# Foundations of Neuroimaging

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(125 pages)

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MAGNET HARDWARE

1 - $\mathbf{B}_0$ field from superconducting magnet
2 - gradient coils
3 - body RF transmit/receive
4 - RF receive-only
5 - shim coils (in gradients)

$\mathbf{B}_0 \rightarrow z$ (longitudinal)
$\mathbf{B}_1 \rightarrow x, y$ (transverse)

(1) $\mathbf{B}_0$ field

max gradient: [80 mT/m, 200 T/m/sec]

(2) Body gradient coils

(3) RF transmit body coil

(4) RF receive-only head coils

3.5 million watt amplifiers to add ramps to $\mathbf{B}_0$ field

RF transmitter (30 kW)

RF receiver

Circularly polarized $\mathbf{B}_1$ field rotating \rotatebox{90}{$\mathbf{B}_0$} at Larmor freq. (B1 is several orders of magnitude smaller than $\mathbf{B}_0$)

$\mathbf{1T} = 10,000$ Gauss
Earth: 0.25 - 0.65 G
25 - 65 mT

$\frac{\mathbf{y}}{2\pi} = 42$ MHz/T
**Spin & Precession**

- Nuclei act like a spinning sphere of matter with an embedded equatorial charge.

  - Moving charge creates magnetic field.

  - Classical picture: current loop from spinning charge (right-hand rule).

  - N.B.: Classically, this would cause EM radiation, spin-down.

- Stern-Gerlach experiment:
  - Pass silver atoms through strong magnetic field → split into just 2 beams.

**Microscopic picture**

- No strong magnetic field: $B\phi = 0$.

- Strong magnetic field, $B\phi =$.

**Macroscopic picture**

- Bulk magnetization: $M^0 = M_z^0$.

**Precession**

- Distinguish precession (slow) from spin (fast).

- Treat classically, like spinning top.

\[ \frac{2\pi}{\omega} \frac{I}{I+\frac{1}{2}} = \gamma B_0 \]

- Larmor frequency: $\omega = \gamma B_0$.

- Gyromagnetic ratio: $\gamma = \text{constant}$.  

**Bulk equilibrium magnetization**

- Parallel to $B_0$.

\[ M_z^0 = \frac{\gamma^2 h^2 B_0 N_s}{4KT_s} \]

- Where $I = \pm \frac{1}{2}$.

**Two non-constants**

- $\gamma = \text{gyromagnetic ratio}$

- $h = \text{Planck's constant}$

- $B_0 \rightarrow M_z^0$ proportional to $B_\phi$ strength.

- $N_s \rightarrow M_z^0$ proportional to number spins.

- $K = \text{Boltzmann constant}$.  

- $T_s = \text{abs. temperature sample}$.
The Bloch equation describes the time-dependent behavior of the density matrix $\rho(t)$ of a system of $N$ identical, weakly coupled two-level quantum systems. The density matrix is a matrix of dimension $2N \times 2N$, where each element $\rho_{ij}$ represents the probability of finding the system in the $i$th state while it is in the $j$th state. The equation is:

$$i\hbar \frac{d\rho}{dt} = -\frac{1}{2} \{ H, \rho \} + \sum_i \mathbf{M}_i(t) \cdot \mathbf{S}_i$$

where $H$ is the Hamiltonian of the system, $\mathbf{M}_i(t)$ are the magnetic moment vectors of the individual systems, $\mathbf{S}_i$ are the angular momentum operators, and $\hbar$ is the reduced Planck constant.

The Bloch equation is a set of coupled differential equations that govern the time evolution of the density matrix. In the presence of an applied magnetic field $\mathbf{B}$, the density matrix evolves according to the equation above. The solution to the Bloch equations allows us to predict the time evolution of the state of the system under various conditions, such as the application of external fields or the presence of interactions between the systems.
**VECTOR ADD, MULTIPLY**

- Adding vectors is easy
  \[ \mathbf{c} = \mathbf{a} + \mathbf{b} = [a_x + b_x, a_y + b_y] \]
  - Just add components (Vector)
  - Applies to complex numbers
  - Generalizes to any \( D \)

  \[ \| \mathbf{c} \| = \sqrt{(a_x + b_x)^2 + (a_y + b_y)^2} \]

- Multiple ways to multiply vectors: here are 3

  **Dot Product**
  (= Inner Product)
  (= "scaled projection onto")
  \[ \mathbf{c} = \mathbf{a} \cdot \mathbf{b} = [b_x, b_y, b_z] \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = a_x b_x + a_y b_y + a_z b_z \]
  - Scalar
  - Generalizes to any \( D \)

  \[ \mathbf{c} = \mathbf{a} \cdot \mathbf{b} = \| \mathbf{a} \| \| \mathbf{b} \| \cos \theta \]

  \[ \mathbf{c} = \mathbf{a} \cdot \mathbf{b} \leftrightarrow \text{zero if } \mathbf{a}, \mathbf{b} \text{ orthogonal} \]

  **Cross Product**
  (= Outer Product)
  (Can be generalized: see "geometric algebra")
  \[ \mathbf{c} = \mathbf{a} \times \mathbf{b} = \begin{bmatrix} 0 & -b_z & b_y \\ b_z & 0 & -b_x \\ -b_y & b_x & 0 \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = [a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_x b_y - a_y b_x] \]
  - Vector
  - Geometric algebra: bivector plane area
  - Skew-symmetric: \( \mathbf{a}^T = -\mathbf{a} \)
  - Unique orthogonal specific to 3D
  - \( \| \mathbf{c} \| = \| \mathbf{a} \| \| \mathbf{b} \| \sin \theta \)
  - Max if orthogonal

  **Complex Multiply**
  (See also quaternions, geometric algebra generalization)
  \[ \mathbf{c} = \mathbf{a} \cdot \mathbf{b} = \begin{bmatrix} b_x & -b_y \\ b_y & b_x \end{bmatrix} \begin{bmatrix} a_x \\ a_y \end{bmatrix} = [a_x b_x - a_y b_y, a_x b_y + a_y b_x] \]
  - cf.
  - Not affected by angle between
  - Angles add
  - Magnitudes multiply
  - Specific to 2D
  - \( \| \mathbf{c} \| = \| \mathbf{a} \| \| \mathbf{b} \| \)
  - Like real nums
**Effects of $\vec{M}$, $\vec{B}$, and $\Theta$ on Precession Freq.**

**Bloch 1st term**
\[
\frac{d\vec{M}}{dt} = \vec{M} \times \vec{B}
\]

**Cross prod. properties review**
\[
\left| \frac{d\vec{M}}{dt} \right| = \left| \vec{M} \right| \cdot \left| \vec{B} \right| \cdot \sin \Theta
\]

**Starting condition**
⇒ now see effects of changing $\left| \vec{M} \right|$, $\left| \vec{B} \right|$, $\Theta$

**Change $\vec{M}$ length**
⇒ $\frac{d\vec{M}}{dt}$ proportionally larger, so cancels
⇒ same precession freq. as starting cond.

**Change $\Theta$ between $\vec{M}$ and $\vec{B}$**
⇒ $\frac{d\vec{M}}{dt}$ goes up (then down) as $\sin \Theta$
⇒ but circumference also goes up as $\sin \Theta$, cancelling again
⇒ same precession freq.

