho(w) e^{-t/T_e} e^{-i\omega(t-T_e)} \, dw
\]

\[
90^\circ_y - 180^\circ_y: \quad S_2(t) = \text{no } \frac{1}{2} \text{ factor}
\]

\[
90^\circ_x - 180^\circ_y: \quad S_2(t) = \text{multiply by } i \rightarrow \text{add } \frac{1}{2} \text{ phase}
\]

etc for \( A_E \) ... like
Echo TRAINS - spin-echo trains

- it's (too) easy to make echoes...

\[ E_n = \frac{3(n-1) - 1}{2} \]

Echoes after end of nth pulse
3 RFs \( \Rightarrow \) 4 echoes (here)
6 RFs \( \Rightarrow \) 121 echoes (!)

Secondary echo: \( SE_{1,2} \) acts like RF pulse
\( \alpha_3 \) makes an echo from it

Stimulated echo: combined effect of 3
\[
\alpha_1: M_L \rightarrow M_T \\
\alpha_2: \text{leftover } M_T \text{ flipped to } M_L \text{ (saved)} \\
\alpha_3: \text{flip saved } M_L \rightarrow M_T \text{ which can then begin to cancel delays (after being held in limbo between 180°, FID2 and FID3); acts like 2-pulse echo}
\]

- a useful multi-echo sequence (CPMG) is a 90° followed by 180° at 2\( T \) spacing

Typically, 90° and 180° applied in different axes (\( x' \), then \( y', y', ... \))
which reduces phase errors due to imperfect 180° pulses
(since slightly-off rotation around \( y' \) affects phase less)
**EXTENDED PHASE GRAPHS**

- using full Bloch eq. solutions is tedious 😊
- pictorial method for visualizing effects of series of α pulses (vs. easier to visualize 90°, 180°)
- problem #1: α pulse rotates a portion of transverse magnetization into a position that results in rephasing and another portion into M₀
- problem #2: third pulse can uncover and rephase transverse magnetization temporarily saved in longitudinal

< diagram showing phase dispersion and RF pulses effect on transverse and longitudinal magnetization>

- rule for effect of α RF pulse on transverse mag
- rule for effect of α RF pulse on longitudinal mag

- each branch is weighted
- each branch decays
  1) T₁ "stored" 
  2) T₂ 
  3) M₀ recovery

- echo when phase path crosses zero

- QM view helps
3 - PULSE ECHO AMPLITUDES

- Assume $M_z = 1$

**RF Transmit**
- $\alpha_1$  
- $\alpha_2$  
- $\alpha_3$

**RF Receive**
- $\tau_1$  
- $T_2$  
- $T_1$  
- $T_1 + T_2$

**Echo**  |  **Time**  |  **Amplitude**
---|---|---
$SE_{1,2}$ ($t = 2\tau_1$) | $\sin \alpha_1 \sin \frac{\alpha_2}{2} e^{-2\tau_2/T_2}$ | $\alpha_1 = 90^\circ, \alpha_2 = 180^\circ$  
$L_2$ special cases

$2^\circ$ ("secondary") ($t = 2T_2$)  
($t = 2T_1 + 2T_3$) | $-\sin \alpha_1 \sin \frac{\alpha_2}{2} \sin \frac{\alpha_3}{2} e^{-2\tau_2/T_2}$ | $2\alpha_1 = \alpha_2$  

$STE$ ("stimulated") ($t = 2T_1 + T_2$) | $\frac{1}{2} \sin \alpha_1 \sin \alpha_2 \sin \alpha_3 e^{-\tau_2/T_1} e^{2\tau_1/T_2}$ | ($N.B.: T_1$)

$SE_{2,3}$ ($t = \tau_1 + 2T_2$) | $[1 - (1 - \cos \alpha_1) e^{-\tau_1/T_1}] \sin \alpha_2 \sin \frac{\alpha_3}{2} e^{-(\tau_1 + 2\tau_2)/T_2}$ | ($N.B.: T_1$)

$SE_{1,3}$ ($t = 2(\tau_1 + T_2)$) | $\sin \alpha_1 \cos \frac{\alpha_2}{2} \sin \frac{\alpha_3}{2} e^{-2(\tau_1 + T_2)/T_2}$ | ($N.B.: T_1$)

- $T_1$-dependence in $STE$ (but also $SE_{2,3}$) from temporary "storage" of $M_T$ in $M_L$, then recovery by third pulse
**HYPER ECHOES**

(N.B.: coord hat to put \( z \) horiz vs. Block notes)

3 solid lines 1 dashed line

1. \( \alpha_y \) -180 y -\( \alpha_y \) = 180 y

2. \( \alpha_y \) -180 y -\( \alpha_y \) = 180 y

3. \( \alpha_y \) -180 y -\( \alpha_y \) = 180 y

**Practical use**

- multi-echo example
- can also use to prepare, then separate read-out
- practical prob: 180° pulses deposit a lot of RF (6x 90°) -> prob at high fields
- by arranging 90° to get big echo in middle of k-space, can get by with much less RF power

- Hennig & Scheffler (2001)
- normalize \( \vec{M} \) amplitude \( \rightarrow 1 \)
- sphere surface defines 2D space for \( \vec{M} \) moved by:
  1) vect. rotation of \( \vec{M} \) around tilted axis in transverse x-y plane by RF, with flip, \( \phi \), and phase, \( \Phi \): \( P(x', y', \phi) \)
  2) rotation around z by phase evolution due to freq. offset, \( \omega \) (3D offset) and time, \( t : \Phi(x', y', \omega, t) \)
- three symmetries:
  - solid lines: phase eval at RF flip or 180°, phase or RF
  - dashed lines: just 180° equiv.

- by combining long sequences observing these symmetries, can generate 90°, even w/ many inserted \( \alpha \)-pulses in between
**Gradient Echoes** - $T_2^*$, GE chains

- **Initial negative gradient** dephases spins
- After $t = T$ of positive gradient, spins rephase
- Does not correct for $T_2^*$ inhomogeneities
  - So echo amplitude is $A_E = e^{-t/T_2^*}$
- The initial "FID" is not "free" since it is being actively de-phased by gradient, so FID decay
  - $\frac{1}{T_2} < \frac{1}{T_2^*} < \frac{1}{T_2^{**}}$ \[ A_E = e^{-t/T_2^{**}} \]

- Key difference between spin-echo (SE) and gradient echo (GE)
  - $B_0$ inhomogeneities not canceled
  - Hence, echoes are $T_2^*$-weighted, not $T_2$-weighted \( \Rightarrow \) more susceptible to inhomogeneities

- Echo trains possible with gradient echo (CPMG-like)

- The faster the gradients are switched, the more echoes you get
- EPI hardware \( \Rightarrow \) 64 echoes
IMAGE CONTRAST

T1 Saturation-recovery (no echo, just FID)

- Contrast (PD, T1, T2, T2*) depends on magnetization not getting back to equilibrium, and then differences in how far away each tissue type is at measurement time.

RF

\[ M_z^{\text{longitudinal}}(0) = M_z^{\text{zero}}(1 - e^{-\frac{TR}{T1}}) + M_z^{\text{transverse}}(1 - e^{-\frac{TR}{T1}}) \]

- "steady state" after here

- Simple saturation/recovery w/ no echo

- Initial conditions:
  \[ M_z \text{ before first pulse} = M_z^{\text{0}} \]
  \[ M_z = 0 \text{ immediately after first pulse (i.e., 90° pulse)} \]

- From Bloch eq, Mz just before second pulse:
  \[ M_z^{(N)}(t) = M_z^{(0)}(1 - e^{-\frac{TR}{T1}}) + M_z^{(n)}(0) e^{-\frac{TR}{T1}} \]

- Given:
  1. 90° pulse
  2. no \( M_{xy} \) left

\[ \rightarrow \text{ pure tip: } M_{xy} = M_z \]

- Tip existing mag
  \[ M_z^{(n)}(O^-) = M_x y^{(O^+)} = M_z^{(0)}(1 - e^{-\frac{TR}{T1}}) \]

- That is, the not-completely-negrown longitudinal magnetization, which depends on T1, but which we cannot record, is completely converted to recordable transverse magnetization.

\[ I(r) = C \rho(r)(1 - e^{-\frac{TR}{T1}}) \]

- Spectral density \( \rho(r) \) at p. density; underlies equilb. \( M_z^{\text{0}} \).
**IMAGE CONTRAST**

Why imperfect 90° takes multiple flips til steady state

- initial fMRI images are usually discarded (why?)
  - because they are brighter than all the rest
  - because multiple flip required before steady state

N.B.: B1 imperfections guarantee this situation will occur
  (e.g. at 3T, flip angle varies almost 25% across brain)

- at 3T, steady state
  - for typical 1-2 sec
  - TR images reached after ≈8 images
IMAGE CONTRAST

IR (still just saturation-recovery - no echo)

- inversion recovery w/ no echo

RF

\[ 90^\circ \quad T_1 \quad 90^\circ \quad T_0 \quad 180^\circ \quad T_1 \quad 90^\circ \quad T_0 \quad 180^\circ \quad T_1 \quad 90^\circ \quad T_0 \]

- steady state after here

\[ M_z \]

\[ M_z^0 \quad \text{longitudinal magnetization} \]

\[ M_z' = -M_z^0 \]

- 180 deg pulse reverses longitudinal magnetization

- recovery to end of first TI from long. part of Bloch eq.

\[ M_z' = M_z^0 \left(1 - 2e^{-T_1/T_1}\right) \rightarrow \text{flipped into transverse by second pulse (180°)} \]

\[ M_z' = M_z^0 \left(1 - e^{-\left(TR-T_2\right)/T_1}\right) \]

- longitudinal then regrows from zero

\[ M_z' = M_z^0 \left(1 - e^{-\left(TR-T_2\right)/T_1}\right) \]

- after second 180°, just change sign again

\[ M_z' = -M_z^0 \left(1 - e^{-\left(TR-T_2\right)/T_1}\right) \]

- apply relaxation eq. again

\[ M_z' = M_z^0 \left(1 - e^{-TI/T_1}\right) - M_z^0 \left(1 - e^{-\left(TR-TI\right)/T_1}\right) e^{-TI/T_1} \]

\[ M_z' = M_z^0 \left(1 - 2e^{-TI/T_1} + e^{-TR/T_1}\right) \]

\[ M_z' = M_z^0 \left(1 - 2e^{-TI/T_1} + e^{-TR/T_1}\right) \]

\[ M_z' = M_z^0 \left(1 - 2e^{-TI/T_1} + e^{-TR/T_1}\right) \]

\[ \rightarrow \text{this is magnetization flipped to transverse, made recordable} \]
- Steady state mag (2nd TR) just before 90°
  \[ M_z(t) = M_0 \left( 1 - 2e^{-\frac{(TR-TE/2)}{T_1}} \right) + e^{-\frac{TR}{T_2}} \]

- The echo signal \( M_z \) unlike in simple saturation-recovery FID
  Has an additional delay before it is recorded, so we have to take account of transverse mag relaxation
  \[ A_E = M_0^2 \left( 1 - 2e^{-\frac{(TR-TE/2)}{T_1}} \right) e^{-\frac{TR}{T_2}} \]

- If we assume TE is much less than TR, then we can simplify:
  \[ A_E = M_0^2 \left( 1 - e^{-\frac{TR}{T_1}} \right) e^{-\frac{TE}{T_2}} \]

  \[ \text{TE controls } T_2 \text{ contrast} \]

- Similar equation for SE-IR
  \[ A_E = M_0^2 \left( 1 - 2e^{-\frac{T1}{T_1}} + e^{-\frac{TR}{T_2}} \right) e^{-\frac{TE}{T_2}} \]
**IMAGE CONTRAST**  
GRE w/ small tip angle

- Use basic longitudinal relaxation from Bloch eq. again
  - Assume $M_x^{(n)}(O-) = 0$ → transverse dephased before next pulse
  - $M_x^{(n)}(O-) = M_x^{(0)}(1 - e^{-TR/T2}) + M_x^{(n-1)}(O+) e^{-TR/T1}$

- Assume we have a small tip angle:
  - $M_x^{(n)}(O+) = M_x^{(n)}(O-) \cos \alpha$
  - $M_x^{(n)}(O-) = M_x^{(n-1)}(O-) = M_x^{ss}(O-)$

- Assume we are in steady state vs.
  - $M_x^{ss}(O-) = M_x^{(n-1)}(O-) = M_x^{ss}(O-)$

- Prepulse steady state

- Longitudinal

- Post-pulse transverse magnetization

- Gradient echo amplitude
  - $A_E = \frac{M_x^{(n)}(1 - e^{-TR/T1}) \sin \alpha e^{-TE/T2*}}{1 - \cos \alpha e^{-TR/T1}}$

- $M_{ss} = M_0 + M_0 \cos \theta$
- $M_{ss} - M_x \cos \theta = M_0$
- $M_{ss}(1 - \cos \theta) = M_0$
- $M_{ss} = M_0 / (1 - \cos \theta)$

- TR contrast mainly depends on flip angle, not TR → $\cos \theta = 1$ → eliminates T1 weighting
- $\sin \alpha e^{-TE/T2*}$
- Saturate, wait for $\text{contrast}_1$, invert, wait for $\text{contrast}_2$, FLASH (center out)

A) $M_z' \left( \text{just after } 90^\circ \right) = 0$ (perfect $90^\circ$)

B) $M_z' \left( \text{after } 90^\circ \right) = M_z^0 \left( 1 - e^{-TD/T2} \right)$ (Blach term #1)

C) $M_z' \left( \text{just after invert} \right) = \cos \phi \ M_z^0 \left( 1 - e^{-TD/T2} \right)$

D) $M_z' \left( \text{after } TI \right) = M_z^0 \left( 1 - e^{-TI/T2} \right) + \left[ \cos \phi \ M_z^0 \left( 1 - e^{-TD/T2} \right) \right] e^{-TI/T2}$

E) $M_z' \left( \text{after } \text{first pulse} \right) = M_z^0 \left[ 1 - \left[ 1 - \cos \phi \left( 1 - e^{-TD/T2} \right) \right] e^{-TI/T2} \right] \sin \alpha$
MAGNETIZATION TRANSFER CONTRAST

- Protons in macromolecules & bind to membranes have wide range of resonant frqs ("bound")
  \[ \Rightarrow T_2 = 1 \text{ msec} \]
  \( \Rightarrow \) i.e., signal but invisible w/ usual TE

- Free protons in blood, CSF, water have narrow range of resonant frqs ("free")
  \[ \Rightarrow T_2 = 50 \text{ msec} \]

- Mag transfer pulse sequence
  1) Off center freq pulse to hit "bound" (but don't hit water too hard)
  2) Wait for magnetization transfer from saturated longitudinal \( M_L \) of "bound" \( \rightarrow M_L \) of "free"
  3) Result of transfer \( \rightarrow \) attenuation

\[ \Rightarrow \text{N.B.: this always happens a little} \]
\[ \text{something to keep in mind if hard pulse} \]
\[ \text{(wide freqs)} \]

- Used to increase contrast in TOF
  \[ \text{TOF (not explained) bright vessels from inflow fresh spins} \]

\[ \text{MT - contrast added: suppress tissue but not blood} \]

- View w/ MIP: maximum intensity projection along lines
  \[ \text{max} \rightarrow \text{view as movie} \]
**Signal-to-Noise, Contrast-to-Noise**