**Change $\vec{B}$ length**
⇒ $\frac{d\vec{M}}{dt}$ goes up, proportional to $\vec{B}$
⇒ but circumference is same at starting cond.
⇒ increased precession freq. ($\omega = r \vec{B}$)
**Simple Matrix Operations**

**Basic Idea**
- A matrix \( \begin{bmatrix} \text{rotates} & \text{scales} \end{bmatrix} \) a vector

\[ \vec{b} = \mathbf{M} \vec{a} \]

**3D Example**
- \[ \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \]

Add translate (after rotate/scale)
- Commonly used "hack" for aligning vols
- A 4D matrix \( \begin{bmatrix} \text{rotates/scales} & \text{then} & \text{translates} \end{bmatrix} \) (4th D = 1)

\[ \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \begin{bmatrix} 3 \times 3 \text{ rot/scale} \\ 0 \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \]

- N.B.: Have to keep track of order!!
  - Rotate/scale then translate ≠ translate then rotate/scale
  - Change rot component: Untranslate, rot, retranslate

3 special cases (3D): Rotate around each major axis without changing length (scale = 1.0)

- Rotate around X-axis:
  \[ \mathbf{R}_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & -\sin \alpha & \cos \alpha \end{bmatrix} \]
  - E.g., 90° flip

- Rotate around Y-axis:
  \[ \mathbf{R}_y(\alpha) = \begin{bmatrix} \cos \alpha & 0 & -\sin \alpha \\ 0 & 1 & 0 \\ \sin \alpha & 0 & \cos \alpha \end{bmatrix} \]
  - E.g., 180° flip
to avoid add 180° phase after 90° flip on x'

- Rotate around Z-axis:
  \[ \mathbf{R}_z(\alpha) = \begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \]
  - E.g., precession with BP along z'

**General Case**
- Rotate around general z'-axis:
  \[ \mathbf{R}_{z'}(\alpha) = \mathbf{R}_z(\Theta) \mathbf{R}_y(\Phi) \mathbf{R}_z(\alpha) \mathbf{R}_y(\Phi) \mathbf{R}_z(\Theta) \rightarrow (\text{quaternions are more efficient}) \]
Solutions to Simple Differential Eq.

**Solution:**

\[ \frac{dM(x,y)(t)}{dt} = \frac{M(x,y)(t)}{T_2} \]

\[ M(x,y)(t) = M(x,y)(0) \cdot e^{-t/T_2} \]

**Goal:**
1. Find eq. whose derivative satisfies diff. eq.
2. Also find soln. (one of many) that passes thru init condition

Since our diff. eq. is:
- derivative of funct. = const. same funct.

Try exponential, since derivative \((e^x)' = e^x\)

**Diff. Eq.**

\[ M(x,y)(t) = \frac{-1}{T_2} \cdot M(x,y)(t) \]

One soln.

\[ M(x,y)(t) = e^{-t/T_2} = e^{\frac{-t}{T_2}} \]

Chain rule deriv. exponent

**Take deriv. to check**

\[ M'(x,y)(t) = \frac{-1}{T_2} \cdot e^{-t/T_2} \]

Another soln.

\[ M(x,y)(t) = \text{const} \cdot e^{\frac{-t}{T_2}} \]

So: any const is OK!

Take deriv. to check

\[ M'(x,y)(t) = \frac{-1}{T_2} \cdot \text{const} \cdot e^{\frac{-t}{T_2}} \]

Const is "initial condition"

Information added to soln. (not from diff. eq.)

\[ M(x,y)(t) = M(x,y)(0) \cdot e^{-t/T_2} \]

Const = \(M(x,y)(0)\)

Magnetization immedi. after RF (B1) ends
Verify Solution to T1 Regrowth

- Slightly more complex T1 sol'n compared to T2 sol'n.

T2 sol'n verify (from prev).

T1 solution verify.

\[
\frac{dM}{dt} = \frac{M_{xy}}{T_2}
\]

Original diff eq.

\[
M'(t) = \frac{-1}{T_2} \cdot M(t)
\]

Make unknown funct. M(t) more visible.

\[
M(t) = M_{xy}(0)e^{-t/T_2}
\]

Proposed solution.

\[
M'(t) = \frac{-1}{T_2} M_{xy}(0)e^{-t/T_2}
\]

Test by take deriv.

\[
M'(t) = \frac{-1}{T_2} (M_{xy}(0) e^{-t/T_2})
\]

- Derivative in original T1 eq. says \(M(t)\) minus \(M_2^0\).

\[
M'(t) = \frac{-1}{T_1} \left( M(t) - M_2^0 \right)
\]

Solution \([M_2^0 - M_2^0 e^{-t/T_2} + M_2(0) e^{-t/T_2}]\).

- Which equals our re-calculated derivative:

\[
M'(t) = \frac{-1}{T_1} \left( -M_2^0 e^{-t/T_2} + M_2(0) e^{-t/T_2} \right)
\]
**Bloch Eq. - Matrix Version**

\[
\frac{d\vec{M}}{dt} = \vec{M} \times \gamma \vec{B}_\Phi
\]

Solution:

\[
\vec{M}(t) = \begin{bmatrix}
M_x(t) \\
M_y(t) \\
M_z(t)
\end{bmatrix} = \begin{bmatrix}
\cos \omega t & \sin \omega t & 0 \\
-\sin \omega t & \cos \omega t & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
M_x(0) \\
M_y(0) \\
M_z(0)
\end{bmatrix} = R_z(\omega t) \vec{M}(0)
\]

Include Relaxation

\[
\frac{d\vec{M}}{dt} = \vec{M} \times \gamma \vec{B}_\Phi - \frac{M_x I + M_y J}{T_2} - \frac{(M_z - M_z^0) K}{T_1}
\]

Solution:

\[
\vec{M}(t) = \begin{bmatrix}
e^{\frac{t}{T_2}} & 0 & 0 \\
0 & e^{\frac{t}{T_2}} & 0 \\
0 & 0 & e^{\frac{t}{T_1}}
\end{bmatrix} \begin{bmatrix}
\cos \omega t & \sin \omega t & 0 \\
-\sin \omega t & \cos \omega t & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
M_x(0) \\
M_y(0) \\
M_z(0)
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
M_z(0)(-e^{-\frac{t}{T_1}})
\end{bmatrix}
\]
EXCITATION IN THE ROTATING FRAME

- Original Bloch eq. in laboratory frame
  \[
  \frac{d\mathbf{M}}{dt} = \mathbf{M} \times \mathbf{B}
  \]

- Add on-resonance B1 to \mathbf{B}_0
  \[
  \mathbf{B} = B1(t)(\cos w_0t \mathbf{z} - \sin w_0t \mathbf{j}) + B_0 \mathbf{k}
  \]

- Lab frame < no gradient
  * basic excite
  lab frame < no gradient
  - Matrix version
  \[
  \frac{d\mathbf{M}}{dt} = \begin{bmatrix}
  \frac{dM_x}{dt} \\
  \frac{dM_y}{dt} \\
  \frac{dM_z}{dt}
  \end{bmatrix} = \begin{bmatrix}
  0 & -w_0 & w_0(t) \\
  w_0 & 0 & -w_0(t) \\
  -w_0(t) \sin w_0t & w_0(t) \cos w_0t & 0
  \end{bmatrix} \begin{bmatrix}
  M_x' \\
  M_y' \\
  M_z'
  \end{bmatrix}
  \]

- Substitution to convert to the rotating frame
  \[
  \mathbf{M} = R_{2}(w_0t) \cdot \mathbf{M}_{rot}
  \]
  \[
  \mathbf{B} = R_{2}(w_0t) \cdot \mathbf{B}_{rot}
  \]

- After substitution any off-resonance appears as residuals \( B_0 \) (\( B_{1x} \))
  (see off-res notes page)

- Rotating frame < on-resonance
  * basic excite, \( B_{1x} \)-only no gradient
  \[
  \frac{d\mathbf{M}_{rot}}{dt} = \frac{d\mathbf{M}}{dt} \times \mathbf{B}_{off}
  \]

- Rotating frame < off-resonance
  * general, \( B_{1x} \)-only incl gradients
  \[
  \frac{d\mathbf{M}_{rot}}{dt} = \begin{bmatrix}
  0 & 0 & w(t) \\
  0 & 0 & -(w_0 - w + w(t)) \\
  0 & 0 & -w(t)
  \end{bmatrix} \begin{bmatrix}
  M_x' \\
  M_y' \\
  M_z'
  \end{bmatrix}
  \]

- Gradient: \( w(t) = R_G z^2 \)
  - Off-res: appears as residual \( B_0 \), lifting \( B_{1x} \) vector
  - This means \( \mathbf{M} \) vector update will contain component that rotates \( \mathbf{M} \) around \( z \)-axis (in rotating coords, \( \xi \)-phase)

- Rotating frame < on-resonance
  * small tip approx.
  \[
  \frac{d\mathbf{M}_{rot}}{dt} = \begin{bmatrix}
  0 & w(t) & 0 \\
  w(t) & 0 & 0 \\
  0 & 0 & 0
  \end{bmatrix} \begin{bmatrix}
  M_x' \\
  M_y' \\
  M_z'
  \end{bmatrix}
  \]

  *small tip \( \Rightarrow \) easier to solve!
### Bloch Eq. Summary

\[
\frac{d\vec{M}}{dt} = \vec{M} \times \vec{\mathcal{B}} - \frac{M_x \hat{i} + M_y \hat{j}}{T_2} - \frac{(M_z - M_{z_0}) \hat{k}}{T_1}
\]

(Lab-frame)

- Vector lengths not to scale!

- Full lab-frame picture is complex:
  - 3 component of \( \frac{d\vec{M}}{dt} \) update vector
  - Larmor freq. component 7-9 orders magnitude larger than \( T_2, T_1 \) decay
  - \( \vec{B}_1 \) is also rapidly wiggling

- Conceptual simplification in 4 stages:

  1) **Lab frame**
     - Just precession
     - \( \vec{M} \) stopped
     - That is, \( \vec{B}_0 = 0 \)

  2) **Rotating frame**
     - \( \vec{M} \) stopped
     - But \( \vec{M} \times \vec{\mathcal{B}} \) still works!!
     - "Preshess" around \( \vec{B}_1 \) axis

  3) **Add \( \vec{B}_1 \)**
     - \( \vec{B}_1 \) also stopped!
     - Slow precess, now around tilted \( \vec{B}_{off} \)

  4) **Off-resonance**
     - Tilted plane
     - Apparent \( B_z \) comp. from residual precess. around \( z \) from off-resonance
**RF Field Polarization**

- Polarization (change of direction)
- Linearly polarized field
  \[ B_1(t) = B_1 \cdot \cos \omega t \hat{x} \]
  Magnetic strength: \([-1, 1] \cdot 1\]

- N.B.: \(B_1\) adds to much larger \(B_0\)

- Cylindrical polarized field (quadrature)
  \[ \vec{B}_{1q}(t) = B_1 (\cos \omega t \hat{x} - \sin \omega t \hat{y}) \]
  \[ = B_1 \cdot e^{-i\omega t} \]

- In the rotating coordinate system, flipping around x-axis vs. y-axis is just difference in phase of RF field

**Torque Stirs**

- Typical 90° flip (around x-axis)
- Typical 180° flip (around opposite y-axis)

**180° Flip**

- 9<x<90
- 2x power at x
**SIGNAL EQUATION**

\[ \Phi(t) = \oint \mathbf{B}(\mathbf{r}) \cdot \mathbf{M}(\mathbf{r}, t) \, d\mathbf{r} \]

- **magnetic flux**
  - thru coil \( \rightarrow \) scalar
  - (integral mag. field perpendicular to area)

- **magn. field**
  - detected \( \rightarrow \) scalar
  - by coil geometry
  - at each point
  - in object

- **local magnetization**
  - of object
  - (time-dependent)

- **position**: \( \mathbf{r} \rightarrow x, y, z \)

- **time deriv. inside**: use Bloch process variable
  - \( \partial \Phi(t)/\partial t \)
  - \( \cos \rightarrow \sin \)
  - \( \sin \rightarrow \cos \)

- **equate**: use complex notation

- **evaluate using free precession eqns.** (solution to Bloch) ignoring relaxation
  - rewriting in complex notation
  - in time-dependence from lab frame Bloch

- **ignore change in z-comp. \( \mathbf{M} \) because so slow**: \( \rightarrow \) i.e., we only see \( M_{xy} \), not \( M_z \)

- **substitute** \( \mathbf{M}(t) \) with lab frame
  - \( \mathbf{M}_{xy}(t) = M_{xy}(0) e^{-i\omega t} \)

- **simplify**: 1) **ignore decay** (assume this \( t=0 \))
  - \( \rightarrow \) so equals 1 (scalar)

- **2) assume phase-sensitive detection**
  - \( \omega \)w (difference from \( \omega_0 \) \( \rightarrow \) rotating frame

\[ \mathbf{S}(t) = \int \mathbf{B}_{xy}(\mathbf{r}) \mathbf{M}_{xy}(0) e^{-i\omega_0 t} \, d\mathbf{r} \]

- **Laboratory frame Bloch solutions**
  - \( M_L \rightarrow \) same
  - \( M_T = M_L(0) e^{i\phi} \)

- **demodulate**

- **spatially-dependent resonant freq.**: \( \omega_0 - \) rotating frame

- **i.e., after subtraction of \( \omega_0 = \gamma \)R.**

**Standard Signal expression**

\[ \mathbf{S}(t) = \int \mathbf{M}_{xy}(\mathbf{r}, 0) e^{-i\omega_0 t} \, d\mathbf{r} \]

- **phase angle in rotating frame**
  - \( \omega_t = \text{rad/\ sec} = \text{rad/s} \quad (\phi = \omega t) \)

- **getting difference converts lab \( \rightarrow \) rotating**
PHASE-SENSITIVE DETECTION

how we get rotating frame

- method for moving very high frequency Larmor oscillations down to tractable frequency range

V(t) ~123 MHz

multiply

Low-Pass Filter

S(t) ~50 kHz
digitization

reference signal (123 MHz)

demodulated signal ∝ RF coil signal - reference (transmitter)

\[ \sin[(\omega_0 + \delta \omega)t] \cdot \sin[\omega_0 t] \]

\[ \frac{1}{2} \cos \delta \omega t - \cos(2\omega_0 + \delta \omega)t \]

filter this one out w/ low pass filter

- this signal
  is digitized

one freq - freq domain

chirp - time domain

Signal
reference
demodulated
after filter
-> rotating frame!!

quadature coil

- two signals are made from a single receiving RF coil

- a quadrature coil can be treated the same way (OK to combine after adding 90° phase, then PSD)

- quadrature coil has better S/N since noise in each part is uncorrelated (1/\sqrt{2} better)

\[ \tilde{S}(t)_{\text{complex}} = M_y e^{-i \delta \omega t} \]
- **FID - FREE INDUCTION DECAY, T2**

- Signal (FID) resulting from RF pulse w/ angle \( \alpha \)

\[
\tilde{S}(t) = \sin \alpha \int_{w=\infty}^{w=0} \rho(w) \cdot e^{-t/T_2(w)} \cdot e^{-i\omega t} \, dw
\]

- An example spectral density ("Lorentzian inhomogeneity")

\[
\rho(w) = M_0^2 \frac{(\gamma \Delta B)^2}{(\gamma \Delta B)^2 + (w - \omega_0)^2}
\]

- [Subst \( \rho(w) \), rearrange to extract \( \omega_0 \), take integral]

\[
\tilde{S}(t) = \pi \cdot M_0^2 \cdot(X \Delta B) \cdot \sin \alpha \cdot e^{-t/(\gamma \Delta B)} \cdot e^{-t/T_2} \cdot e^{-i\omega_0 t}
\]

- [Combine T2 + static terms]

\[
\tilde{S}(t) = \pi \cdot M_0^2 \cdot(X \Delta B) \cdot \sin \alpha \cdot e^{-t/T_2} \cdot e^{-i\omega_0 t}
\]

\[
\frac{1}{T_{2}^{*}} = \frac{1}{T_2} + \frac{1}{T_{2}'^*} \quad \text{e.g., extra decay from B0 offsets}
\]

- N.B. center freq., not original integration variable

- \( \rho(w) = \frac{1}{c^2 + w^2} \)

- \( \rho(w) = \frac{1}{\gamma \Delta B} \) (not Lorentzian)

- \( \rho(w) = \) (N.B., not Lorentzian!)