- Signal-to-noise defined as: \( \text{SNR} = \frac{\sigma_B}{\sigma_N} \)
- Temporal SNR: \( \epsilon_{\text{SNR}} = \frac{1}{\sigma_B} \)
- "Contrast" is a difference
- Contrast-to-noise ratio:

\[
CNR_{AB} = \frac{S_A - S_B}{\sigma_N}, \quad \text{or} \quad \frac{S_A}{\sigma_N} - \frac{S_B}{\sigma_N} = \text{SNR}_A - \text{SNR}_B
\]

### Spin-Echo

\[
A_E = M_0 \left(1 - e^{-TR/T1}\right) e^{-TE/T2}
\]

### Gradient Echo

\[
A_E = \frac{M_0 \left(1 - e^{-TR/T1}\right) \sin \alpha e^{-TE/T2}}{I - \cos \alpha e^{-TR/T1}}
\]

### General Rules: Spin-Echo, Long TR GE

- Proton-density weighted
- \( TR \rightarrow \) (no T1 diffs)
- \( TE \rightarrow \) (no T2 diffs)

<table>
<thead>
<tr>
<th>Proton-density weighted</th>
<th>( TR ) (no T1 diffs)</th>
<th>( TE ) (no T2 diffs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T1-weighted</td>
<td>( TR \approx T1 ) (big T1 diffs)</td>
<td>( TE \approx T2 ) (big T2 diffs)</td>
</tr>
<tr>
<td>T2-weighted</td>
<td>( TR ) (no T1 diffs)</td>
<td>( TE \approx T2 ) (big T2 diffs)</td>
</tr>
</tbody>
</table>
SIGNAL-TO-NOISE S/N

- generalized dependence of SNR on 3D imaging parameters

\[
\text{SNR/voxel} \propto \frac{\Delta x \Delta y \Delta z}{\sqrt{N_{x} N_{y} N_{z} \Delta t}}
\]

- size (volume) of voxels (with the number of voxels held constant), linear effect on S/N

  \[
  \Delta x \Delta y \Delta z \quad \text{e.g.,} \quad 3\times3\times3 \text{ mm} \rightarrow 4\times4\times4 \text{ mm} \rightarrow \frac{64}{27} = 2.37 \text{ times better S/N}
  \]

- more voxels (with size of voxels, \(\Delta t\) per read/step constant), \(\sqrt{n}\) effect on S/N

  \[
  \text{e.g.,} \quad 64\times64 \rightarrow 128\times128 \rightarrow \frac{\sqrt{128\times128}}{\sqrt{64\times64}} = 2 \text{ times better S/N}
  \]

- \# acquisitions \(\sqrt{n}\) better S/N

  \[
  \text{e.g.,} \quad 1 \text{ acq} \rightarrow 2 \text{ acq} \rightarrow \frac{\sqrt{2}}{1} = 1.41 \text{ times better S/N}
  \]

- longer timestep during readout, \(\sqrt{\Delta t}\) better S/N

  \[
  \Delta t = \frac{1}{BW_{\text{read}}} \quad \text{digitization timestep during echo acquisition}
  \]

- \(BW_{\text{read}}\) determined by cutoff freq, analog low-pass filter
- \(\Delta t\) controls BW because low-pass cutoff has to be set higher for smaller (higher freq-detecting) \(\Delta t\)
- must filter out freq's > \(f_{\text{max}} = \frac{1}{2\Delta t}\) because they alias
**Complex Algebra**

**Addition**: \((a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)\)

**Multiplication**: \((a_1, b_1) \times (a_2, b_2) = (a_1a_2 - b_1b_2, a_1b_2 + a_2b_1)\)

**Angle/Phase**

**Addition**: \((A_1, \phi_1) + (A_2, \phi_2) = (A_1 \cos \phi_1 + A_2 \cos \phi_2, A_1 \sin \phi_1 + A_2 \sin \phi_2)\)

**Multiplication** (non-commutative): \((A_1, \phi_1) \times (A_2, \phi_2) = (A_1A_2, \phi_1 + \phi_2)\)

**Division** (non-commutative): \((A_1, \phi_1) / (A_2, \phi_2) = (A_1/A_2, \phi_1 - \phi_2)\)

**Complex to Real Power**: \((A, \phi)^n = (A^n, n\phi)\)

\[e^{i\phi} = \cos \phi + i\sin \phi\]

**Fourier Transform**

\[H(f) = \int h(t) e^{-2\pi ift} dt\]

\[F[G(x) \ast h(x)] = G(k) \ast H(k)\]

**Convolution**

\[f(x) = g(x) \ast h(x) = \int g(z) \cdot h(x-z) dz\]

\[F[f(x)] = F[g(x)] \cdot F[h(x)]\]

**Convolution Theorem**

- Don't confuse freq, angle!
- How to convert:

\[r, i \leftrightarrow (A, \phi)\]

\[A = \sqrt{r^2 + i^2}, \phi = \arctan (i/r)\]

\[r = A \cos \phi\]

**Fourier Transform**

- Shorthand for a unit vector (2b)
- fors arbitrary amplitude, multiply
- Phase is integral of freq. variable

\[\phi = \int \omega dt\]
**Fourier Transform (1)**

- How to calculate $H(f)$ for one $f$ ($f=3$):
  - (real signal: only need 2 correlations)

- $\hat{h}(t) \cdot e^{-j\frac{2\pi ft}{\text{cyc/sec}}}$
- Complex multiply
- $\int_{-\infty}^{\infty} h(t) \cdot e^{-j\frac{2\pi ft}{\text{cyc/sec}}} dt$
- Sum across $t$ (complex add)

- Complex frequency domain
- Cartesian $(A, \phi)$
  - Amplitude
  - Phase
- Real frequency domain
  - $f$
  - $\hat{h}(f)$
- Imaginary frequency domain
  - $\hat{h}(f)$

$\theta t$ is a phase angle
$\theta \cdot t = \theta t$
$cyc/sec \cdot sec = cyc$

Fourier integral terms written out

- $h(t) \cos(\pi t f) + \hat{h}(f) \sin(\pi t f)$
- $- \hat{h}(f) \sin(\pi t f) + h(t) \cos(\pi t f)$

4 correlations

- $F$ (real frequency domain)
- $F$ (imaginary frequency domain)

- $A$
- $\phi$

Like correlating with $\sin$ and $\cos$ (at each freq) so we get phase (at each freq.)
**Fourier transform (1b)**

\[ e^{i\phi} = \cos \phi + i \sin \phi \]
\[ e^{-i\phi} = e^{i(-\phi)} \]
\[ = \cos(-\phi) + i \sin(-\phi) \]
\[ = \cos \phi - i \sin \phi \]

- **cos** is an "even" function, **sin** is an "odd" function

- cos is an even function, sin is an odd function

**An orthogonal decomposition**

- think of discretely sampled sin(bx), cos(bx) as vectors
- \( \text{Corr}(\vec{v}_1, \vec{v}_2) \equiv \text{projection of } \vec{v}_1 \text{ onto } \vec{v}_2 \equiv \vec{v}_1 \cdot \vec{v}_2 \)

\[
\begin{align*}
\text{Corr} (\cos bx, \sin bx) &= 0 \\
\text{Corr} (\sin bx, \sin bx) &= 0 \\
\text{Corr} (\cos bx, \sin bx) &= 0
\end{align*}
\]

- in the continuous case, orthogonal functions defined as:

\[
\int_{x=hi}^{x=lo} f(x) g(x) \, dx = 0
\]
UNDERSTANDING INVERSE FOURIER TRANSFORM AS ANOTHER CASE OF CORR W/ COS, SIN

- start with spike in image domain
- take example of spike at \( x = 0 \)
  \[
  [\cos(x), \cos(2x), \cos(kx)] \text{ all freqs correlate w/ spike at } x = 0
  \]
- if spike is moved away from zero, the frequency spectrum obtained by correlating cos with spikes oscillates

W.B. opposite direction sin spike are on imaginary axis

- to see why this is, since spike is all zero except at spike, the dot product for a given frequency only depends on the value of the \( e^{-j2\pi kx} \) cos and sin at location of spike

Successively higher freqs

real component of FT

- pos. pair (real) spikes same dist from origin
- pos/neg. pair (imaginary) spikes same dist from origin
- one spike at distance from origin

\( \rightarrow \) this is one way of thinking about what one point in k-space means, via correlating it w/ cos's, sin's to get periodic result in image space (inverse FT)
FOURIER TRANSFORM OF AN IMAGE (2)

1. **Real image** & **Imaginary image**
   - Real image: 
     - (Zero)
   - Imaginary image: 
     - (Zero)

   - Fourier Transform
   - Inverse Fourier Transform

2. **Amplitude image** & **Phase image**
   - Amplitude image: 
     - (Zero)
   - Phase image: 
     - (Zero)

   - View complex vectors directly

3. **Complex vectors**
   - Complex vectors

---

3 equivalent representations of image & spat. freq. space
FOURIER TRANSFORM OF REAL IMAGE (2)

- What a single k-space point looks like for real image (polar coordinates $A, \phi$ instead of $r, \theta$)

Image space

- Offset of stripes is k-space phase
- Brightness of stripes proportional to k-space amplitude

K-space (spatial freq. space)

- Distance from center is stripe spacing
- Angle of point perpendicular to angle of stripes

Amplitude

Phase

(Should be all zero, not same as "stripe phase" above)

Inverse Fourier transform

(Image recon.)

Cartesian dimension of k-space — x- and y- spatial freq.

N.B.: each dimension of spatial freq. space (k-space) from correlation w/ sin & cos — don't confuse $k_x, k_y$ w/ sin & cos!

N.B.: increasing one 1D component increases the spatial freq of the 2D wave and rotates it
FOURIER TRANSFORM OF IMAGE (4)

- 3 equivalent representations of complex numbers
  in image space and spatial-freq. space (k-space)
- example: cosine on in image space, then shifted in x-dir

REAL IMAGE

I(x,y) = \cos(x)

FT of I(x,y) = \cos(x)

I(x,y) = \cos(x - \frac{\pi}{4})

FT of I(x,y) = \cos(x - \frac{\pi}{4})

Real component less than above because of rotation

N.B.: an example of the "Fourier Shift Theorem" (see below)

45° rot compared to complex above

Phase now 45° at k_x = 1, k_y = 0
(-45° at k_x = -1, k_y = 0)
**Fourier Transform of Image (5)**

- (cont.) Center of k-space (real image)
- Complex image

### Real Image

\[ I(x, y) = \frac{1}{2} \left( 1 + \cos(x) \right) \]

\[ I(x, y) = \begin{cases} i & \text{zero} \\ i & \text{max} \end{cases} \]

\[ H(k) = \int h(x) \cdot e^{-i2\pi kx} \, dx \]

Center of k-space:

- Avg image brightness \( I(\text{real}) \)

\[ \begin{array}{c}
\text{FT} \quad \downarrow \\
\text{FT}^{-1}
\end{array} \]

- Positive center k-space

### Complex Image

\[ I(x, y) = \cos(x) - i \sin(x) = e^{-i\varphi} \]

- Complex

\[ I(x, y) = \begin{cases} \text{flat} & \text{zero} \end{cases} \]

FT, FT⁻¹

- “Missing” spike results in single spike correlating with \( \cos \) and \( \sin \)

N.B.: This k-space is non-Hermitian:

- k-space will only have Hermitian symmetry if image is real:
- Hermitian symmetry when complex conjugate (complex num w/ sign flipped in image part) is equal to function w/ \( \phi \) arg:

\[ 1D: H(k) = H^*(k) \]

\[ 2D: H(-k_x, k_y) = H^*(k_x, k_y) \]

N.B. this is also exactly what gradient does to image space!
FOURIER TRANSFORM OF IMAGE (G)

- (cont.) x- and y-spatial freqs.
- special case: real image from sum of reals

REAL IMAGE

\[ I(x, y) = \cos(x) + \cos(y) \]

N.B. adds but doesn't rotate stripes

\[ \begin{array}{c}
\text{rotates stripes!} \\
\end{array} \]

N.B.: the k-space phase will affect offset of real-valued image space cosinusoid

- therefore for real-valued image, we can visualize inverse FT as real-valued sum of offset real-valued cosinusoids

- N.B. cannot do this with MRI k-space data since phase errors (incl. multiple wraps) mess up real component — must use amplitude img
**Gradient Coils**

- Gradient coils for x, y, z generate approximately a linear gradient in the strength of the z-component of the magnetic field \( B_0 \).

- For example, the x gradient coil induces a ramp in the z-component of the magnetic field when moving in the x-direction:

\[
B_{G,z} = G_x x
\]

*Since a pure linear gradient of \( B_{G,z} \) in only the x, y, or z directions is not possible according to Maxwell equations, each gradient coil also induces a magnetic field that has components in the x- and y-directions (\( B_{G,x} \) and \( B_{G,y} \)).

- The other magnetic field components are usually ignored because they are so small relative to \( B_{G,z} \), since \( B_{G,z} \) is added to \( B_0 \), and since \( B_0 \) is much stronger than \( B_{G,x} \), \( B_{G,y} \), and \( B_{G,z} \).

- Since standard reconstruction methods assume the existence of "non-Maxwellian" gradient fields, spatial distortion is introduced.

- The Maxwellian terms \( B_{G,x} \) and \( B_{G,z} \) are known; they can be included in the重建 process.
SLICE SELECTION ($G_z$)

- slice select gradient on during RF stim

\[ \mathbf{B}_z = \frac{\chi}{\gamma_1} \left( \mathbf{B}_0 + \mathbf{B}_{G_z} \right) \]

- protons here can only be excited by a narrow band of radio frequencies

- to apply a pulse containing a narrow band of frequencies, we use a sinc pulse envelope (Fourier transform of a narrow freq. band)

\[ f(t) = \frac{\sin(\beta)}{\beta} \]

- this excites protons in a narrow slab

- since the slice-selection gradient introduces (space-dependent) phase shifts (see freq encode) these have to be removed by a post-excitation rephasing $z$-gradient

- approximation from assuming tip occurs instantaneously in middle

- valid for small tip: $90^\circ \rightarrow 52\%$

- in practice: adjust to max, use crusher to kill spurious transverse on $180^\circ$
PULSES FOR SLICE SELECTION

- Fourier transform approach to slice-selective pulse (linear approx., even though tipping is non-linear)

$$\hat{B}_1(t) \propto \int_{f=-\infty}^{f=\infty} p(f) \cdot e^{-i2\pi f t} \, df$$

- Time-dependent RF stimulation (complex)
- Frequency selection function
- Solve with: \( p(f) = \) frequency band:

$$\hat{B}_1(t) = A \cdot f_w \cdot \text{sinc}(\pi f_w t) \cdot e^{-i2\pi f_c t}$$

- Amplitude controlling flip angle (controls slice width)
- Sinc envelope determined by freq. width, \( f_w \), (N.B. wider \( f_w \) is narrower sinc)
- Larmor oscillation at center freq.
- Modulation (complex)

---

Fourier Transform Pairs, Rules

- Multiplication in one domain equals convolution in other:

$$F \left[ g(t) \cdot h(t) \right] = G(f) \otimes H(f)$$

- Convolution with delta function impulse moves other function to impulse center

---

Fourier Transform Solution to: \( \frac{\partial}{\partial t} \)
**SLICE SELECT RF PULSES**

Interleaved Acquisition \(\rightarrow\) better S/N b/c imperfect slice profile

Common RF pulses

- non-selective pulse ("hard" pulse)
- standard slice select sinc
- Gaussian

\(\rightarrow\) pulses need to be "apodized" (have "foot" removed)
\(\rightarrow\) multiply by function so begin/end of pulse is differentiable

**Fat Saturation**

- fat protons have chemical shift causing resonant freq offset
- add phase offset not due to gradients, RF
- fix by off-water-resonance 90° (saturation) pre-pulse centered on fat freq
- need high quality (narrow freq) pulse to avoid saturate water!