- Spectral density, \( \rho(w) \), inhomogeneous \( B0 \), recovery, unobservable
**Echoes** — spin echo

90° - τ - 180°, T2/T2* & echo

Rotating coords

- Just after 90° x' pulse, f₀ + f₀ have same phase.

- Relaxation + phase dispersion of f₀ + f₀.

- Just after 180° y' pulse (y' pulse like x' pulse but RF has +90° phase).

- Echo caused by re-phasing of f₀ + f₀ (w/ decay due to T2).

- Remember brief RF just tips vectors while retaining length.

- Relaxation includes tips and shrinks (Mₜ) and grows (Mₜ echo).

- 180° x' pulse works, too, but echo will have +π phase (left side in figs above).

- Echo generated even if second pulse not 180° (see next).

- FID decay (and echo growth/decay) described by T₂*

- Reduction in height of echo compared to initial described by T₂, echo fixes the star.

- FID transmit

- TE

- RF receive signal ampl.
**ECHOES — Spin echo**

$\alpha_1 - T - \alpha_2 - T$ (both pulses along $y'$ for simplicity)

**effect of $\alpha$ pulse**

$M_x' \rightarrow M_x \cos \alpha - M_z \sin \alpha$

$M_y' \rightarrow M_y'$

$M_z' \rightarrow M_x' \sin \alpha + M_z' \cos \alpha$

(etc. for $\alpha_1, \alpha_2$)

**general transforms** (operators)

$M_x \rightarrow (M_x \cos \omega t + M_y \sin \omega t) e^{-\gamma T/2}$

$M_y \rightarrow (-M_x \sin \omega t + M_y \cos \omega t) e^{-\gamma T/2}$

$M_z \rightarrow M_z(1 - e^{-\gamma T/2}) + M_x' e^{-\gamma T/2}$

**effect of $T$ delay**

$\gamma$ (slow precession in rotating frame)

**decay**

Immediately after $\alpha$ pulse

$M_x'(\omega, 0) = -M_z(\omega) \sin \alpha,$

$M_y'(\omega, 0) = 0,$

$M_z'(\omega, 0) = M_z(\omega) \cos \alpha$

For one isochromat of freq. $\omega$

**after $T$ delay**

$M_x'(\omega, T) = -M_z(\omega) \sin \alpha, \cos \omega t e^{-\gamma T/2}$

$M_y'(\omega, T) = M_z(\omega) \sin \alpha, \sin \omega t e^{-\gamma T/2}$

$M_z'(\omega, T) = M_z(\omega) [1 - (1 - \cos \alpha) e^{-\gamma T/2}]$

**immediately after $\alpha_2$ pulse** (no effect on $M_y'$; rewrite $y'$; combine $x$ and $y$ eqs.)

$M_x'y'(\omega, T) = M_z(\omega) \sin \alpha_1 \left(\sin^2 \frac{\alpha_2}{2} e^{i\omega t} - \cos^2 \frac{\alpha_2}{2} e^{-i\omega t}\right) e^{-\gamma T/2}$

$M_z(\omega) [1 - (1 - \cos \alpha) e^{-\gamma T/2}] \sin \alpha_2$

**time dependent free precession around $z'$ (rewrite $M_x'y'(\omega, T')$)**

For a large num of freq's:

$M_x'y'(\omega, t) = M_x'y'(\omega, T') e^{-(t-T)/\gamma} e^{-i\omega(t-T)}$

$= M_z(\omega) \sin \alpha_1 \sin \omega t / 2 e^{-\gamma t/2} e^{-i\omega(t-T)}$

$- M_z(\omega) \sin \alpha_1 \cos \omega t / 2 e^{-\gamma t/2} e^{-i\omega t}$

$- M_z(\omega) [1 - (1 - \cos \alpha) e^{-\gamma T/2}] \sin \alpha_2 e^{-(t-T)/\gamma} e^{-i\omega(t-T)}$

Terms:

1. $\alpha_2 = 180^\circ$

2. $\alpha_2 = 90^\circ$

3. $\alpha_2 = 0^\circ$

Terms 2 & 3 are dephasing $\rightarrow$ FID of echo

Term 1: rephasing $\rightarrow$ rephase at $t = 2T$

**echo signal from 1**

$S(t) = \sin \alpha_1 \sin^2 \frac{\alpha_2}{2} \int_{-T}^{0} \rho(w) e^{-(t-T)/\gamma} e^{-i\omega(t-T)} dw$

$A_E = \sin \alpha_1 \sin^2 \frac{\alpha_2}{2} \int_{-\infty}^{0} \rho(w) e^{-TE/2} e^{-i\omega(TE/2)} dw = M_z \sin \alpha_1 \sin^2 \frac{\alpha_2}{2} e^{-TE/2}$

$S_1(t) = \frac{1}{2} \int_{-\infty}^{0} \rho(w) e^{-(t-T)/\gamma} e^{-i\omega(t-T)} dw$

$S_2(t) = \frac{\text{no } 1/2 \text{ factor}}{\text{multiply by } i \rightarrow \text{add } 1/2 \text{ phase}}$

$90^\circ, -90^\circ, 90^\circ, -180^\circ, 90^\circ, -180^\circ, 90^\circ, -180^\circ, 90^\circ, -180^\circ$

$90^\circ, -180^\circ, 90^\circ, -180^\circ$
Echo TRAINS - spin-echo trains

- It's (too) easy to make echoes...

\[ E_n = \frac{3^{(n-1)} - 1}{2} \]

Echoes after end of nth pulse:
3 RFs → 4 echoes (here)
6 RFs → 121 echoes (!)

Secondary echo: \( SE_{1,2} \) acts like RF pulse \( \alpha_3 \) makes an echo from it

- Two more conventional two-pulse spin echoes

Stimulated echo: combined effect of 3

\[ \alpha_1: M_L \rightarrow M_T \]
\[ \alpha_2: \text{leftover } M_T \rightarrow \text{flipped to } M_L \text{ (saved)} \]
\[ \alpha_3: \text{flip saved } M_L \rightarrow M_T \text{, which can then begin to cancel delays (after being held "in limbo" between } \]
180° \( \text{FID}_2 \text{ and } \text{FID}_3 \); acts like 2-pulse echo

- A useful multi-echo sequence (CPMG) is a 90° followed by 180° at 2\( \tau \) spacing

- Typically, 90° and 180° applied in different axes (\( x', y', z' \), etc.), which reduces phase errors due to imperfect 180° pulses (since slightly-off rotation around \( y' \) affects phase less)
EXTENDED PHASE GRAPHS

- Using full Bloch eq. solutions is tedious 😊
- Pictorial method for visualizing effects of series of $\pi$ pulses (vs. easier to visualize $90^\circ$, $180^\circ$)
- Problem #1: $\pi$ pulse rotates a portion of transverse magnetization into a position that results in rephasing and another portion into $M_L$
- Problem #2: third pulse can uncover and rephase transverse magnetization temporarily saved in longitudinal

$\Rightarrow$ rule for effect of $\pi$ RF pulse on transverse mag

$\Rightarrow$ rule for effect of $\theta$ RF pulse on longitudinal mag

- Echo when phase path crosses zero

- Each branch is weighted
- Each branch decays
  1) $T_1$ "stored"  
  2) $T_2$  
  3) $M_L$ regrowth
3 - Pulse Echo Amplitudes

- Assume $M_z^0 = 1$

RF transmit

RF receive

<table>
<thead>
<tr>
<th>Echo</th>
<th>time</th>
<th>amplitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>SE$_{1,2}$</td>
<td>$t = 2T_1$</td>
<td>$\sin \alpha_1 \sin^2 \frac{\alpha_2}{2} e^{-2\gamma/T_2}$</td>
</tr>
<tr>
<td></td>
<td>$t = 2T_1 + T_3$</td>
<td>$\sin \alpha_1 \sin^2 \frac{\alpha_2}{2} \sin^2 \frac{\alpha_3}{2} e^{-2\gamma/T_2}$</td>
</tr>
</tbody>
</table>

2\(^\circ\) ("secondary")

- $t = 2T_2$
- $t = 2T_1 + 2T_3$

- $\sin \alpha_1 \sin^2 \frac{\alpha_2}{2} \sin \frac{\alpha_3}{2} e^{-2\gamma/T_2}$

STE ("stimulated")

- $t = 2T_1 + T_2$

- $\frac{1}{2} \sin \alpha_1 \sin \frac{\alpha_2}{2} \sin \frac{\alpha_3}{2} e^{-\gamma/T_2}$

- $1 - (1 - \cos \alpha_1) e^{-\gamma/T_2}$

$\sin \alpha_2 \sin^2 \frac{\alpha_3}{2} e^{-(\gamma + 2T_3)/T_2}$

- $T_1$-dependence in STE (but also SE$_{2,3}$) from temporary "storage" of $M_T$ in $M_L$, then recovery by third pulse
**HYPER ECHOS**

(N.B.: coordinate system used vs. left-handed in Block 4.0 notes)

- Hennig & Schelller (2001)
- Normalize $\mathbf{M}$ amplitude $= 1$
- Sphere surface defines 2D space for $\mathbf{M}$ moved by:
  1) Vector rotation of $\mathbf{M}$ around tilted axis in transverse x-y plane by RF with flip angle and phase, $\phi = P(x', \phi)$
  2) Rotation around z by phase evolution due to frequency offset, $\omega$ (BP offset) and time, $t = \phi/\omega$

- Three symmetries:
  - Solid lines: phase evolution, $\mathbf{M}$ flip or RF
  - Dashed lines: just $180^\circ$ equivalent

**Practical use**

- Multi-echo example
- Can also use to prepare, then separate read-out

- Practical prob: $180^\circ$ pulses deposit a lot of RF (6x 90°)
  \[ \Rightarrow \text{prob at high fields} \]

- by arranging to get big echo in middle of k-space can get by with much less RF power
GRADIENT ECHOES - \( T_2^* \), GE chains

- Initial negative gradient dephases spins
- After \( t = T \) of positive gradient, spins rephase
- Does not correct for \( T_2^* \) inhomogeneities
  
  \[
  A_E = e^{-t/T_2^*}
  \]

- The initial "FID" is not "free" since it is being actively de-phased by gradient, so FID decay
  
  \[
  \frac{1}{T_2} < \frac{1}{T_2^*} < \frac{1}{T_2^{**}} \Rightarrow A_E = e^{-t/T_2^{**}}
  \]

- Key difference between spin-echo (SE) and gradient echo (GE)
  
  is that \( B_0 \) inhomogeneities not canceled
  
  \[
  \Rightarrow \text{hence, echoes are } T_2^* \text{-weighted, not } T_2 \text{-weighted} \Rightarrow \text{more susceptible to inhomogeneities}
  \]

- Echo trains possible w/gradient echo (CPMG-like)

- The faster the gradients are switched, the more echoes you get
- EPI hardware
  
  \[
  \Rightarrow 64 \text{ echoes}
  \]
- **IMAGE CONTRAST**

**T1 Saturation-recovery (no echo, just FID)**

- Contrast \((PD,T1,T2,T2^*)\) depends on magnetization not getting back to equilibrium, and then differences in how far away each tissue type is at measurement time

- RF

\[
\begin{array}{cccc}
& 1 & 2 & 3 & 4 \\
T & 70^\circ & \text{TR} & & \\
M_z^0 & \rightarrow & M_z^o (1 - e^{-TR/T1}) + \text{[ignored]} & \\
M_z^0 \rightarrow & \text{"steady state" after here} & \\
\end{array}
\]

- Simple saturation/recovery \(\inf\) no echo

- Initial conditions:
  - \(M_z\) before first pulse = \(M_z^0\)
  - \(M_z = 0\) imm. after first pulse \(\text{(i.e., 90^\circ pulse)}\)

- From Bloch eq, \(M_z\) just before second pulse:

\[
M_z^{n+1} (O^-) = M_z^1 \left(1 - e^{-TR/T1}\right) + M_z^0 (O^+) e^{-TR/T1}
\]

\(M_z^0\) before current pulse \(M_z^{n+1}\) "regrowth-from-zero" term 

M_z^1 "left-immed.-after-pulse" term (U.S. decaying)

- Given:
  1. 90° pulse
  2. no \(M_{xy}\) left

\(\rightarrow\) pure tip: \(M_{xy} = M_z\)

- Tip existing mag

\[
M_z^{(n)} (O^-) = M_{xy} (O^+) = M_z^0 \left(1 - e^{-TR/T1}\right)
\]

Longitudinal mag just before pulse

Transverse we can record after pulse

Transverse mag depends on \(T1\)!

- That is, the not-completely-regrown longitudinal magnetization, which depends on \(T1\), but which we cannot record, is completely converted to recordable transverse magnetization

\[
I(r) = C \rho(r) (1 - e^{-TR/T1(r)})
\]

Assume immediate recording of signal

Spectral density \(\rho(r)\) at point density underlies equilb. \(M_z^0\)
IMAGE CONTRAST: Why imperfect 90° takes multiple flips til steady state

- initial fMRI images are usually discarded (why?)
  - because they are brighter than all the rest
  - because multiple flip required before steady state

N.B.: B1 imperfections guarantee this situation will occur
  (e.g. at 3T, flip angle varies almost 25% across brain)

- at 3T, steady state
  for typical 1-2 sec
  TR images reached
  after ~8 images

\[ M_z \]

\[ M_z^0 \to \]

\[ \cdots \]

\[ \text{TR} \]

\[ \text{etc} \]

\[ \text{similar} \]

\[ \text{still not steady state} \]
**IMAGE CONTRAST**

IR (still just saturation-recovery — no echo)

- inversion recovery w/ no echo

\[
\begin{align*}
M_z^0 & \quad \text{longitudinal magnetization} \\
M_z^{-} & = -M_z^0
\end{align*}
\]

- 180° pulse reverses longitudinal magnetization

\[
M_z' = M_z^0 (1 - 2 e^{-\Delta T_1/T_1}) \quad \text{flipped into transverse by second pulse (90°)}
\]

- longitudinal then regrows from zero

\[
M_z' = M_z^0 (1 - e^{-(TR-T_1)/T_1}) \quad \text{from first Bloch term only}
\]

- after second 180°, just change sign again

\[
M_z' = -M_z^0 (1 - e^{-(TR-T_2)/T_2})
\]

- apply relaxation eq. again

\[
M_z' = M_z^0 (1 - e^{-\Delta T_1/T_1}) - M_z^0 (1 - e^{-(TR-T_1)/T_1}) e^{-\Delta T_2/T_2}
\]

\[
M_z' = M_z^0 (1 - 2 e^{-\Delta T_1/T_1} + e^{-\Delta T_2/T_2})
\]

- this is magnetization flipped to transverse, made recordable
**IMAGE CONTRAST**

**SE, IR-SE**

- Steady state mag (2nd TR) just before 90°
  \[ M_z^a (0) = M_z^0 \left(1 - 2e^{-\frac{(TR-TE/2)}{T_1}}\right) + e^{-TR/T_2} \]

- The echo signal \( M_z^e \) unlike in simple saturation-recovery FID. Has an additional delay before it is recorded, so we have to take account of transverse mag relaxation.
  \[ A_E = M_z^0 \left(1 - 2e^{-\frac{(TR-TE/2)}{T_1}}\right) + e^{-TR/T_2} \] e^{-TE/T_2}