**HOWTO**

1. fat sat pulse
2. wait T2 so fat signal decays, but no T1 regrowth of fat
3. RF stim for water "protons of interest"

**Adding Another Gradient Tilts Slice**

- with 3 gradients on, can excite arbitrary angle plane
- translate plane by changing either gradient amplitude or RF freq band: \(\vec{B}_1\)
WHY "FREQUENCY-ENCODING" IS A MISNOMER

- comes from original analogy in Lauterbur (1973):

  Spectroscopy
  1) chemical shift change freq  \rightarrow\  gradient change freq.
  2) stimulate w/ broadband RF  \rightarrow\  same
  3) time-sample FID containing multiple freqs  \rightarrow\  same
  4) FT of FID to get spectrum
      $\Delta f$ of $\Delta f$ offsets
      $\rightarrow\  \frac{\Delta f}{c}$

  - this is technically correct (FT of FID) but highly misleading
  - e.g., phase-encoding (turning a different gradient ON and OFF before recording FID) seems to be something completely different since OFF gradient can't affect freqs in FID
  - the "k-space" perspective is a "Copernican Turn"
  - idea is that data is not a set of samples of a time domain signal generated by multiple chemical-shift like frequencies
  - rather, it is a set of samples of a frequency-domain signal, each sample generated by multiple spatial locations, (which are analogous to multiple time points)
  - i.e., the 'direction' of the FT (Fourier transform) is reversed:

<table>
<thead>
<tr>
<th>spectroscopy</th>
<th>MRI</th>
</tr>
</thead>
<tbody>
<tr>
<td>samples of oscillations in time-domain</td>
<td>samples of spatial freq. in freq. domain</td>
</tr>
<tr>
<td>FT $\rightarrow$ frequency-domain spectrum of shifts</td>
<td>FT $\rightarrow$ spatial object (like a time-domain signal)</td>
</tr>
</tbody>
</table>

- the original analogy only 'works' because $FT \cong FT^{-1}$ (except sign change)
FREQUENCY ENCODING (1)

- Frequency encode gradient ($G_x$) causes precession rates to vary linearly in $x$-direction.

- Different frequency signals are mixed together and recorded as a 1-D signal over time.
  - Correct, but remember, we are recording summed local magnetization vectors after de-modulation.

- A Fourier transform, which can convert back and forth between $x$-position (cf. time) and spatial frequency (cf. temporal freq.) is done on signal.
  - Correct.

- Spatial frequencies get confused/conflicated with precession frequencies.
  - Wrong!!

- Therefore, the Fourier transform is used to convert position-dependent precession frequencies into spatial position.

* N.B.: Gradient record does not exactly match record.

** Conceptually wrong!!

→ FT actually converts spatial frequencies to spatial position.
  → The spatial frequency increases for each time point in the readout.
  → The precession freq. ramp is constant each timestep.
FREQUENCY ENCODING (2)

- "Frequency"-encode gradient ($G_x$) turned on during
  during echo causes precession rates
to immediately vary with x-position

- at beginning of gradient on, the phase of
  signal coming from each x-position is the same
  summed phase angle is what we measure

- early after gradient on, phase advances (because
  of faster precession frequency) arise with greatest
  phase advance at longest x-position

- later during gradient on, phase advances cause
  multiple wraparounds of phase angle across space

- in practice, the lowest spatial frequency ($\phi_0$)
  occurs in the middle of the gradient on time
  because the phase is "rounded" negatively by
  a preparatory gradient (to the highest negative
  spatial frequency) before data collection occurs

$\phi_0$ is spatial frequency

$G_x$ in x-direction

$G_x$ levels
(= slope)

early after gradient on, phase advances (because
of faster precession frequency) arise with greatest
phase advance at longest x-position

- later during gradient on, phase advances cause
  multiple wraparounds of phase angle across space

- in practice, the lowest spatial frequency ($= 0$)
  occurs in the middle of the gradient on time
  because the phase is "rounded" negatively by
  a preparatory gradient (to the highest negative
  spatial frequency) before data collection occurs

$\phi_0$ is spatial frequency

$G_x$ in x-direction

$G_x$ levels
(= slope)
FREQUENCY ENCODING (3) why each datapoint is 1 spatial freq.

Standard Fourier transform: (Temporal freq. \(\leftrightarrow\) time)

\[
H(\omega) = \int_{t=-\infty}^{t=\infty} h(t) \cdot e^{-i \omega t} \, dt
\]

"k" is often used instead of "f" for the frequency variable

Imaging equation: (Spatial freq. \(\leftrightarrow\) space)

\[
S(\omega) = \int_{x=-\infty}^{x=\infty} I(x) \cdot e^{-i \omega x} \, dx
\]

Sum across x of object

- Oscillations come from readout phase wrapping, where f is single spatial freq (e.g., 5)
- And x goes across object

To make image, do inverse Fourier transform of recorded signal S(\(\omega\))

Don't confuse with instantaneously changed precession freqs which are constant across entire readout time (for each x position)
**Alternate Derivation (incl. effects of $G_x$) Signal EQ**

- Oscillators at $w = fB$ at each position (just $x$ for now)

\[ S(t) = m(x) e^{-i\phi(x)} dx \]

- By definition, freq, $w$ is rate of change of phase, $\phi$

\[ \frac{d\phi(x,t)}{dt} = w(x,t) = fB(x,t) \quad \text{and}\quad \phi(x,t) = \int_0^t w(x,t) dt = f\int_0^t B(x,t) dt \]

- Assuming phase initially 0, $B$ affected by gradients

\[ B(x,t) = B_0 + G_x(t) \cdot x \]

\[ \phi(x,t) = x\int_0^t B_0 dt + \left[ f\int_0^t G_x(t) dt \right] x \]

\[ = w_0 t + 2\pi k_x(t)x \]

- Demodulation removes the $B_0$-caused carrier frequency $e^{-i w_0 t}$ from the first equation

\[ S(t) = \int x m(x) e^{-i 2\pi k_x(t) x} dx \]

- Amplitude of each oscillator, gradient-controlled phase
Phase-EncodE GRADIENT Gy

- Turn on gradient after excitation but before readout
- Different levels of Gy
  \[ B_{\text{y},y} \]
  \[ y \]
  \[ y \]
  \[ y \]
- Higher levels of Gy (slope of \( B_z \) in y-direction!)
  \[ \Rightarrow \] higher spatial freq. (more phase wraps) in y-direction
- Phase wraps persist after phase-encode gradient off
- Read-out gradient (Gx) phase wraps then add to phase-encode phase

2D Imaging Equation

\[
S(k_x, k_y) = \left( \sum_{x} \left( \int_{y} I(x, y) \cdot e^{-i2\pi (k_x x + k_y y)} \, dy \right) \right)
\]

- Signal recorded at single time point (one readout point)
- Complex signal (from phase-sensitive detection)
- Done by RF coil
- Scalar (what we try to reconstruct)
- Phase angle (of scalar magnetization!) in rotating frame, set by gradients

Ignoring relaxation, spatial frequency \( k_x \) and \( k_y \) have no "inertia" — they stay wherever the gradients last left them
3-D IMAGING - two phase-encode gradients

- use $z$-gradient for 2nd phase-encoding instead of slice selection

- excitation of whole slab (slice-select is whole brain)

- simple spin echo example (in real life, usu. done with echo trains [FSE] or small flip angle to allow short TR)

\[ S(x, y, z) = \int I(x, y, z) \, dx \, dy \, dz \]

\[ I(x, y, z) = e^{-i2\pi (k_x x + k_y y + k_z z)} \]

- i.e., freq-encode phase, first phase-encode phase, and second phase-encode phase all just add (= 3D rotation of phase stripes)

- S/N much better than 2-D because each entire excited volume contributes to signal from each pulse instead of just slice

\[ \text{phase stripes created throughout volume vs. slice:} \]

\[ \text{one readout point in image space} \]

N.B. this ignores relaxation effects for now
PHASE & FREQ, 2D & 3D

Since the phase-encode gradient and the freq-encode gradient both affect phase, the result is a rotation of phase "stripes" when the two add.

N.B.: Stripes have sharp edges from phase wrap (not sinusoid since φ from 2-comp quadrature!)

Stripes here represent complex value
phase of whole image summed to one (complex) number by RF coils

Successive readout steps:

- Small phase encode G_y
- Large phase encode G_y
- TE

- Higher spatial freq.
- More rotation

Readout G_x

... t_1, t_2, t_3...

3D phase encode w/ G_y and G_z starts rotated in y-z plane

Large phase encode G_z
Small phase encode G_y
GRADIENTS MOVE K-SPACE LOCATION OF DATA POINT

- k-space (spatial-frequency space) location is set by integral of gradient over time up to recording point

\[ k = \mathcal{F} \int_0^{t_{\text{rec}}} G(t) \, dt \]

Spatial freq recorded at strength as function of t

simple form of integral w/ boxcar gradient

\[ k = Gt \]

(k is area under curve)

record data point here

all of the following gradients end up at the same point in k-space:

Frequency-encode FID

RF \[ 90^\circ \]

\[ G_x \]

\[ k_y \]

samples

Frequency-encode gradient echo

RF \[ 90^\circ \]

\[ G_x \]

\[ k_y \]

\[ k_x \]

\[ \neg G_x \]

Frequency-encode spin-echo (plus gradient echo!!)

RF \[ 90^\circ \] \[ 180^\circ \] \[ \tau \] \[ T_E \]

\[ G_x \]

\[ k_y \]

\[ k_x \]

\[ \text{RF } 180^\circ \text{ to conjugate point} \]

Phase-encode then frequency encode gradient echo

RF \[ 90^\circ \]

\[ G_y \]

\[ G_x \]

\[ k_y \]

\[ k_x \]

\[ G_x + G_y \]
**IMAGE RECONSTRUCTION**

\[
S(k_x, k_y) = \sqrt{\sum_x \sum_y I(x, y) e^{-i 2\pi (k_x x + k_y y)}}
\]

- Signal (complex)
- RF coil sums
- Brain spin density (real)
- Gradient caused phase maps (complex)
- In transverse magnetization
- N.B.: assumes perfect sinusoids! (they're not)

\[
I(x, y) = \sqrt{\sum_{k_x} \sum_{k_y} S(k_x, k_y) e^{-i 2\pi (x k_x + y k_y)}}
\]

- Screen image
- Imagery is real
- In practice, complex
- Use amplitude image:
  \[
  \frac{A}{\sqrt{\sum_{k_x} \sum_{k_y} S^2(k_x, k_y)}}
  \]

Adding exponents same as multiplying two \( e^{i 2\pi k_x} \)’s

\[
\text{same as two sequential 1D FFTs (actual code)} = \sqrt{\sum_{k_x} \left[ \sum_{k_y} S(k_x, k_y) e^{-i 2\pi x k_x} e^{-i 2\pi y k_y} \right] dk_x dk_y}
\]

In practice, finite number of samples, \( N \) and \( M \), are collected

\[
I(x, y) = \sum_{m = -M/2}^{M/2-1} \left[ \sum_{n = -N/2}^{N/2-1} S(n, m) e^{i 2\pi n \Delta k_x x / M} e^{i 2\pi m \Delta k_y y / N} \right]
\]

sampling interval in k-space

- \( \Delta k_x = 1 / (M \Delta x) \)
- \( \Delta k_y = 1 / (N \Delta y) \)
**Sampling**

- must consider effects of sampling
- limited points in k-space
- limited range of frequencies sampled ($k_{\text{min}} \rightarrow k_{\text{max}}$)
- limited in rate of sampling ($\Delta k$)

- N.B., aliasing less familiar when result of limited frequency domain sampling than limited space or time domain sampling

- correct reconstruction
- as above with blurring, ringing
- aliased occurs in spatial domain
- replicas overlap, causing wraparound

- infinite frequency range
- infinite fine sampling

- finite frequency range
- finite spacing of samples

Thus, finer sampling of same range of spatial freqs increases FOV
UNDER/OVER SAMPLE

\[ \text{FOV}_x = \frac{1}{\Delta k_x} \]

\[ s_x = \frac{\text{FOV}_x}{N} = \frac{1}{N \Delta k_x} \]

**FOV** (distance to repeat) is reciprocal of spatial frequency sampling interval

Pixel size is FOV divided by K-space sample count

---

3 more examples (not incl. less samples to same spat. freq. [bottom last page])

**Basic Image**

**Same num samp. to 2x spat. freq.** (i.e. gradients stronger or time ON longer)

**2x num. samples to same spat. freq.** (i.e. gradients weaker or time ON shorter)

**2x number samples to 2x spat. freq.** (i.e. gradients stronger or time ON longer)

---

N = 10
\[ k_x = 5 \]
\[ \Delta k_x = 1 \]
\[ \text{FOV} = 1 \]
\[ s_x = 0.1 \]

N = 10
\[ k_y = 10 \]
\[ \Delta k_y = 5 \]
\[ \text{FOV} = 2 \]
\[ s_x = 0.05 \]

N = 20
\[ k_x = 5 \]
\[ \Delta k_x = 0.5 \]
\[ \text{FOV} = 2 \]
\[ s_x = 0.05 \]

N = 25
\[ k_y = 10 \]
\[ \Delta k_y = 1 \]
\[ \text{FOV} = 1 \]
\[ s_x = 0.05 \]

- basic image
- Square pix
- X-pix half width
- Replicas intrude
  - Scanner makes square image
  - "wrap" occurs
- Square pix
  - twice X-pix count so FOV = 2X
  - this is "phase oversamp"
- Scanner crops to square
- Replicas move out
- X-pix half width
  - twice X-pix count
  - same FOV
  - this is decrease pixel size w/o change FOV
**Fourier Transform Solution to Replicas**

1. image/brain space
2. sampled data spatial frequency

\[ \ast \text{convolve} \]

\[ \rightarrow \text{x multiply} \]

\[ = \text{equals} \]

- Limit approach to Fourier transform of conv

\[ \Delta k = \frac{1}{\text{FOV}} \]

**Useful FTs**

- Rect
  \[ \text{Rect} \left( \frac{x}{w} \right) \xrightarrow{\mathcal{F}} W \cdot \text{sinc}(\pi W k) \]

- Gaussian (special case)
  \[ e^{-\pi x^2} \xrightarrow{\mathcal{F}} e^{-\pi k^2} \]
  \[ \text{larger} \rightarrow \text{narrower} \]

- Gaussian (adj. width)
  \[ e^{-ax^2} \xrightarrow{\mathcal{F}} \sqrt{\frac{\pi}{a}} e^{-\frac{\pi k^2}{a}} \]

- Comb
  \[ \sum_{n=-\infty}^{\infty} \delta(x - \frac{n}{\Delta k}) \xrightarrow{\mathcal{F}} \Delta k \sum_{p=-\infty}^{\infty} \delta(k - p \Delta k) \]
**POINT-SPREAD FUNCTION**

\[
\hat{I}(x) = \Delta k \sum_{n \in \mathbb{N} \setminus 2} s(n \Delta k) e^{i 2\pi n \Delta k x}
\]

- Set true image to \(s\) function, then measured signal is:
  \(s(m \Delta k) = 1\)

- Substitute into \(\text{recon}\) to get PSF:
  \[h(x) = \Delta k \sum_{n \in \mathbb{N} \setminus 2} e^{i 2\pi n \Delta k x}\]

- Simplify
  \[h(x) = \Delta k \frac{\sin \left( \pi N \Delta k x \right)}{\sin \left( \pi \Delta k x \right)} \Rightarrow \text{periodic}\]

- That is, image is reconstructed from a sum of sinc's, because the FT of a boxcar pixel in \(k\)-space is an \text{image sinc}. All outside center at sinc zeros-crossing. If \(k_{\text{max}} \leq x_{\text{max}}\)

---

**Image**
- How PSF modifies ideal (infinite \(k\)) image
  \(\ast\) convolve = ringing

**FT**
- \(\text{FT}\) (under rect narrower sinc)
- \(\text{FT}\) (acquisition window, truncated high spat, 6)

**Spatial freq. data**
- Multiply
**GENERAL LINEAR INVERSE RECON FOR MRI**

\[ S(k_x) = \int \frac{I(x) e^{-i 2\pi k_x x}}{x} dk_x \]

Signal eq. \( \Rightarrow \) fwd problem

\[ I(x) = \int_{k_x} S(k_x) e^{i 2\pi k_x x} dk_x \]

Recon eq. \( \Rightarrow \) inv. problem

\[ s = F \hat{c} \]

\[ s = \frac{1}{\sqrt{K_y}} \begin{bmatrix} F_{i} \end{bmatrix} \begin{bmatrix} i \end{bmatrix} \]

Linear "forward solution"

Matrix vectors have complex entries

Can build in any measurable priors

\[ F_{x,y,t} = g(x,y) e^{-i \phi(x,y)} e^{-(nT \pm m \Delta z + TE)/T_2} e^{-i \Delta \phi(x,y) T \pm m \Delta t} - i \Delta m (k_x x + \Delta k_y y) \]

Cal gain at this location

Cal phase

T2 decay

Error (x,y dep.)