  - If we assume \( TE \) much less than \( TR \), then we can simplify:
  \[ A_E = M_z^0 \left(1 - e^{-TR/T_1}\right) e^{-TE/T_2} \]

  - Similar equation for SE-IR
  \[ A_E = M_z^0 \left(1 - 2e^{-TI/T_1} + e^{-TR/T_2}\right) e^{-TE/T_2} \]
**IMAGE CONTRAST**

GRE w/ small tip angle

- Use basic longitudinal relaxation from Bloch eq. again

\[ M_{2'}^{(n)}(O) = M_{2}^{0}(1 - e^{-TR/T1}) + M_{2}^{(n-1)}(O+) e^{-TR/T1} \]

(\text{long TR or spoiler})

- Assume we have a small tip angle:

\[ M_{2}^{0} \cos \alpha \Rightarrow M_{2}^{(n)}(O+) = M_{2}^{(n)}(O-) \cos \alpha \]

- Assume we are in dynamic equilibrium:

\[ M_{2}^{(n)}(O) = M_{2}^{(n-1)}(O-) = M_{2ss}^{ss}(O-) \]

- Steady state

\[ M_{2ss}^{ss}(O) = \frac{M_{2}^{0}(1 - e^{-TR/T1})}{1 - \cos \alpha e^{-TR/T1}} \]

\[ M_{x'y'}^{ss}(t) = \frac{M_{2}^{0}(1 - e^{-TR/T1}) \sin \alpha e^{-TR/T1}}{1 - \cos \alpha e^{-TR/T1}} \]

gradient echo amplitude

\[ A_{E} = \frac{M_{2}^{0}(1 - e^{-TR/T1}) \sin \alpha e^{-TE/T2}}{1 - \cos \alpha e^{-TR/T1}} \]

T1 contrast mainly depends on flip angle, not TR → \( \cos \theta = 1 \) → eliminates T1 weight since denominator equals numerator
**IMAGE CONTRAST MDEFT / 3D FLASH**

\[90^\circ \text{ TD} \rightarrow 180^\circ \text{ TI} \rightarrow T \rightarrow \alpha (\text{spiral phase}) \rightarrow \alpha \rightarrow 90^\circ \text{ TD} \rightarrow 180^\circ \text{ TI}\]

- Saturate, wait for contrast \(_1\), invert, wait for contrast \(_2\), FLASH (center out)

A) \(M_z' \text{ (just after } 90^\circ) = 0 \) (perfect \(90^\circ\))

B) \(M_z' \text{ (after } \text{TD}) = M_z^0 \left(1 - e^{-\text{TD}/T_2}\right) \) (Bloch term #1)

C) \(M_z' \text{ (just after invert)} = \cos \phi \ M_z^0 \left(1 - e^{-\text{TD}/T_2}\right) \)

D) \(M_z' \text{ (after } \text{TI}) = M_z^0 \left(1 - e^{-\text{TI}/T_2}\right) + \left[\cos \phi \ M_z^0 \left(1 - e^{-\text{TD}/T_2}\right)\right] e^{-\text{TI}/T_2} \)

\[= M_z^0 \left[1 - \left[1 - \cos \phi \left(1 - e^{-\text{TD}/T_2}\right)\right] e^{-\text{TI}/T_2}\right] \text{ after preparation}\]

**Special case \(T_1 = T_2\):** 
\[M_z^0 \left[1 - e^{-\text{TI}/T_2}\right]^2 \]

→ using hard \(130^\circ\) 'inversion' coil cancel hard alpha \(B_1\) inhomogeneities (Thomas et al., 05)

- After the first X pulse:

E) \(M_z' \text{ (just after } \text{TD}) = M_z^0 \left[1 - \left[1 - \cos \phi \left(1 - e^{-\text{TD}/T_2}\right)\right] e^{-\text{TI}/T_2}\right] \sin \alpha \)
MAGNETIZATION TRANSFER CONTRAST

- Protons in macromolecules & bound to membranes have wide range of resonant freqs ("bound")
  \[ T_2 = 1 \text{ msec} \]
  i.e., signal but invisible w/ usual TE

- Free protons in blood, CSF, water have narrow range of resonant freqs ("free")
  \[ T_2 = 50 \text{ msec} \]

- Mag transfer pulse sequence
  1) Off center freq pulse to hit "bound" (but don't hit water too hard)
  2) Wait for magnetization transfer from saturated longitudinal \( M_L \) of "bound" \( \rightarrow M_L \) of "free"
  3) Result of transfer \( \rightarrow \) attenuation

> N.B. this always happens a little (cf. TI-weighted, T2-weighted)
  something to keep in mind if hard pulse (wide freq)

- Used to increase contrast in TOF
  TOF (not explained) bright vessels from inflow fresh spins
  MT - contrast added: suppress tissue but not blood

- View w/ MIP: maximum intensity projection along lines

\[ \max \rightarrow \text{view as movie} \]
**SIGNAL-TO-NOISE, CONTRAST-TO-NOISE**

- Signal-to-noise defined as: \[ \text{SNR} = \frac{\text{avg obj signal}}{\text{s.d. non-object region}} \]
- Temporal SNR: \[ \text{SNR}_{\text{temp}} = \frac{S_{A} - S_{B}}{\text{s.d. GM - WM}} \]
- "Contrast" is a difference
- Contrast-to-noise ratio:

\[ \text{CNR}_{AB} = \frac{S_{A} - S_{B}}{\text{s.d. GM - WM}} = \text{SNR}_{A} - \text{SNR}_{B} \]

**Spin-echo:**

\[ A_{E} = M_{0} e^{-(1 - e^{-TR/T1})} e^{-TE/T2} \]

**Gradient echo:**

\[ A_{E} = \frac{M_{0} e^{-(1 - e^{-TR/T1})} \sin \alpha e^{-TE/T2\alpha}}{1 - \cos \alpha e^{-TR/T1}} \]

**General rules:** Spin-echo, long TR GE

<table>
<thead>
<tr>
<th>proton density weighted</th>
<th>TR long (no T1 diff)</th>
<th>TE short (no T2 diff)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T1-weighted</td>
<td>TR ~ T1 (big T1 diff)</td>
<td>TE ~ T2 (no T2 diff)</td>
</tr>
<tr>
<td>T2-weighted</td>
<td>TR ~ T2 (no T1 diff)</td>
<td>TE ~ T2 (big T2 diff)</td>
</tr>
</tbody>
</table>
**SIGNAL-TO-NOISE S/N**

- Generalized dependence of SNR on 3D imaging parameters

\[
\text{SNR/voxel} \propto \frac{\Delta x \Delta y \Delta z}{\sqrt{N_a^2 N_x N_y N_z \Delta t}}
\]

- Voxel size (in same number)
- Number of repeats
- Number of voxels (in same size)
- Read timestep

- Size (volume) of voxels (with the number of voxels held constant), linear effect on S/N
  \(\downarrow\) e.g., \(3 \times 3 \times 3 \text{mm} \rightarrow 4 \times 4 \times 4 \text{mm}\) \(\frac{64}{27} = 2.37 \text{ times better S/N}\)

- More voxels (with size of voxels, \(\Delta t\) per read step constant), \(\sqrt{N}\) effect on S/N
  \(\downarrow\) e.g., \(64 \times 64 \rightarrow 128 \times 128\) \(\frac{128 \times 128}{64 \times 64} = 2 \text{ times better S/N}\)

- # acquisitions \(\sqrt{N}\) better S/N
  \(\downarrow\) e.g., \(1 \text{ acq} \rightarrow 2 \text{ acq}\) \(\frac{\sqrt{2}}{1} = 1.41 \text{ times better S/N}\)

- Larger timestep during readout, \(\sqrt{\Delta t}\) better S/N
  \[\Delta t = \frac{1}{2\text{BW}_{\text{read}}}, \text{ digitization timestep during echo acquisition}\]