Freq + phase (complex)

Multi-coil

\[ \begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} k_y \end{bmatrix} \begin{bmatrix} F_{\text{coil 1}} & F_{\text{coil 2}} \end{bmatrix} \begin{bmatrix} i \end{bmatrix} \]

Naturally incorporates undistorted field map

Different sensitivity function for each coil

Contains additional info about some lox

But, need reference scan, low-res OK

(need phase corrections for each coil)

\[ i = F^s \]

Over-determined

\[ F^+ = (F^TF)^{-1}F^T \]

More PnentRo inverse

\[ = F^T(FF^T)^{-1} \]

Small

\[ (x,y)^2 \Rightarrow 16 \times \text{bigger} \]

\[ \text{for 4 coils} \]

Slice-by-slice

Assume slice select swamps other

\[ i = [(F^T)^{-1}F^T]s \]
**FAST SPIN ECHO (FSE)**

- One 90° pulse followed by multiple 180° pulses (e.g., 8) each with a different phase-encode gradient.

- Each phase "wind" is "unwound" because leftover phase would be re-focused away by 180° (e.g., EPI where it persists between blips).

- The "effective TE" is the TE when center of k-space is collected (largest effect on contrast, largest echo).

- Each subsequent echo has more T2 decay: \( E_n = e^{-nTE/T2} \), \( n = 1, 2, \ldots M \).

- By arranging to collect \( k_y = 0 \) early, PD-weighted instead of T2-weighted.

- Possible to correct different T2-weighting of echoes by estimating T2 curve from \( G_y = 0 \) echo train.

- 3DFSE — like 2D except wind/unwind added to thick slice select (w/emphasis on 180°).
MULTI-SLAB 3D FSE, PROBLEMS

- echoes die out quickly by $e^{-t/12}$
- since echoes after $90^\circ$ limited to $<30$, can't fill 3-D k-space in a reasonable time
- SAR constraint $SAR \propto B_0^2 \theta^2 \Delta f$
  $\Rightarrow 180^\circ$ pulses deposit 4-6x power of $90^\circ$
- "multi-slab" is halfway between slices and single-slab

- problem at slice boundaries — esp. movement
- multislab requires slice selective RF pulses $\Rightarrow$ longer than non-selective 'hard' pulses

- $4\text{ ms RO}$
  hard to get under 8 msec inter-echo spacing

- $TE_{eff}$ is time from $90^\circ$ to echo thru center of k-space

- $G_z$ is "partition"
- $G_y$ is "phase encode"
- $G_x$ readout needs no pre-wind since $180^\circ$ does it

etc to fill 3D k-space

**FAST GRADIENT ECHO** (GRASS / FLASH / MP-RAGE)

- Small tip so TR can be greatly reduced (e.g. 10 msec, less than T2)
- 'leftover' undecayed transverse magnetization "unwound" and re-used "spoiled" before next shot

### STEADY-STATE COHERENT (GRASS, FISP)

- Unwind phase from phase-encode Mx before next pulse (there because TR<TE)
- Unwind read gradient, too

\[ S = k \sin x \left[ \frac{1}{1 \cos \alpha + (1-\cos \alpha) \frac{T_2}{T_1}} \right] e^{\frac{-T_1}{T_2}} \]

- T2/T1-weighted contrast (bright CSF)

### STEADY-STATE SPOILED (SPGR, FLASH)

- Spoil with random gradient (but this still allows some T1 refocusing)
- Spoil with gradient plus incremented phase of RF pulses (RF spoiling)
- Good gray-white contrast (T1-weighted)

### NON-STEADY STATE, MAGNETIZATION-PREP (MP-RAGE)

- Preparation pulse \( \alpha \) strong T1-weighting
- Contrast varies in spatial-freq-dependent way
- Longitudinal mag. not affect much by low angle pulses
Motivation

- Image values are arbitrary/relative (cliffseg's, manufacturers)
- Uncorrected coil fall-off (receive inhomogeneity) can result in 2-3x differences in voxel brightness
- Uncorrected variation in local B1 field can cause contrast variation
  - At 3T, B1 can vary by 25% across the brain
  - This can invert contrast in a fast gradient echo

Pre-scan normalise

- Collect low-res GE image, receive with body coil (no coil fall-off)
- Set PARMs to get low GM/WM contrast
- Collect data scan (e.g. MPRAGE) w/surface coils, strong GM/WM
- Use ratio between scans to generate smooth correction field

T1 divided by T2

- MPRAGE ➔ strong T1-Contrast
- SPACE ➔ T2-weighted (no T1 weighting)
- T1/T2 removes coil fall-off
- Distortion derived in GE(MPRAGE) and SE(SPACE)
- Problems < noise in regions of low signal

MP2RAGE

RF

160°

1st volume ➔ PD-weighted

2nd copy of volume ➔ more T1-weighted

1st to 6th, two k-spaces

- N.B. SSFP-like in partition, phase-encode directions
- Convert to -0.5 to 0.5 image: \( S = \text{real} \left( \frac{\mathbf{S}^*_{T1} \cdot \mathbf{S}^*_{T2}}{||\mathbf{S}^*_{T1}||^2 + ||\mathbf{S}^*_{T2}||^2} \right) \)
- Calc. PD & T1 from above

ref. 2 flip angles
QUANTITATIVE T1 - HELMS 2-FLIP ANGLE METHOD

- Start with gradient echo signal e.g., dropping T2-decay \( e^{-TE/T2} \)

\[
S_{\text{Ernst}} = A \cdot \sin \alpha \cdot \frac{1 - e^{-TR/T1}}{1 - \cos \alpha \cdot e^{-TR/T1}}
\]

Ernst eq.

\( \max: \cos \alpha_E = e^{-TR/T1} \)

"Ernst angle"

\( \alpha_E = \cos^{-1}(e^{-TR/T1}) \)

- Simplify/linearize/estimate

\( TR \ll T1 \)

Linear approx. of exponentials

Taylor expansion simplification of sin, cos, drop small terms

Helms et al. (2006)

\[
S = A \cdot \sin \alpha \cdot \frac{TR/T1}{\alpha^2/2 + TR/T1}
\]

- Solve for TD and

\( A \) (proton density) given signals from 2 diff flip angles

\( \max: \alpha^2/2 = TR/T1 \)

2 flip angles

\[
T1_{\text{est.}} = 2TR \left( \frac{S_1/\alpha_1 - S_2/\alpha_2}{S_2\alpha_2 - S_1\alpha_1} \right)
\]

\[
A_{\text{est.}} = S_1S_2\left( \frac{\alpha_1/\alpha_2 - \alpha_1/\alpha_2}{S_2\alpha_2 - S_1\alpha_1} \right)
\]

- Problem: Flip angle varies a lot at 3T (e.g., 25%) from nominal/requested (e.g., Flip series)

- acq: Spin-echo and stimulated echo (EVI)

RF in

\( \alpha = 90^\circ \) \( TE/2 \rightarrow 2\alpha \cdot \) \( TE/2 \rightarrow \alpha \cdot TE/2 \)

Slice

Phase

Read

RF in

SE

STE

\( T2^* \)

- Add EPI-like echo train to each FLASH excitation.

- Tiny error for flip \( \leq 15 \) deg.

Estimate

- Plot of signal vs. flip angle

- B1 map

\( \alpha = 90^\circ \) \( TE/2 \rightarrow 2\alpha \cdot \) \( TE/2 \rightarrow \alpha \cdot TE/2 \)

Partition (3D)

- Solve for \( 2\alpha \) insert (EV1)

\[
S_e = k \cdot \sin^3 \alpha \cdot e^{-TE/T2}
\]

\[
S_{\text{SE}} = k \cdot \sin^2 \alpha \cdot \sin 2\alpha \cdot e^{-TE/T2}
\]

\[
S_{\text{STE}} = \frac{k}{2} \cdot \sin^2 \alpha \cdot \sin 2\alpha \cdot e^{-TE/T2}
\]

- Jiru & Klose (2006)
Echo Planar Imaging (EPI) (another fast gradient echo)

- Single shot EPI collects all k-space lines (e.g., 64) after a 90° RF pulse using a train of gradient echoes.

- Since there is only one RF pulse per slice, spins never get reset to all-the-same (= zero freq, center of k-space).
- Therefore, the recording point (Δt) in k-space (= spin phase stripe pattern) stays wherever the x and y gradients last left it.

- That explains why successive y phase-encode steps are achieved without changing the size of the Gy "blips".
- Echoes are T2*-weighted (gradient echo).
- Contrast mainly determined by echoes near center of k-space, which are only recorded after about 32 echoes.

Fat saturation - before main 90° RF res, 90° fat slice select, fat spoiler and fast readout.

Fat saturation makes the loud "beep".
**SPIN ECHO EPI**

**why SE - BOLD may be selective for capillary bed**

- Standard EPI is a gradient echo method, which results in $T_2^*$-weighting
- Deoxyhemoglobin is paramagnetic, which reduces signal in a $T_2^*$-weighted image due to greater dephasing
- The excess of oxygenated hemoglobin (probably the result of the need to drive $O_2$ into tissue, which requires more $O_2$ in the blood than is actually used) leads to the positive BOLD effect
- Spin echo corrects (cancels) static $T_2^*$ ($T_2$) dephasing, incl. deoxy
- If all spins stayed in the same position, spin echo would eliminate the BOLD effect by eliminating dephasing
- Diffusion exposes spins to different fields (reducing gradient echo dephasing)
- Magnetic field gradients produced by large vessels are smoother across space than those produced by small vessels
- For $TE \approx 100$ ms, spins diffuse 10's of mm, which is larger than diameter of small capillary, meaning that spins will likely experience different fields over time

- Therefore, spin echo will be less successful at canceling BOLD effect near small vessels. (BOLD effect will be reduced near large vessels where diffusion is less likely to expose spin to different fields here)

**N.B.: this argument applies to the extravascular signal**

- This argument only works for extravascular spins. Intravascular signal in BOLD is large (despite being only 4% by volume) because large gradient produced around red blood cells

\[ \text{measure intra/extra w/ bipolar pulse which kills signal in faster moving blood in moderate and larger vessels} \]

**Over half of SE-BOLD at LST is venous...**
SPIN ECHO EPI

- EPI is a multi-gradient echo pulse sequence.

- "Spin-echo EPI" uses a 180° pulse to add a
  single spin echo to the contrast-controlling
  gradient echo through the center of k-space.

- "asymmetric spin-echo EPI" arranges for the spin echo
  to occur 1 msec before the gradient echo, which
  gives more T2*-weighting (for ky=0 echo).

- The 180°-pulse rephasing reduces the T2* signal, which
  is why the partially rephased asymmetric spin echo
  has been more commonly used.

- At higher fields, spin echo EPI is more promising.
  
  - Signal to noise is higher so we can take spin echo hit
  
  - Contribution from venous blood is reduced, since blood
    T2 is so short, we can let it decay away before recording.
- coil fall-off intuitively contains info about location if same brain location imaged by different coils w/ diff. fall-offs

\[ \text{but what does this look like in } k\text{-space?} \]

- slow variation in RF-field fall-off (e.g., 1-4 cycle/FOV) causes a blur in acquired data in k-space

\[ \text{N.B., not addition!} \]

- to see this, consider multiplication by coil fall-off function in image space, which equals convolution (w/ FT of that function) in k-space— at all spatial frequencies!!

- simple example w/ "brain" consisting of one spatial freq.

\[ \text{Image domain} \quad \begin{array}{c}
\text{FT} \\
\text{Spatial freq. domain}
\end{array} \quad \begin{array}{c}
\text{multiply} \\
(\ast) \text{convolve}
\end{array} \quad \begin{array}{c}
\text{FT} \\
\text{Acquired image}
\end{array} \]

- N.B. inverse FT of k-space data "smeared" in spatial freq.

\[ \text{Space is sharp image w/ fall-off (not blurred image).} \]

- "smear" means normally orthogonal spatial freq.'s "leak" to adj. freqs.

- GRAPPA - construct k-space "kernel" to fill in missing k-space lines by training on fully-sampled data from near k-space center across multi-coils

- SENSE - general linear inverse approach

- N.B.: neither would work unless normally orthog. spatial freqs. blurred!
- Excite multiple slices at once
- Function of $G_z$ blips is to shift slices in $G_y$ direction
- This occurs because for given slice, a phase pedestal is added to the entire slice
  \[ \text{[N.B. different than } B_0 \text{ defect-induced incremented phase errors]} \]
  \[ \Rightarrow \text{ "Fourier Shift Theorem"} \]
- Problem w/ all up $G_z$ blips $\Rightarrow$ phase error builds up
  \[ \text{\underline{trick#1}} \]
  - Start w/2 slices, one at $z=0$, other above
  \[ \Rightarrow \text{if } 180^\circ \text{ phase shift used, blip up/down same! (no effect at } z=0) \]
  \[ \Rightarrow \text{i.e., move top or bottom replica} \]
  \[ \text{\underline{trick#2}} \]
  - For multiple slices not all at $z=0$, phase no longer same for even/odd
  \[ \Rightarrow \text{but can add phase to equilibrate to } k\text{-space before recom.} \]
  \[ \text{\underline{trick#3}} \]
  - For more than 2 slices:
    \[ \begin{array}{cccc}
    \text{1st} & \text{even} & \text{odd} & \text{even}
    \end{array} \]
    \[ \text{etc} \]
MULTI BAND/BLIPPED CHIP

Relation between leave-one-out aliasing and nominally fully-sampled SMS

- Leave alternate lines out wraps image
- SENSE/GRAPPA to fix block coil view smears K-space data
- Nominally, w/ SMS we record every line of K-space
- But equivalent to leave alternate out b/c our multi-slice FOV was not big enough

- Slice GRAPPA
  - reg GRAPPA → recon missing lines
  - slice GRAPPA → recon multiple K-spaces
    For each overlapped slices by training on fully-sampled data at beginning of scan

- Interslice "leakage block"
  - when training GRAPPA kernel on fully-sampled data,
    also minimize interslice leakage (split to slice GRAPPA)
  - can also do regular GRAPPA on top of this
    reason: for diffusion, loss in S/N from undersample cancelled by shorter TE readout
  - gain from reduced image distortion from shorter readout
ECHO-VOLUME IMAGING EVI

- multi-shot (like FLASH) but acquiring one plane of 3-D k-space per shot (can do spin echo, too)

- entire k-space must be filled before 3D image is reconstructed

- main issue is movement artifact since data assembled from many shots over several secs

- breathing-induced B0 problems in different partitions may cause blur
SPIRAL IMAGING

- by using smoothly changing gradients (sinusoids) less gradient power required than w/trapezoids (less eddy currents)

earlier EPI hardware like this: sinusoidal gradient waveform from resonant circuit w/non-uniform sampling to get constant Δk_x

- sinusoids in both G_x and G_y give spiral k-space trajectory

- constant angular velocity goes too fast at large k_x, k_y
- constant linear velocity better but impractical near k_x=0, k_y=0
- compromise: start constant angular, end constant linear

Constant angular velocity

w(t) = w_0 t

k(t) = A t e^{i w_0 t}

\[ G(t) = \frac{1}{i} \frac{d}{dt} k(t) \]

= A e^{i w_0 t} + i A t w_0 e^{i w_0 t}

\[ G_x(t) = A \cos w_0 t - A t w_0 \sin w_0 t \]

\[ G_y(t) = A \sin w_0 t + A t w_0 \cos w_0 t \]

Constant linear velocity

w(t) = w_0 T_e

k(t) = A T_e e^{i w_0 T_e}

\[ G(t) = \frac{1}{i} \frac{d}{dt} k(t) \]

= \frac{A}{2T_e} e^{i w_0 T_e} + \frac{A}{2} w_0 e^{i w_0 T_e}

\[ G_x(t) = \frac{A}{2T_e} \cos w_0 T_e + \frac{A}{2} w_0 \cos w_0 T_e \]

\[ G_y(t) = \frac{A}{2T_e} \sin w_0 T_e + \frac{A}{2} w_0 \sin w_0 T_e \]
SPIRAL 3D IR FSE

- 3D: block select vs. slice select
- FSE: multiple echoes from one 90°
- spiral: multiple spirals vs. multiple lines
- interleaved spirals (like FSE interleaves)
- true IR (vs. MPRAGE) all echoes after 90° derive from mag w/ same TI contrast (vs. non-steady-state)
- possible to preserve sign
- high, uniform contrast, but lots of waiting (TI), high BW

RF

180° (prep_1) T1 ~ 700 msec // 180° // 180° // M_L // M_L // M_L // M_L // 180° (prep_2)

Gz

Gy

Gx

Sig.

3D k-space

("stack of spirals")

Loop order

spiral interleaves

k_x interleaves

k_z echoes

echoes → (after one 90°)
Phase Errors & Echo-Centering Errors

- anything that causes a deviation of the $B_z$ field strength from the expected value $(B_{0,z} + G_{x,z} x + G_{y,z} y + G_{z,z} z)$ changes precision frequency and therefore, expected phase angle
- incorrect phase of spatial frequency stripes results in a shift in space in the magnitude image after reconstruction

**Phase Errors**

- Phase stripes in image domain (one k-space point)

**Fourier shift theorem**

Phase shift in spatial freq. domain causes spatial shift in image domain

$$I(x-x_0) = \int e^{-i2\pi k_x (x-x_0)} S(k_x) e^{i 2\pi k_x x} dk_x$$

- First defense: freq. prescan
- Refine w/shimming and $B_0$-mapping/phase unwrapping before reconstruction

**Echo Centering Error**

- If realignment of all spins ($k_x = k_y = 0$) doesn't occur at the middle of read gradient, echo is shifted
- Since echo is in spatial frequency domain, this is frequency shift

- Spatial frequency shift results in wrapping in phase image after reconstruction
  \[ \Rightarrow \text{magnitude image unchanged} \]

**Fourier freq. shift theorem**

Freq. shift in freq. domain causes phase shift in spatial freq. space

$$S(k_x - k_{x_0})$$
FAST SCAN ARTIFACTS  EPI vs. Spiral

brain-induced field defects lead to phase errors

EPI
- $G_x$ readout gradient strong → field defects smaller percentage
  less deformation of $K_x$ (vertical stripe components)

- $G_y$ "blips" are weak and total $G_y$ record time
  much longer (5 times) than standard readout (50 ms vs. 10 ms)

- an extra gradient in the x-direction, for example, maps
  and unmaps phase as a function of x-position

- but $G_x$ big, so effect on freq.-encode direction is much
  less than on phase-encode direction, where error accumulates

- for a given x-position, the strength of the spurious gradient
  is constant, so the accumulation of phase error results
  in a shift in the y-direction ($=K_x$-space spin-stripe disp.)

- the phase error causes a shift in the y-direction
  proportional to x-gradient strength (≠ shear) but no blurring
  (N.B. Shift varies w/x-position) $y \xrightarrow{x} y$

Spiral
- with center-out spirals phase errors accumulate
  in a radial direction

- thus, higher spatial frequencies have more error (≠ more shearing)

- for spurious x-direction gradient as above, there is
  a radial blurring, rather than a vertical shift

- for defects with more complex contents in the y-direction
  (than linear, as above) the vertical shifts (in EPI) will
  vary with y-position, and may result in signals from different
  y-positions being reconstructed on top of each other
**IMAGE-SPACE VIEW OF LOCALIZED Bφ DEFECT, EFFECT ON RECON**

- Localized Bφ defects often arise from air pockets embedded in tissue.
  - Air in middle/outer ear → indentation in inferior temporal lobe
  - Air in olfactory epithelium → orbitofrontal c.f., ant. thal. compression

- Collect one data (k-space) point:
  - 4 cycles of phase in y-dir (a-p position)
  - Localized Bφ defect

- Complex multiply:
  - = correlate sin/cos with brain

- Brain structure sampled with distorted stripes:
  - One complex number

- Reconstruction from distorted data points:
  - ... + [undistorted stripes used by inverse FFT]

- Same defect makes leftward dent in vertical phase stripes

- Spatial information can be lost when continuous changes in phase are flattened by Bφ defect

- Shifts can pile multiple pixels on top of each other into one bright pixel

- Local estimates of ΔBφ needed to correct images:
  1) Fieldmap method: < multiple TEs to estimate local ΔBφ from ΔTE slope
  2) Point-spread-function: < extra phase encode to estimate PSF (should be & function)
     -> deconvolve distorted image in phase-encode direction

- Local upward displacement image phase (phase encode dir)

**N.B.:** Image shift only occurs if shift spatially occurs sampled in successively later echoes (see next page)
LOCALIZED BØ DEFECT, EFFECT ON RECON (2)

- when local BØ defect disturbs image space phase stripes during signal acquisition, estimates of local spatial freq. are affected (compressed stripes = higher spatial freq.)

- if each successive ky line recorded w/ same echo time (e.g., w/ single line phase encoding), this will correspond to constant spat. freq. offset in k-space

- a k-space freq. offset only results in image space phase shift (Fourier freq. shift theorem), which is invisible in amplitude image (cf., echo cont. error)

- however, with w/EPI, static BØ defect causes more and more local displacement of image phase stripes for each additional ky line

  - that is, later lines have greater spat. freq. offset
  - effectively stretches k-space in ky direction
  - same num samples to higher spatial freq. shrinks FOV (squishes voxels — see FOV page)

- when image is reconstructed, region with local BØ defect shifted oppositely

  - Thus, local shift effect due to combination of 3 things:

    1) static local ΔBØ defect

    2) successive increases in phase error for successive spat. freq. measurements during long EPI readout

    3) small size of ky phase encode blips relative to BØ defect (much less of this effect in freq. encode direction)

- respiration (which affect BØ) in 3D FLASH might cause similar effect within k₂ partition (if successive spat. freqs.)
GRADIENT NON-LINEARITIES

- ideally the \( G_x, G_y, \) \( \text{and} \) \( G_z \) gradient coils attempt to impress a linear variation onto to \( Z \)-component of the \( B \) Field — \( B_z \) — in the \( x, y, \) \( \text{and} \) \( z \)-directions

- in practice, gradient coils are non-linear (esp. printed-circuit-like)

- non-linearities are worse in smaller coils, but also in higher performance coils, designed for post-processing correction of distortions

- non-linearities result in phase errors, which result in 3-D image distortion

- a non-linear slice-select gradient will excite a curved slice

- non-linear phase and frequency encode gradients will distort in-plane features

- some scanners correct these differently for 3-D scans (all directions), 2-D scans (just in-plane), and EPI scans (no corrections!)

- this can result in errors approaching \( 1 \text{ cm} \) in function-structure overlays

- different coil designs have different directions of distortion (!)

- the assumption of non-Maxwellian gradients results in additional phase errors

- these can also be corrected since the \( B_x \) and \( B_y \) components are known

- that is, the assumption that gradients cause no field in the \( B_x \) and \( B_y \) direction
SHIMMING AND $B_0$-MAPPING

- Passive iron shims inserted to flatten $B_0$ field
- Additional coils (usu. in the gradient coils) can be statically energized in an attempt to flatten the $B_0$ field (a few ppm good)

- Primary use is to compensate for defects in flatness present without a sample in the magnet (geometric imperfections, impurities in metal, etc) (= several hundred ppm)

- Linear shim coils impose gradients in $x$, $y$, and $z$
- Higher order shims impose higher order spherical harmonic field components (e.g. $z^2$)

- Secondary use is to compensate for inhomogeneities caused by introducing the sample into the $B_0$ field

- Local resonance offsets caused by $B_0$ defects estimated from images
  \[ \text{e.g., sample phase at multiple echo times} \]

- Fit defective field using combination of fields generated by shim coils, then add these corrections to base shim currents
  \[ \text{this only corrects spatially gradual field defects} \]
  \[ \text{local defects due to air in sinuses much higher order than shims} \]

- After shimming, field map measured again

- Image voxel displacements calculated from resonance offset map are used to un-warp the reconstructed magnitude image

- For EPI images, assume displacements all in phase-encode direction (since freq-encode gradient is strong relative to defects)
Navigators Echoes

1D Navigator

- 1D navigator
- Bφ drift problem
- Slow up/down drifts in Bφ continuously occur
- A pedestal in Bφ is a pedestal in phase (not gradient)
- Which causes spatial shift (Fourier shift theorem)
- In EPI, mainly affects phase-encode dir b/c of small slip
- Result is successive volumes drift in phase-encode dir

Gradient Balance Problem

- Unequal L/R readout gradients cause L/R shift in position of even/odd lines in k-space
- Causing N/2 (Nyquist) ghosting → another phase error

3D Navigator: Collect 3D sphere in k-space

- Rotation of object = rotation of k-space amplitude pattern
- Translation of object = phase shift of k-space phase (Fourier shift)
- Sample at sufficient radius to pick up high spatial freq features
- N.B.: Excite whole volume
- Do N,S hemispheres separately (less T2*, cancel EPI-like error accumulation)

Walsh et al. (2002) MRM

\[
\begin{align*}
\text{RF} & \quad 90^\circ \\
G_z & \\
G_y & \\
G_x & \\
\text{Sig} & \quad 0 \quad 4 \quad (\text{msec}) \quad 8 \quad 12
\end{align*}
\]

\[
\begin{align*}
z(n) &= \frac{2n - N - 1}{N} \\
y(n) &= \cos\left(\frac{\pi n}{2}\right) \\
x(n) &= \sin\left(\frac{\pi n}{2}\right)
\end{align*}
\]

- Can be used for prospective motion correction (rotate, translate w/ gradients)
- Better estimate, because of speed, than full TR & EPI images (27 ms vs. 2.4 sec)
- May need to smooth rot, trans estimates across time (e.g., Kalman filter)
RF FIELD INHOMOGENEITIES $B_0$ inhomogeneities

- Receive coil inhomogeneities alter the amplitude of the received signal, altering the reconstructed proton density in a spatially varying way.
  - Variations can be used (e.g. GRAPPA, SENSE) and/or corrected.

- Transmit coil inhomogeneities affect the flip angle in a spatially varying way (can affect contrast: FLASH).
  - Potentially worse (why local transmit is still in progress).
  - Usual fixed by using a large transmit coil (e.g. body coil).

- RF penetration at higher fields ($\propto$ higher RF frequencies)
  is less uniform:
  1) Decreased RF wavelength (closer to size of head) at higher freq.
  2) Increased permittivity ($\varepsilon$) and conductivity ($\sigma$) at higher field.

- 2nd advantage of the fall-off in signal recorded with a small, receive-only RF coil is better signal-to-noise.
  (less noise received from other parts of brain)

- Different sensitivity functions from different coils can be used to scan less lines in k-space (GRAPPA/SENSE/SPACE-RIP).

  - Record lo-res volume ($b$/coil fall-off is smooth) through both body coil and small coil(s).
  - Divide small coil body coil at each voxel to determine receive field.
  - Use receive field to normalize main image(s).