- \(\text{BW}_{\text{read}}\) determined by cutoff freq, analog low-pass filter
- \(\Delta t\) controls BW because low-pass cutoff has to be set higher for smaller (higher free-detecting) \(\Delta t\)
- Must filter out freg's \(> f_{\text{max}} = \frac{1}{2\Delta t}\) because they alias
COMPLEX ALGEBRA

**real/imaginary**

- **add**: \((r_1, i_1) + (r_2, i_2) = (r_1 + r_2, i_1 + i_2)\)
- **mult**: \((r_1, i_1) \times (r_2, i_2) = (r_1r_2 - i_1i_2, r_1i_2 + i_1r_2)\)

**angle/phase**

- **add**: \((A_1, \phi_1) + (A_2, \phi_2) = (A_1 \cos \phi_1 + A_2 \cos \phi_2, A_1 \sin \phi_1 + A_2 \sin \phi_2)\)
- **multiply** (commutative): \((A_1, \phi_1) \times (A_2, \phi_2) = (A_1A_2, \phi_1 + \phi_2)\)
- **divide**: \((A_1, \phi_1) \div (A_2, \phi_2) = (A_1/A_2, \phi_1 - \phi_2)\)

**complex to real power**: \((A, \phi)^n = (A^n, n\phi)\)

\[e^{i\phi} = \left[ \begin{array}{c}
\text{expand as series} \\
\text{recognize} \cos, \sin \text{ series}
\end{array}\right]
\]

- the real "e" to "purely imaginary" power

\[e^{i\phi} = \cos \phi + i \sin \phi = \cos \phi, \sin \phi = \text{vector on unit circle}\]

\[e^{i\phi}^n = (\cos \phi + i \sin \phi)^n = \cos n\phi + i \sin n\phi\]

**Fourier Transform**

\[H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt\]

\[H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt\]

**Convolution Theorem**

\[F[g(x) \cdot h(x)] = G(k) \ast H(k)\]

\[\text{because of FFT, faster if kernel not small}\]

**Convolution**

\[f(x) = g(x) \ast h(x) = \int_{-\infty}^{\infty} g(z) \cdot h(x-z) dz\]

\[F(f(x)) = F(g(x)) \ast F(h(x)) = \int_{-\infty}^{\infty} F(g(z)) \cdot F(h(x-z)) dz\]

**Fourier transform of two functions multiplied by each other equals the convolution of the Fourier transform of each function**
Fourier transform (1)

\[ H(f) = \int_{-\infty}^{\infty} h(t) \cdot e^{-i \frac{2\pi ft}{\text{cyc/sec} \cdot \text{sec} = \text{cyc}}} \, dt \]

- How to calculate \( H(f) \) for one \( f \) \((f=3)\):

real signal

imaginary signal (zero)

\[ \begin{bmatrix} \cos \\ -\sin f t \end{bmatrix} \]

\[ \begin{bmatrix} \cos (2\pi 3t) \\ \sin (2\pi 3t) \end{bmatrix} \]

complex multiply

integrate/sum these multiplies across all \( t \)

real frequency domain

imaginary frequency domain

like correlating with \( \sin \) and \( \cos \) (at each freq) so we get phase (at each freq.)
Fourier transform (1b)

\[ e^{i\phi} = \cos \phi + i \sin \phi \]
\[ e^{-i\phi} = e^{i(-\phi)} \]
\[ = \cos(-\phi) + i \sin(-\phi) \]
\[ = \cos \phi - i \sin \phi \]

- \( \cos \) is an "even" function, \( \sin \) is an "odd" function

An orthogonal decomposition

- think of discretely sampled \( \sin(bx) \), \( \cos(bx) \) as vectors
- \( \text{Corr}(V_1, V_2) \equiv \text{projection of } V_1 \text{ onto } V_2 \equiv \frac{V_1 \cdot V_2}{|V_1||V_2|} \)

\[
\begin{align*}
\text{Corr} (\cos bx, \sin bx) &= 0 \\
\text{Corr} (\sin bx, \sin bx) &= 0 \\
\text{Corr} (\cos bx, \sin bx) &= 0
\end{align*}
\]

- in the continuous case, orthogonal functions defined as:

\[
\int_{-\infty}^{\infty} f(x) g(x) \, dx = 0
\]
UNDERSTANDING INVERSE FOURIER TRANSFORM AS ANOTHER CASE OF CORR W/ COS, SIN

- Start with spike in image domain
- Take example of spike at \( x = 0 \)

\[ \begin{bmatrix} \cos(x) , \cos(2x) , \cos(kx) \end{bmatrix} \text{ where all freq's correlate w/ spike at } x = 0 \]

- If spike is moved away from zero, the frequency spectrum obtained by correlating cos with spikes oscillates

N.B. Opposite direction sin spikes are on imaginary axis

- To see why this is, since spike is all zero except at spike, the dot product for a given frequency only depends on the value of the \( e^{-2\pi ikx} \cos \) and \( \sin \) at location of spike

Successively higher freqs

- Pos. pair (real) spikes same dist. from origin
- Pos/neg. pair (imaginary) spikes same dist orig.
- One spike at distance from origin

\( \rightarrow \) This is one way of thinking about what one point in \( k \)-space means, via correlating it w/ cos's, sin's to get periodic result in image space (inverse FT)
FOURIER TRANSFORM OF AN IMAGE (2)

1. Real image → Imaginary image
   - (Zero) point in both spaces
   - Fourier Transform

2. Amplitude image → Phase image
   - (Zero) point in both spaces
   - View complex vectors directly

3. Complex vectors
   - Zero vector
   - Equivalent representations of image & spatial frequency space
FOURIER TRANSFORM OF REAL IMAGE (3)

- what a single k-space point looks like for real image (polar coordinates $A, \phi$ instead of $r, \theta$)

**image space**

**k-space** (spatial freq. space)

offset of stripes is k-space phase

brightness of stripes proportional to k-space amplitude

distance from center is stripe spacing

angle of point perpendicular to angle of stripes

(N.B.: need conjugate point, too)

(N.B.: need conjugate point, too)

value from 0 to 360°

inverse Fourier transform

(image recon.)

(Should be all zero; not same as 'stripe phase' above)

(N.B.: each dimension of spatial freq. space (k-space) from correlation w/ sin, cos — don't confuse $k_x, k_y$ w/ sin, cos!)

- Cartesian dimension of k-space — x- and y- spatial freq.

N.B.: increasing one 1D component increases the spatial freq of the 2D wave and rotates it
FOURIER TRANSFORM OF IMAGE (4)

- 3 equivalent representations of complex numbers in image space and spatial-freq. space (k-space)
- Example: cosinusoid in image space, then shifted in x-dir

REAL IMAGE

$I(x, y) = \cos(x)$

FT OF REAL IMAGE

$\text{FT of } I(x, y)$

$I(x, y) = \cos(x - \pi/4)$

$\text{FT of } I(x, y)$

N.B.: an example of the "Fourier Shift Theorem" (see below)

$45^\circ \text{ rot compared to complex above}$

$90^\circ \text{ rot compared to complex above}$

real component less than above because rot:
FOURIER TRANSFORM OF IMAGE (5)

- (cont.) center of k-space (real image)
- complex image

REAL IMAGE

I(x,y) = 1 + \cos(x)

FT OF REAL IMAGE

center of k-space:

\[ H(k) = \int h(x) \cdot e^{-2\pi i k \cdot x} dx \]

avg image brightness \[ \leftarrow I(\text{real}) \]

positive center k-space

COMPLEX IMAGE

I(x,y) = \cos(x) - i \sin(x)

FT OF COMPLEX IMAGE

FT, FT^{-1}

"missing" spike results in single spike correlating with cos or sin

N.B.: this k-space is non-Hermitian.

k-space will only have Hermitian symmetry if image is real.