  [See also: $\varepsilon T_1$, MP2RAGE, $T_1/T_2$.]
DIFFUSION - WEIGHTED IMAGING

Simple diffusion weighting

- RF: A°
- Gz: select
- Gy: phase encode
- Gx: prepare
- b = y^2 G^2 Δτ^2 (T-Δτ/3)

- "apparent diffusion coefficient" → calculate from b=0 image and at least 6 b=large (e.g., 1000) images
- after obtaining 3x3 tensor, calc eigenvalues/values to find orientation & shape of diffusion ellipsoid
- two useful scalar values from 3 eigenvalues:
  - MD
  - FA

- to get large b, need large G, Δτ, T

1) Anisotropic Diffusion (Gaussian)

- measure D along multiple axes
- have to measure tensor, not scalar
even for determining one primary direction
- isotropic: D = δD
- since D is symmetric, need minimum of 6 different measurement directions
- without 3rd number (S)
- x, y projections same

D = [u_x, u_y, u_z] [D_xx, D_xy, D_xz] [u_x, u_y, u_z]

D = u^T · D · u

scalar diffusion

Diffusion Surface (non-Gaussian)

- need to measure diffusion in many directions (>6) to properly characterize even 2 main directions

2) Length Scale by multiple b-values

- form line to semi-log signal as function of b
- if not straight line: multi-exponential, e.g.,
- hi ADC/short extra vs. low ADC/long/intracellular

- assume Gaussian diffusion
- spins acquire phase during first Δτ
- if spins diffuse (move) along gradient by time T, signal is lost because negative Δτ doesn't re-phase
- attenuation:
  \[ A(D) = \frac{S_{b=0}}{S_{b}} = e^{-bD} \]

where
  \[ b = y^2 G^2 \Delta \tau^2 (T-\Delta \tau/3) \]

in z dir

3D: σ_x = \sqrt{6} D \Delta τ
ID: σ_x = \frac{\sqrt{2}}{2} D \Delta τ

\[ e.g., \ brain : D = 0.001 \ mm^2/sec \]

B_0 x

only detect project onto x

- classical diffusion coefficient
- time for diffusion

10 mm

 vant Hoff delay

Summarizes gradient

100 mm

Axial anisotropy

\[ FA = \frac{(\lambda_1-\lambda_2)^2 + (\lambda_2-\lambda_3)^2 + (\lambda_3-\lambda_1)^2}{2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)} \]

tract tracing

1) Markov process
2) crossing fibers
3) "freeway ramp" prob
4) sharp turns into gray

voxels

[mulcahy, 1999]

log signal

oe assume A_2 = 0.5

vs.

fit to biexponential curve

(e.g., A_1 = A_3 = 0.5)

(e.g., D = 0)

- b = 0

- b = 10

- b = 30

- b = 100

- b = 200

- b = 1000

- b = 5000

- b = 10000
PRACTICAL DIFFUSION-WEIGHTED PULSE SEQ

- spin-echo 'Stejskal-Tanner' EPI seq. (standard on scanners)
  - allows longer TE
  - flips $M_2$ so rephase gradient + sign as de-phase

- eddy-currents are long time-constant currents in metal of scanner that distort B field → spatial image distortion

- "doubly-refocused" spin echo sequence (DSE) can cancel the effects of eddy currents (w/partic. time constants)
  - (also, keep cuscshers orthogonal to diffusion-encoding gradients)

Nagy et al. (2014) MRM

$\Psi_{\text{TRSE}} = 0 = \Psi_1 - \Psi_2 - \Psi_3 + \Psi_4$

RF

- phase dispersion (δ echo)

$\Psi_1$, $\Psi_2$, $\Psi_3$, $\Psi_4$

RF in

90°

PD°

PD°

PD°

causing phase dispersion $\Psi$

other

diffusion

nav

TE

SE

navigators

$SE_{1,2}$

$SE_{2,3}$

$SE_{1,3}$

twice refocused Spin-echo (for center K-space)

etc

et al
**PERFUSION - ARTERIAL SPIN LABEL**

- **basic idea:**
  - tag blood below area of interest
  - collect control & tagged images
  - assume directional input flow

- **continuous ASL (CASL)**
  - continuously tag a plane
  - greatest on, blood gets adiabatically inverted as it passes through location w/condensed resonant tag

- **pulsed ASL (PASL)**
  - e.g., EP STAR, FAIR, PICORE, QUIPPE II
  - tag block of tissue below slice(s)

- small diffs between control and tag (≈1%)
  - requires accurate balancing of control & tag images, control mag. transfer

- contrast problems:
  - transit delays — biggest confounding factor
  - relaxation rate diff.
  - venous clearance (vs. microspheres, which get stuck!)

- solutions for quantitative
  - in short delay so all spins arrive into low-velocity capillaries
  - kill end of tag to reduce spatial variation of tag

**QUIPPE II - quantitative perfusion**

1) pre-saturate spins in target slices
2) tag - 180° pulse below slices
3) control - 180° pulse above slices (to control off-resonance)
4) EPI or spiral images of target slices (TI)

\[ \Delta M \approx \text{flow} \times \left[ 2M_0 \cdot TI, e^{-\frac{TI_2}{T1A}} \right] \]

- can extract flow and BOLD

**BOLD**

1) alternate tag and control, GRE TE=30 ms
2) dual echo spiral

- k=0 early =⇒ hi S/N flow
- TE=30 ms =⇒ BOLD
PERFUSION - pCASL

- Original CASL (continuous arterial spin labeling) requires RF on continuously to adiabatically invert blood flowing through one plane
  - can only image one slice (ble dephasing from gradient)
  - hard to keep RF on continuously on modern scanner (esp. EC)
  - can use special purpose RF transmit (separate xmt channel)

A) Original CASL

RF
\[ G_z \]

\[
\begin{array}{c}
\text{image formation} \\
\text{module (readout)} \\
\end{array}
\]

\text{[multiple possibilities]}

B) pCASL - pseudo continuous arterial spin labeling  
  Dai, Alsop (2008)

\[
\begin{array}{c}
\text{RF} \\
\text{G}_z \\
\end{array}
\]

\[ \Delta t \]

\[ \delta \]

\[ \text{begin readout} \]

- Problem: multiple pulses create aliased slice planes
  - \( \text{RF}(t) = \frac{1}{\Delta t} \text{comb}(t/\Delta t) \otimes \text{rect}(t/8) \)
  - \( \text{Comb}(t) = \sum_{n=-\infty}^{\infty} \delta(t-n) \)
  - \( \text{Rect}(t) = \begin{cases} 1 & \text{if } |t| < \frac{1}{8} \\ 0 & \text{otherwise} \end{cases} \)

\[ \text{F}[\text{RF}(t)] = \text{Comb}(6\Delta t) \cdot \delta \text{sinc}(\pi \delta) \]

\[ \text{aliased labeling planes at } \delta = n/\Delta t \text{ in frequency space, modulated by broad sinc() i.e. spacing inversely proportional to } \delta \]

- Use Hamming or hyperbolic secant to reduce replicas

\[
\begin{array}{c}
\text{Hamming} \\
\text{FT} \\
\end{array}
\]

\[
\begin{array}{c}
\text{Hyperbolic secant (c.f. Gaussian)} \\
\text{FT} \\
\end{array}
\]

C) pCASL w/ shaped gradients

\[
\begin{array}{c}
\text{RF} \\
\text{G}_z \\
\end{array}
\]

\[ 0.8 \text{ msec} \]

\[ \text{tag} \]

\[ \text{control} \]

\[ \text{readout} \]

\[ \text{EPI} \]

\[ \text{FLASH} \]

\[ \text{SMS} \]

\[ \text{Stack spirals 3D} \]

\[ \text{-tag pulses have phase offset respecting gradient} \]

\[ \text{-control identical except every other has } +\pi \text{ phase} \]

\[ \text{-no net flip} \]
Off Resonance Excitation

- **Main idea**: Examine evolution of \( \vec{M} \) vector in rotating coord syst set to "off-resonance" \( \vec{B}_1 \)-field freq (\( \omega_{rf} \)), not Larmor freq of \( \vec{M} \) (\( \omega_0 \)).

- Normally, if rotating coord syst freq set to Larmor freq (\( \omega_{rf} = \omega_0 \)), an actually precessing \( \vec{M} \) will be stationary (ignoring decay) \( \Rightarrow \) implies effective \( B_z = 0 \) in rotating.

- Now, move \( \vec{M} \) to rotating coord syst at \( \vec{B}_1 \)-freq lower than \( \omega_0 \) (assume \( \vec{B}_1 = 0 \) to left) \( \Rightarrow \) existing \( \vec{M} \) will now appear to precess around \( z \)-axis:

\[
\Delta \omega_0 = \omega_0 - \omega_{rf}
\]

- Thus, viewing \( \vec{M} \) vector in off-resonance rotating coord syst makes it look like additional \( \vec{B}_2 \)-field is causing "extra" precession.

- "Extra" \( \vec{B}_2 \)-component is proportional to \( \Delta \omega_0 \) offset \( \Rightarrow \) can be pos or neg: high \( \omega_{rf} \) \( \Rightarrow \) pos \( \vec{B}_2 \), low \( \omega_{rf} \) \( \Rightarrow \) neg \( \vec{B}_2 \).

- Extra \( \vec{B}_2 \) adds to \( \vec{B}_1 \) resulting in slow precession around tipped axis: \( \vec{B}_{eff} \) (effective).

- Extra \( \vec{B}_2 \) from any gradient \( \Rightarrow \) same effect on \( \Delta \omega_0 \) (changes \( \omega_0 \) instead of changing \( \omega_{rf} \)).

\[
\vec{B}_{eff} = (\frac{\Delta \omega_0}{\gamma}) \hat{k} + B_{2z} \hat{k} + B_1 \hat{i}
\]

- Effective \( \vec{B} \) in rotating frame set to \( \vec{B}_1 \)-freq (pos or neg)
- Apparent "extra" \( \vec{B}_2 \) from Larmor-B1 freq mismatch

---

**adiabatic RF pulse**

Off-Res., opposite way

RF: sweep freq \( \omega_0 \) : constant freq \( \omega_0 \) : sweeps because spins flow along gradient direction

---

Off-Res., opposite way
SPECTROSCOPY + IMAGE

- chemical shift: small displacement resonant freqs due to variable shielding of target nucleus (e.g., \(^1H\)) by surrounding electron orbitals

- e.g., acetic acid:
  - oxygen attracts electron so less shielding of target nucleus
  - 3 of these H's (more shielded)
  - 1 of these H's (less shielded)

- how we get chemical shift spectrum:
  - RF stim
  - RF record
  - \( \Delta f = f_{\text{reso}} + \Delta f\text{, ppm} \)
  - Larmor oscillations are multiplied (PSD) by center freq to obtain \( \Delta f \) (not MHz, high freq)
  - data before FT is a series of time-domain samples of the mix of shifted-freq offsets
  - FT turns data into "shift spectrum"

Pulse Sequence

- since we are already using phase (f-reso) encoding for space, we need an "extra dimension" w/ all gradients OFF!

- use spin-echo to undo built-up chemical shifts, then record chemical-NMR-like signal \( \rightarrow \) and FT-it like chemists do!
PRESS, MEGA-PRESS

- usu. single voxel by using 3 orthogonal slice selects
  (100 can add PE gradients & more excitations to get multiple vox)

PRESS — 3 orthogonal slice select

MEGA-PRESS — add "editing" RFs to suppress solvent (water)

G1, G3 — asymmetric spoilers to dephase spins in bandwidth of selective MEGA pulses

FT to get shift spectrum

FT to get shift spectrum
PHASE-ENCODED STIMULUS & ANALYSIS

Periodic stimuli (phase-encoded) - e.g., 8 cycles at 64 sec/cycle

Calculate significance
- Ratio between amplitude at stimulus frequency (= signal) and average of amplitudes at other frequencies (= noise)
- Ignore harmonics, low freq (= movement)

Smooth
- Vector average of complex significance (A, φ) with that at nearest neighbor surface points

Display
- Plot phase using hue and saturation to indicate significance

Delay correction
- Record responses to opposite directions of stimulus (ccw/cw, in/out, up/down)
- Vector average after reversing angle of one penalizes inconsistent more than just avg of angles
Convolution

\[ h(x) = f(x) \ast g(x) = \int_{-\infty}^{+\infty} f(z) \cdot g(x-z) \, dz \]

- Definition of convolution: \((f \ast g)(x)\)
- Commutative

Graphically:
- Place kernel at \(x\)
- Reverse kernel \(\Rightarrow\) mult. sum
- Current \(x\)

Why reverse makes sense:
- b/c commutative, can think like: \(h(x) = \int_{-\infty}^{+\infty} g(z) \cdot f(x-z) \, dz\)

Impulse response function (HDR)
- Impulses (exp. design)

**Intuitive non-reversed view & convolution output**

- How to calculate convolution output for this time point (3 terms in sum, all others zero)
- N.B. cross-correlation same as convolution except no reversed
- \(g(x+z)\) instead of \(g(x-z)\)
- N.B. auto-correlation same, except no-reversed and use same function for both \(f, g\)
**General Linear Model (GLM)**

\[
\hat{y} = \hat{X} \hat{h} + \hat{S} \hat{b} + \hat{n}
\]

- **Data**: design \( \times \) HDR + drifts \( \times \) weights + noise
- **Goal**: solve for the hemodynamic response functions, \( \hat{h} \)

\[
y = Xh + Sb + n
\]

- **Temporal convolving**
  - \( t \) for on, \( \bar{t} \) for off
  - \( \text{Experimental design} \)
  - \( \text{Assumed white noise} \)

- **Multiple conditions**
  - Cond1 occurs
  - Cond2 occurs
  - Cond1 re-occurs

\[
\begin{bmatrix}
\mathbf{h}_1 \\
\mathbf{h}_2 \\
\vdots \\
\mathbf{h}_N
\end{bmatrix}
\]

- **Basic vectors of signal space**
- **Basic vectors of interference space**

**Maximum likelihood estimate**

1. Assume white noise, solve for \( \hat{h} \)
2. \[
\hat{h} = (X^T P_s X)^{-1} X^T P_s y
\]
   where \( P_s = I - S (S^T S)^{-1} S^T \)
   or \[
   \hat{h} = (X_{\perp}^T X_{\perp})^{-1} X_{\perp}^T y
   \]
   where \( X_{\perp} = P_s X \)
3. Significance (how to construct F-ratio)

\[
F = \frac{N-K-L}{K} \left[ \frac{y^T (P_{ks} - P_s) y}{y^T (I - P_{ks}) y} \right]
\]

- **Projection matrix** that removes part of vector that lies in \( S \) space
- **Design matrix** with nuisance effects removed from cols

**Projection matrices**

- \( P_{ks} \) - projects data onto basis
- \( P_s \) - projects data onto nuisance subspace

**See diagram next page for geometric interp**
GEOMETRIC INTERPRETATION OF GENERAL LINEAR MODEL

- With no nuisance functions (S), we could look at orthogonal projection of data onto experimental design and compare that to error to determine significance.

\[ \hat{y} = \hat{X} \hat{h} + \hat{\varepsilon} \]

\[ \hat{X} \hat{h} = P_X \hat{y} \]

Projection matrix, \( P_X \), operates on \( \hat{y} \) to give projection of data into experiment space, \( \hat{X} \).

- When nuisance functions, \( S \), are considered, problem: \( S \) may not be orthogonal to \( X \).

For example: linear trend not orthogonal to std. block design.

\[ \text{Orthogonal projection:} \]

\[ \begin{align*}
\text{Orthogonal projection onto \( \hat{X} \):} & \\
\text{Calc. dot prod.} & \\
\text{sum: not 0}
\end{align*} \]

\[ \text{Oblique projection:} \]

\[ \begin{align*}
\text{Oblique projection onto \( \hat{X} \):} & \\
\text{Data orthogonal to nuisance.} & \\
\text{Error (\( \varepsilon \)) not explained by \( \hat{X} \).} & \\
\text{\( [I - P_X] \hat{y} \)} & \\
\text{Same as projection onto reference only in special case where} & \\
\text{\( S \perp X \).} & \\
\end{align*} \]

WHY USE SURFACES?

- raw MRI data is a 2D flat slice or a 3D volume
  \[ I(x,y) \text{ or } I(x,y,z) \]
  but...

1) the neocortex (and cerebellar cortex) are thin, folded 2D sheets
   - cortex starts as smooth "balloon" →
   - major sulci, temporal lobe form →
   - great size increase, "crinkles" form →

2) neocortex contains many topological maps along its surface
   - retinotopy
   - tonotopy
   - somatotopy
   - musculotopy
   - plus higher level maps → ~2/3 of its area

3) surface displays allow seeing (almost) all of data at once
   - only 30% exposed → everything visible

4) differences in major sulci make 3D-based alignment difficult
   - e.g. STS, monkey-like IPS vs. postcentral
   - requires mapping → extremely anisotropic def.

5) idiosyncratic sulcal crinkles
   - these introduce additional noise into alignment in 3D
   - exact position of crinkles unlikely to have functional implications (the 3D align might respect them)
1) MNI auto-Talairach → generates 4x4 matrix
   - make average brain target (blurry)
   - blur target (further), blur single brain (a lot), gradient descent on xcorr
   - repeat w/ less blurring of avg target and current brain
   - problems: variable neck cut off, only 2 points near center of brain!
     → but much better than standard! < fit to bounding box

2) Intensity Normalization (output: "T1")
   - histogram of pixel values in 10 mm thick T1R slices
   - smooth histogram
   - peak find to get initial estimate of white matter
   - discard outlier peaks across slices
   - fit splines to peaks across slices
     → interpolate scaling factor 1 to T1R
   - scale each pixel so WM peak is 110
   - refine estimate to interpolate in 3D
     → find points in 5x5x5 within 10% of WM, get near scale for them
     → to scale to T1R, interpolate scales over above
     → soap-bubble smooth Voronoi boundaries (3 iterations)
     → re-scale each voxel

3) Skull Stripping (output: "brain")
   "shrink-wrap" algorithm
   - start with ellipsoidal template
     → minimize brain penetration and curvature
     - curvature: spring force
       (from center-to-neighbor vector sum)
     → decompose into 1 and tangential (local normal from summed normal cross products)
     → apply force along surface normal that prevents surf.
       face from entering gray matter
SEGMENTATION & SURFACE RECON

- Implementing a "force" is like directly constructing the operator that minimizes something (without first defining the 'something')
- More formally, we would define cost function, then take its derivative (gradient) to minimize it

Shrinkwrap update e.g. (skull strip, original Dale & Sereno surface refinement)

\[ \mathbf{r}_{\text{center}}(t+1) = \mathbf{r}_{\text{center}}(t) + \mathbf{F}_{\text{smooth}}(t) + \mathbf{F}_{\text{MRI}}(t) \]

**rule for each vertex, \( \mathbf{r}_{\text{center}} \)**

\[ \mathbf{F}_{\text{smooth}} = \lambda_{\text{tan}} \sum_{\text{neigh}} \left( \mathbf{I} - \mathbf{n}_{\text{center}} \mathbf{n}_{\text{center}}^T \right) \cdot \left( \mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}} \right) \]

Identity 3x3

\[ \mathbf{n}_n = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} \]

\[ \lambda_{\text{normal}} \left[ \sum_{\text{neigh}} \left( \mathbf{n}_{\text{center}} \mathbf{n}_{\text{center}}^T \right) \cdot \left( \mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}} \right) \right] - \frac{1}{\#\text{vertices}} \sum_{\forall \text{neigh}} \left( \mathbf{n}_{\text{neigh}} \mathbf{n}_{\text{neigh}}^T \right) \cdot \left( \mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{v}} \right) \]

\[ \mathbf{F}_{\text{MRI}} = \lambda_{\text{MRI}} \mathbf{n}_{\text{center}} \left[ \frac{30}{40} \max \left[ 0, \tanh \left[ I \left( \mathbf{r}_{\text{center}} - d \mathbf{n}_{\text{center}} \right) - I_{\text{thresh}} \right] \right] \right] \]

\[ \text{d sample points into brain along the direction of normal} \]

\[ \text{outside (dark)} \]
\[ \text{skin (light)} \]
\[ \text{skull (dark-light-dark)} \]

Snapshot of surface and "core sample" from one vertex
4) Non-isotropic filtering (output: "win") — "floss" and "spackle"
   - preliminary hard thresholds
     - find ambiguous/boundary voxels
       \[ \text{if } 20\% \text{ or more of } 26 \text{ immediate neighbors different} \]
     - find plane of least variance
   
   - for each ambiguous voxel:
     - consider 5x5x5 volume around 1 voxel
     - find plane of least variance in this hemisphere
     - medium filter w/ hysteresis
     - if 60% of within-slab differ, reverse classification
     - "flosses" sulci without blurring

5) Find cutting planes
   - callosum, to separate hemispheres (SAH)
   - midbrain, to avoid fill into cerebellum (THA)

6) Region-growing to define connected parts (output: "filled")
   - inside-out, outside-in, inside-out — for each hemisphere
   - up/down cycles within each plane
   - plane-by-plane
   - "wormhole filter" (3x3x3 = center + 26)
     - fill (unfilled) voxel if 66% neighbors differ
     - eliminates structures within, 1-D structure
7) **Surface Tessellation** (output: rh.orig, lh.orig)

- Variable num neighbors possible!
- Quads to triangles
- Find filled voxels bordering unfilled
- Make ordered list of neighboring vertices
  \[ \rightarrow \text{so cross-products oriented properly} \]
- Long list of values associated with each numbered vertex
  
  e.g. 
  
  - Position (orig, morphed)
  - Area (orig, morphed)
  - Curvature (intrinsic, Gaussian)
  - Sulccluseness (summed L movement during unfolding)
  - Cortical thickness
  - fMRI data \[ \rightarrow \text{EEG/MEG dipole strength} \]

- Separate fMRI data set must be aligned, sampled

  \[ \text{fMRI voxels larger} \]

  \[ \text{Sample at each surface vertex} \]

  \[ \text{nearest-neighbor "soap bubble" smoothing} \]

  \[ \text{to interpolate data onto hi-res mesh} \]

- Some quantities only well-defined on surface

  \[ \rightarrow \text{gradient of magnitude of cortical map measure (e.g., eccentricity)} \]
SEGMENTATION & SURFACE RECON
Smooth, inflate, final surfaces

- smoothing/inflation/WM,pial done as derivative of energy functional

\[ J = J_\text{tangent} + \lambda_\text{normal} J_\text{normal} + \lambda_\text{image} J_\text{image} \]

\[ J_\text{normal} = \frac{1}{2 \, \#\text{vert}} \sum_\text{centers} \sum_\text{neighbors} \left[ n_\text{center} \cdot (r_\text{neigh} - r_\text{center}) \right]^2 \]

\[ J_\text{tangent} = \frac{1}{2 \, \#\text{vert}} \sum_\text{centers} \sum_\text{neighbors} \left[ t^x_\text{center} \cdot (r_\text{neigh} - r_\text{center}) \right]^2 + \left[ t^y_\text{center} \cdot (r_\text{neigh} - r_\text{center}) \right]^2 \]

\[ J_\text{image} = \frac{1}{2 \, \#\text{vert}} \sum_\text{centers} \left[ I_\text{targ} - I(r_\text{center}) \right]^2 \]

- take directional derivative of energy functional (to find steepest uphill)
- move each vertex in the opposite (negative) direction w/ self-interest test

\[ \frac{\partial J}{\partial r_\text{center}} = \lambda_\text{image} \left[ I_\text{targ} - I(r_\text{center}) \right] \nabla I(r_\text{center}) \]

N.B.: eq. 9 in Deo, Fischl & Sereno different - and incorrect!!

HOW TO derive:
- constants:
  \[ \frac{\partial J}{\partial r_\text{center}} = \lambda_\text{image} \left[ I_\text{targ} - I(r_\text{center}) \right] \nabla I(r_\text{center}) \]

- var:
  \[ \frac{\partial J}{\partial r_\text{center}} = \lambda_\text{image} \left[ I_\text{targ} - I(r_\text{center}) \right] \nabla I(r_\text{center}) \]

- \[ \nabla I(r_\text{center}) \]

- use gradient on image (first blur w/ Guassian)

- scaled by unit normal vector

- \[ \sum_{\text{neighbors}} \left[ t^x_\text{center} \cdot (r_\text{neigh} - r_\text{center}) \right] (t^x_\text{center}) \]

- \[ \sum_{\text{neighbors}} \left[ t^y_\text{center} \cdot (r_\text{neigh} - r_\text{center}) \right] (t^y_\text{center}) \]

- x-component of tangential

- y-component of tangential
**SULCUS-BASED CROSS-SUB. ALIGN**

- Use summed perpendicular vertex move during inflation as vtx measure of "sulcus-ness".
- Add term to error function, J: "sulcus-ness" error

\[ J_{sulc} = \frac{1}{\#vtx} \sum_{vtx} \left[ S_{\text{subj}} - S_{\text{target}} (\text{Reader}) \right]^2 \]

- Find neg of steepest uphill direction of change in sulcus-ness of target

\[ \frac{\partial J_{sulc}}{\partial \phi_{\text{target}}} = \lambda_{sulc} \left[ S_{\text{subj}} - S_{\text{target}} (\text{Reader}) \right] (-\nabla S_{\text{target}} (\text{Reader})) \]

- Icosahedral (5-fold symmetry)
- 7th triangular sub-tessellation

- Smooth wm
- Inflated
- Sphere
- Registered sphere

Sub₁ ➔ inflate ➔ sphere ➔ morph ➔ registered sphere

Sub₂ ➔...

Subₙ ➔...

- Each sub's native surf has diff # vertices
- Interpolate values (coords, surface measures, stats) to each icosahedral vertex from neighboring vertices of native mesh (dashed lines)

- Average surface made from folded/inflated avg coords
  - Folded: loses area from sulcal wrinkles
  - Inflated: retains orig area, correct sulc/gyru ratio ("inflated-avg")

- Can use sampled-to-icos individual subj coords to draw icosahedral surface in shape of indiv. brain

\[ \text{N.B.: morph will have changed local vertex density compared to more uniform native mesh (use native for subj.)} \]
SOURCE OF EEG/MEG

PSPs
- anisotropic cables
  + aligned spatially
  + coherent/biased stim
- one end
- no distant signal from axon spike
- too close

N.B.: spikes only detected by 15μm microelectrode in gray matter!

Head
1) - local dipole
2) - EEG through skull, skin
3) - swelling because skull 1/80 conductivity of brain

MEG
- radial dipoles lost
- tangential dipole generates Gabor-like scalp distrib. of B field

the effect on current flow is like a higher order (larger) "cable"
**INTRACORTICAL CIRCUITS & ORIGIN OF EEG**

**Cell types**
- Excitatory (spiny)
  - Pyramidal
  - Spiny stellate (e.g., V1 layer 4C)
- Inhibitory (smooth)
  - Basket
  - Double bouquet
  - Chandelier
  - Clutch

**Circuits**
- Huge complexity
- First principal components: Input $\rightarrow$ Layer 4 $\rightarrow$ Layer 2/3 $\rightarrow$ Feedforward $\rightarrow$ Layer 5/6 $\rightarrow$ Output feedback
- Microelectrode recording (e.g., 10 um tip):
  - High pass $\rightarrow$ Spikes
  - Low pass $\rightarrow$ Local field potentials
- Spikes only recordable in gray matter
- White matter spikes only recordable with pipette with very fine tip b/c inward & outward currents so spatially close in axon/spike (> 1 um)

**Intra/Inter cortical connections cartoon**

- "Lower" (e.g., V1)
  - Layer 2/3 feedforward
  - Layer 4 input
  - Layer 5 motor output
  - Layer 6 feedback
  - Ascending input (e.g., dLGN) output to sup. collic.
- "Higher" (e.g., V2)
  - Layer 2/3 feedback
  - Layer 4
  - Layer 5
  - Layer 6
  - Motor striatum

Spike has opposite polarity here!
**GRADIENT, DIVERGENCE, CURL**

**Gradient (\(\nabla\))** (generalized derivative)

\[
\nabla \mathbf{F} = \frac{\partial F_x}{\partial x} \hat{i} + \frac{\partial F_y}{\partial y} \hat{j} + \frac{\partial F_z}{\partial z} \hat{k}
\]

- Turns scalar function defined at each \(x, y, z\) point, \(r\)
- Scalar function becomes vector field
- Charge of \(F\) in \(x\)-direction at point \(r\)
- Unit vector in \(x\)-direction

**Divergence (\(\nabla \cdot \mathbf{F}\))** (derivative, dot product)

\[
\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}
\]

- Turns vector field into scalar field
- Charge of \(x\)-component of \(\mathbf{F}\) in \(x\)-direction at point \(r\)

**Curl (\(\nabla \times \mathbf{F}\))** (derivative, cross product)

\[
\nabla \times \mathbf{F} = \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{i} + \left( \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \hat{j} + \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{k}
\]

- Turns vector field into another vector field
- Charge of \(z\)-component of \(\mathbf{F}\) in \(y\)-direction at point \(r\)

**Vector identities**

\[
\nabla \times \nabla \mathbf{F} = \mathbf{0}
\]

Curl of the gradient of any scalar field is zero

\[
\nabla \cdot (\nabla \times \mathbf{F}) = 0
\]

Divergence of the curl of any vector field is zero

\[
\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}
\]
Potential \( (\Phi) \), Electric Field \( (\nabla \Phi) \) \& CSD. \( (\nabla \cdot (-\nabla \Phi) = \nabla^2 \Phi) \)

**Low-Frequency Field Approximation**

- Electric fields uncoupled from magnetic (vs. electromagnetic radiation)
- Pre-Maxwellian approx. (EEG freqs \( \ll 1 \text{ MHz} \))
- Calculate electric fields as if magnetic fields don't exist
- Calculate magnetic fields strictly from distribution of currents
- Ignore capacitive effects, too

**Scalar Potential**, \( \Phi \) (what we measure with electrodes)

\[
\mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t} = - \left[ \frac{\partial \Phi}{\partial x} \mathbf{i} + \frac{\partial \Phi}{\partial y} \mathbf{j} + \frac{\partial \Phi}{\partial z} \mathbf{k} \right] + \mathbf{0}
\]

- Gradient of scalar field

\( \mathbf{E} \) defined as force (vector)

acting on unit charge at a given point in space (as result of arbitrary distribution of other charges)

CSD is Laplacian of \( \Phi \) (\( = \text{div} \, \mathbf{E} \))

- Divergence

\[
\nabla \cdot (-\nabla \Phi) = \text{scalar field} = - \left[ \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} \right] \equiv -\nabla^2 \Phi
\]

Meaning of \( \nabla \cdot \mathbf{E} \):

- Find radiating and contracting regions
- \( \nabla \cdot \mathbf{E} \) turns vector field into scalar field

3D CSD gold standard (rat BAER paper)

- \( \Phi \) data \( \rightarrow \) \( \nabla \Phi \) \( \rightarrow \) \( \nabla \cdot (-\nabla \Phi) \)

Scalar field source/sink movie as function of \( \tau \)

**Note:** Vectors in y dir not vector along circle
# 1D and 2D Current Source Density Expts.

## 1D CSD
- Raw, event-related
- Signal relative to ground, $\frac{\Delta}{\Delta t}$ (eg. skull)

![Diagram of 1D CSD](image)

**Rationale:** CSD changes much more slowly parallel to cortex than perpendicular to cortical sheet.

- Assume approx constant ($\approx 0$) parallel to ctx
- Recording sites (eg., by slowly withdrawing electrode tip)

**LFP** (local field potential)

- Both types of data can be recorded from same electrode

## 2D CSD
- 2D array of electrodes on pial surface or on scalp

![Diagram of 2D CSD](image)

**Rationale:** All electrodes record along same surface, so assume depth profiles are constant

- $\nabla^2$ means find spatial (i.e., 1D depth) curvature of potential
- Discrete approx: center $= \frac{\text{above} + \text{below}}{2}$
- N.B. in example above, even though all 3 potentials are positive, smaller value of center point implies sink!

**For scalp recordings:**
- Sources and sinks are at the scalp (not a depth loc method unless done in 3D)
- Can tell curvature from sign of potential:
  - Concave $\rightarrow$ Sink
  - Convex $\rightarrow$ Source

**LFP** at one timepoint

- For scalp recordings, sources and sinks are at the scalp
- Can tell curvature from sign of potential:
  - Concave $\rightarrow$ Sink
  - Convex $\rightarrow$ Source

- One possible underlying source is one tangential

- (Don't confuse dim of calc. w/ always a second deriv.)
INTRACORTICAL C.S.D.

e.g. click evoked rat A-I
(Sukov & Barth, 1998)

P1

N1

P2

- phase-locked CSD $\rho$

gamma shifts with each cycle
MAXWELL EQUATIONS

Electrostatics, Magnetostatics
low freq limit

\[ \nabla \cdot (\sigma \nabla \phi) = \nabla \cdot \mathbf{J} \]

\[ \nabla \times \mathbf{B} = \mu_0 (\mathbf{J} - \sigma \nabla \phi) \]

\[ \nabla \cdot \mathbf{B} = 0 \]

- propagation of potentials, magnetic fields instantaneous (no capacitance)
- simultaneous Eqs to solve: J are sources, \( \phi, \mathbf{B} \) are data
- linear

Potential (\( \phi \)) and magnetic fields (\( \mathbf{B} \)) produced by a weighted sum of two current source distributions are equal to weighted sum of fields produced by each current source distribution by itself.
WHY WE CAN IGNORE MAGNETIC INDUCTION

\[ \vec{E} = -\nabla \Phi - \frac{\partial \vec{A}}{\partial t} \]

(from Nunez, 1981)

Field component due to charge distribution

Field component due to coupling between electric & magnetic fields

\[ \vec{B} = \nabla \times \vec{A} \]

"vector potential"

\[ \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \]

Currents (in given medium)

Time varying electric fields (in given medium)

\[ \vec{H} = \mu \vec{B} \]

Permeability \( \mu \)

Conductivity \( \sigma \)

\[ \nabla \times \vec{E} = -\nabla \frac{\partial B}{\partial t} \]

Take \( \nabla \times \) of both sides

Use \( \vec{B} = \mu \vec{H} \)

Substitute this into \( \nabla \times \vec{H} \)

\[ \nabla \times \nabla \times \vec{E} = -\mu \frac{\partial}{\partial t} \left( \vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \]

If linear in conductivity and dielectric, too, and fields periodic w/\( \chi \)

\[ \nabla \times \nabla \times \vec{E} = -2 \pi \frac{\mu}{\varepsilon} (\sigma + 2 \pi \mu \varepsilon) \vec{E} \]

To neglect: \( \frac{2 \pi \mu}{\varepsilon} (\sigma + 2 \pi \mu \varepsilon) |\vec{E}| < 1 \)

1) \( |\nabla \times \nabla \times \vec{E}| \propto |\vec{E}|/L^2 \) where \( L \) is dist over which \( \vec{E} \) varies significantly

2) \( \mu \) of tissue similar to empty space

3) Assume conservative (large) \( \sigma \), dielectric unit, and EEG freq

\( \approx \frac{1}{10} \) number is about \( 10^{-6} \Rightarrow \) small
MONPOLE, DIPOLE FORWARD SOL'N

\[ \Phi_1 = \frac{S}{4\pi \sigma r} \]

potential recorded for source monopole

\[ \Phi_2 = \frac{S}{4\pi \sigma} \left( \frac{1}{r_1^2} - \frac{1}{r_2^2} \right) \]

dipole source to measuring distance

dipole sink to measuring distance

approximations for "far enough away" measurements (subtracting two 1/r gives inverse square)

\[ \vec{B}_2 \approx \left( \frac{\mu_0}{4\pi} \right) \frac{S \vec{d} \times \vec{r}}{r^3}, \quad r \gg d \]

now: also explicitly include dipole orientation and measuring point in equations

**Since \( \vec{r} \) also in numerator, this now inverse square

N.B.: both assume inside infinite isotropic conductor

\[ \vec{\Phi}_e(t) = e_i s_i(t) \]

\[ \vec{b}_i(x) = \vec{m}_i s_i(t) \]

\[ \vec{b}_i(x) = m_i s_i(t) \rightarrow \text{Squid's measure} \]

\[ \vec{X}(t) = \sum_j \vec{G}_{ij} \vec{s}_j(t) \]

Linear superposition with fixed electrodes and sensors
Forward Solution

- well-posed (one answer)
- linear: \( b(A) + b(B) = b(A+B) \)
- approximations due to unknown electrical properties of head

- 3-shell spherical analytic
  - skull conductivity of brain
  - "smearing" (cf. cable theory)
  - remember, we only need to be able to calc. weight for each dipole/electrode pair independently

- 3-shell boundary element
  - for magnetic, only need one shell b/c currents thru skin/skull too small to make sig. \( B \)
  - calc. for one head
  - matrix of correction factors

- finite element
  - most general
    - computational intensive w/ small grid
    - many unknown parameters to estimate
**Forward Sol'n**

\[ V_i = \sum_j E_{i:j} + \eta_i \]

(matrix form)

\[
\begin{bmatrix}
V \\
S
\end{bmatrix}
= \begin{bmatrix}
E \\
S
\end{bmatrix}
+ \begin{bmatrix}
\eta
\end{bmatrix}
\]

Lower case bold -> vector
Upper case bold -> matrix

\[
\hat{x} = \hat{A}\hat{s} + \hat{n}
\]
**WHY LOCALIZE?**

- Most of ERP literature is based (indeed) on **temporal components**

**But:**

1. Underlying local cortical generators (from microelectrode LFP, CSD)
   - Extended in time (400 msec), visible from every scalp electrode
   - Multiphasic in every cortical area
   - Temporally non-static depending on stimulus
     - E.g., simple contrast, brightness cliffs can modulate retinal delay by 50 msec!

2. Thus, any "component" consists of sum of activity from multiple cortical areas at different hierarchical levels

3. Stimulus manipulations will change temporal overlap
   - May cause "component" peak to disappear without changing cortical areas being activated

4. Verified by **intra-cortical LFP/CSD** (Schroeder et al., 1998)

   - Stim 50 100 150 200 250 300
   - LFPs from approx. layer 4 in cortex ("input layer")
   - Psychologists now identify a few temporal "component" peaks...
   - But each peak comes from every one of these cortical areas!!

- By contrast, the spatial signature of a signal from one cortical area is static — a better area-based "component"

- Temporal "components" should be retired!!
- Their original reason for being no longer relevant!!
- Easier to record now temporal points (EEG started w/ few electrodes, many time points)
- Easier to "paste" high level psychological functions onto a few waveform deflections

*Inverse φ*

**macaque monkey**

- *Intracortical data*

  - These areas span the visual system from bottom to top, accounting for roughly 50% of the entire macaque monkey cortex
Derivation of Ill-posed Inverse
(from Dale & Sereno, 1993)

\[ x = As + n \]
\[ A = \text{forward sub matrix } (E + B) \]
\[ s = \text{source vector} \]
\[ n = \text{sensor noise vector} \]

\[ \text{Err}_w = \langle \| Wx - s \|^2 \rangle \]
\[ = \langle \| W(As + n) - s \|^2 \rangle \]
\[ = \langle \| WA - I \| s + Wn \|^2 \rangle \]
\[ = \langle \| Ms + Wn \|^2 \rangle \]
\[ = \langle \| Ms \|^2 \rangle + \langle \| Wn \|^2 \rangle \]
\[ = \text{diag is noise variance (already squared)} \]
\[ = \text{tr} (MRMT) + \text{tr} (WCWT) \]
\[ = \text{trace is sum of diag elements} \]

\[ \text{[re-expand]} = \text{tr} (WARATW^T - RAT^T W^T - WAR + R) + \text{tr} (WCWT) \]

Explicitly minimize by taking derivative w.r.t. \( W \), set to zero, solve for \( W \)

\[ 0 = 2WARAT - 2RAT^T + 2WC \]

\[ WARAT + WC = RAT^T \]

\[ W(A + C) = RAT^T \]

\[ W = RAT^T (ARA^T + C)^{-1} \]
Inverse Sn (2)

$W = R A^T (A R A^T + C)^{-1}$

$ightarrow$ "minimum norm" solution
(find $\delta$ w/ smallest norm $= \|\delta\|_1$)

- the minimum norm solution appropriately downplays deeper (= weaker scalp signal) sources since these are more likely to fall into the noise floor

- "problems" of minimum norm:
  - deeper sources get displaced to the surface
  - small superficial sources "win" because of approx inverse square form of true solution
    $\rightarrow$ smaller norm of distributed superficial soln,
    - can't fix by increasing priors of deep sources!!
    - that will give deep sources given noise as input!!
  - achieves smaller norm w/ equivalent fit/error
    - also tiny dipoles in correct place
      (see noise normalization)
**Inverse Solutions to Ill-Posed Compared**

\[
\mathbf{S} = \mathbf{W} \mathbf{x}
\]

- How to use the inverse solution, \( \mathbf{W} \)
- Same \( \mathbf{W} \) for all time points

"Minimum norm" solution
i.e., norm \( \| \mathbf{s} \| \) of solution
is smallest of infinitely many alternate solutions

**Linear inverse operator**

\[
\mathbf{W} = \mathbf{R} \mathbf{A}^T (\mathbf{A} \mathbf{R} \mathbf{A}^T + \mathbf{C})^{-1}
\]

- From error minimization derivation
- Easier inverse

\[
\begin{bmatrix} \mathbf{W} \end{bmatrix} = \begin{bmatrix} \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{A}^T \end{bmatrix} \begin{bmatrix} \mathbf{A} \end{bmatrix} + \begin{bmatrix} \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{A}^T \end{bmatrix} \begin{bmatrix} \mathbf{C} \end{bmatrix}
\]

\[
\begin{bmatrix} \mathbf{A} \mathbf{R} \mathbf{A}^T \end{bmatrix} \Rightarrow \text{Square in } \# \text{ of sensors (small)}
\]

**Alternate, algebraically equivalent Bayesian derivation (w/ bigger inverses!):**

\[
\begin{bmatrix} \mathbf{W} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^T \end{bmatrix} \begin{bmatrix} \mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{A} \end{bmatrix} + \begin{bmatrix} \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{A}^T \end{bmatrix} \begin{bmatrix} \mathbf{C} \end{bmatrix}
\]

\[
\begin{bmatrix} \mathbf{A}^T \mathbf{C}^{-1} \mathbf{A} \end{bmatrix} \Rightarrow \text{both square in } \# \text{ of sources (large)}
\]

\[
\begin{bmatrix} \mathbf{A}^T \mathbf{C}^{-1} \mathbf{A} \end{bmatrix} \Rightarrow \text{hard inverses}
\]
**PROBLEMS W/ SURFACE NORMAL**

- Since nearby points on surface often have different orientation, surface normal constraint can help (since fwd soln A, B very different)

- but, since point spread function typically extends across sulci, artificial sign reversals occur

  ![Diagram]

  - Positive current at A indistinguishable from negative current at B, or pos at C

  → after unfolding →

- **Solutions**

  1) Ignore sign → saves useful orientation info!

  2) Solve onto 3 orthogonal dipoles at each critical point instead of a single oriented dipole

    → more appropriate when averaging across subjects, since detailed variations vary a lot

    → also fills in bottom of sulci (else unsigned stripes)
**FMRI Constrained Inverse**

- insert FMRI values for $R_{ii}$'s
- but still allow other sites to have non-zero $R_{ii}$'s
- pathologies occur if solution restricted completely to FMRI points by setting non-FMRI $R_{ii}$'s to zero

- this allows extracting time course from sources visible in EEG/MEG and FMRI

- N.B.: sources that are only visible in EEG/MEG will be dispersed to small distributed values at a large number of vertices visible in both EEG/MEG and FMRI

visible only in EEG/MEG and not FMRI → distributed at small amplitude across many vertices
Noise Sensitivity Normalization

\[ W = R A^T (A R A^T + C)^{-1} \]

- Multiply inverse operator by noise sensitivity matrix, \( D \) (diagonal)

\[ D_{ii} = \frac{1}{\text{diag} \cdot \sqrt{W C W^T}} \]

\[ W_{\text{norm}} = D W \]

\[ s_i = (W_{\text{norm}} x)_i = (D W x)_i = \frac{W_i}{\sqrt{(W C W^T)_i}} \]

- Normalize each row of inverse operator by noise sensitivity at that location

If assume Gaussian white noise, noise covariance, \( C \), is multiple of \( I \), so

\[ W_{\text{norm}} = \frac{W_{\text{orig}}}{\| W_{\text{orig}} \|^2} \]

- Scale each row of \( W \) by single value - the norm of that row

\[ \text{row of } W \text{ is } \frac{W_{\text{orig}}}{\| W_{\text{orig}} \|^2} \]

Inverse solution coefficients \( W \) \Rightarrow scale (divide) by norm of this row

That is, if inverse soln for deep source is reduced by interaction of inverse square nature of Fred and min norm, dividing by norm of row of inverse (same) will increase / rescue deep source
NOISE SENSITIVITY NORMALIZATION

Shallow source (unit strength)
- Small input (inv small)
- Large spread (fwd big)

Deep source (unit strength)
- Small input (inv reduced)
- Large spread (fwd small)

\[ s_i = \frac{\mathbf{w}_i \cdot \mathbf{x}}{\| \mathbf{w}_i \|_{\text{orig}}} \]

- Effect on inverse solution:
  - More like significance vs. actual power
- Effect on point-spread function is to equalize shallow & deep
  - Shallow spread out more than min norm
  - Deep shrunk to same as shallow

Point spread functions

[Diagrams of point spread functions with noise normalized]

Therefore, fixed estimate increased relative to shallow
**Conclusions**

- More EEG or more MEG better

- EEG better than MEG (cf. radial) (EEG far more currently less accurate)

- Biggest gain from adding small # EEG (or MEG) (e.g. 20) to many MEG (or EEG) (e.g. 150)

- Easier to add many MEG, so: optimal < 30 EEG < 300 MEG

- EEG/MEG forward-solution-scaling-factor error causes more cross-talk
**MUSIC**

(1) (from Dale & Sereno, 1993) (cf. Mosher & Leahy)

- Using **Sensor Covariance**

\[
D = \langle xx^T \rangle = \sigma^2 I + \sum \sum \sigma_i \sigma_j C_{i,j} A_i A_i^T
\]

\[
\sim \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^T / n
\]

\[
D = U \Lambda U^T = \begin{bmatrix} U_1 & U_2 & \cdots & U_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} U_1 & U_2 & \cdots & U_n \end{bmatrix}^T
\]

**Columns of U matrix are orthogonal basis vectors of "spatial pattern" space (One per is spatial pattern across sensors)**

= Find, order most significant **spatial patterns** in **sensors**, over time

Project **forward solutions** onto these **spatial patterns**

("Project" = dot product = similarity) for each point in brain (fwd defines spatial sensor pattern for unit source)

\[
\mathbf{\Sigma} = A_i^T U \Lambda U^T A_i \Rightarrow \text{big single number if forward solution looks like } U_i's
\]
(2) how to weight the minimum norm inverse

\[ R_{ii} \approx \frac{A_i^T A_i}{A_i^T U \Lambda U^T A_i} \]

- like parallel resistance
- \[ R_{\text{parallel}} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \ldots} \]

- any low resistance (\(R_i\)) decreases overall resistance (small \(R_{ii}\))

- i.e., if toward soln, has appearance like any low eigenvalue spatial pattern, it gets devalued

\[ W = R A^T (A R A^T + C)^{-1} \]
- How it works: Take advantage of spatial information that changes over time

N.B.: Problem if assumption about lack of perfect correlation is violated

E.g., if two widely separated sources (i.e., different weld spots) are highly correlated, MUSIC will eliminate both since no single weld spot will look like that "Z-separatd dipole" pattern (e.g., LRIA-1)

*"Dual MUSIC" back to fix...

- How it fixes min norm problem

Equiv soln's: min norm picks superficial b/c smaller norm

If sources really were superficial, these spatial patterns would appear in D

If not, superficial sources would be eliminated, leaving deep