Hermitian symmetry when complex conjugate (complex num w/ sign flipped in image part) is equal to func value w/ mag arg:

1D: H(k) = H^*(k)

2D: H(-k_x, k_y) = H^*(k_x, k_y)

[ N.B. this is also exactly what a gradient does to image space! ]
FOURIER TRANSFORM OF IMAGE (G)

- (cont.) x- and y-spatial freqs.
- special case: real image from sum of reals

REAL IMAGE

\[ I(x, y) = \cos(x) + \cos(y) \]

**N.B.** adds but doesn't rotate stripes

\[ I(x, y) = \cos(x+y) \]

rotates stripes!

FT of real image

- Remember, single k-space point transforms to complex img.
- But if Hermitian symmetric, imaginary components cancel
- Since all we want in image space reconstruction is real component, can just add real components
  of complex vectors at each image space point
  for every complex image corresponding to each k-space point
- **N.B.** the k-space phase will affect offset of real-valued image space cosine/sine
- Therefore for real-valued image, we can visualize
  inverse FT as real-valued sum of offsets
  real-valued cosine/sine
- **N.B.** cannot do this with MRI k-space data since phase errors
  (incl. multiple wraps) mess up real component. Must use amplitude img.
**Gradient Coils**

Gradient coils for x, y, z generate approximately a linear gradient in the strength of the z-component of the magnetic field \( B_z \).

- For example, the x gradient coil induces a ramp in z-component of the magnetic field when moving in the x-direction:

\[
B_{G,z} = G_x x
\]

*Since a pure linear gradient of \( B_{G,z} \) in only the x, y, or z directions is not possible according to Maxwell equations, each gradient coil also induces a magnetic field that has components in the x- and y-direction (\( B_{G,x} \) and \( B_{G,y} \)).

*The other magnetic field components are usually ignored because they are so small relative to \( B_{G,z} \), since \( B_{G,x} \) is added to \( B_0 \), and since \( B_0 \) is much stronger than \( B_{G,z} \), \( B_{G,y} \), and \( B_{G,x} \).

Since standard reconstruction methods assume the existence of "non-Maxwellian" gradient fields, spatial distortion is introduced.

The Maxwellian terms \( B_{G,x} \) and \( B_{G,y} \) are known; can be included in the reconst process.

\[ \Delta \phi_{G_x}(x) \approx -\frac{x^2 G_x^2 x}{2B_0} \]
SLICE SELECTION \( (G_z) \)

- slice select gradient on during RF stim

\[
B_z = \frac{x}{2\pi} G_z z
\]

- protons here can only be excited by a narrow band of radio frequencies

- to apply a pulse containing a narrow band of frequencies, we use a sinc pulse envelope (Fourier transform of a narrow freq. band)

in practice, Gaussian pulse envelope good too

\[
\text{Fourier Transform} \quad \frac{\sin(x)}{x}
\]

- this excites protons in a narrow slab

- since the slice-selection gradient introduces (space-dependent) phase shifts (see freq encode) these have to be removed by a post-excitation rephasing \( z \)-gradient

- approximation from assuming tip occurs instantaneously in middle

- valid for small tip: \( 90^\circ \rightarrow 52\% \)

- in practice: adjust to max, use crusher to kill spurious transverse on \( 180^\circ \)
Fourier Transform pairs, rules

- Multiplication in one domain equals convolution in other:
  \[ \mathcal{F} \{ g(t) \cdot h(t) \} = G(f) * H(f) \]

- Convolution with delta function (impulse) moves other function to impulse center

Fourier Transform Solution to: \[ \frac{1}{\sqrt{f}} \]

- Time
  \[ \mathcal{F} \{ \text{time} \} = \frac{1}{\sqrt{f}} \]

- Frequency
  \[ \mathcal{F} \{ \text{frequency} \} = \text{time} \]

- Convolve
  \[ \mathcal{F} \{ \text{convolve} \} = \text{impulse} \]
SLICE SELECT RF PULSES

Interleaved Acquisition $\rightarrow$ better S/N b/c imperfect slice profile
spin history prob if motion

Common RF pulses

- non-selective pulse ("hard" pulse)

- standard slice select sinc

- Gaussian

$\rightarrow$ pulses need to be "apodized" (have "foot" removed)
$\rightarrow$ multiply by function so begin/end of pulse is differentiable

Fat Saturation

- fat protons have chemical shift causing resonant freq offset
- add phase offset not due to gradients, RF
- fix by off-water-resonance 90° (saturation) pre-pulse centered on fat freq
$\rightarrow$ need high quality (narrow-freq) pulse to avoid saturate water!

HOWTO
1) fat sat pulse
2) wait $T_2$ so fat signal decays, but no $T_1$ regrowth of fat
3) RF stim for water "protons-q" interest"

Adding Another Gradient Tilts Slice

- $G_z$, $G_z + G_y$

- Plane of constant $B_z$

- with 3 gradients on, can excite arbitrary angle plane

- translate plane by changing either gradient amplitude or RF freq band: $\vec{B}_0$
WHY "FREQUENCY-ENCODING" IS A MISNOMER

- comes from original analogy in Lauterbur (1973):

1) chemical shift change freq  \( \rightarrow \)  gradient changes freq.
2) stimulate w/ broadband RF  \( \rightarrow \)  same
3) time-sample FID containing multiple freqs  \( \rightarrow \)  same
4) FT of FID to get spectrum  \( \rightarrow \)  FT of FID to get \( \Delta x \) offsets

\[ \text{FT of} \quad \text{FT of} \quad \text{FT of} \quad \text{FT of} \]

- this is technically correct (FT of FID) but highly misleading
  - e.g., phase-encoding (turning a different gradient ON and OFF before recording FID) seems to be something completely different since OFF gradient can't affect freqs in FID

- the "k-space" perspective is a "Copernican turn"
  - idea is that data is \( \neq \) a set of samples of a time domain signal generated by multiple chemical-shift like frequencies
  - rather, it is a set of samples of a frequency-domain signal, each sample generated by multiple spatial locations,
    (which are analogous to multiple time points)

- i.e., the 'direction' of the FT (Fourier transform) is reversed:

<table>
<thead>
<tr>
<th>spectroscopy</th>
<th>samples of oscillations in time-domain</th>
<th>( \text{FT} \rightarrow \text{frequency-domain} ) spectrum of shifts</th>
</tr>
</thead>
</table>
| MRI           | samples of spatial freq. in freq-domain| \( \text{FT}^{-1} \rightarrow \text{spatial object (like a time-domain signal)} \)

- the original analogy only 'works' because \( \text{FT} \approx \text{FT}^{-1} \) (except sign change)
**FREQUENCY ENCODING (1)**

-- Frequency encode gradient ($G_x$) causes precession rates to vary linearly in $x$-direction

\[ \text{precession} \uparrow \frac{B_0}{\text{in} \ x-\text{direction}} \]

\[ \text{frequency} \]

\[ \text{correct} \quad \text{(remember that strength of} \ G_x \text{causes variation of slope of} \ B_0 \text{in} \ x-\text{direction)} \]

-- different frequency signals are mixed together
and recorded as a 1-D signal over time

\[ \text{correct} \quad \text{but remember, we are recording summed}
\text{local magnetization vectors after de-modulation} \]

-- a Fourier transform, which can convert back and forth
between $x$-position (cf. time) and spatial frequency
(cf. temporal freq) is done on signal

\[ \text{correct} \]

-- spatial frequencies get confused/conflicted with
precession frequencies

\[ \text{wrong} \]

-- therefore, the Fourier transform is used to convert
position-dependent precession frequencies into

\[ \text{spatial position} \]

\[ \text{conceptually wrong} \]

\[ \text{inverse} \]

\[ \text{FT actually converts spatial frequencies}
\text{to spatial position} \]

\[ \text{the spatial frequency increases for each}
\text{time point in the readout} \]

\[ \text{the precession freq ramp is constant each timestep} \]
FREQUENCY ENCODING (2) connect intuition – why phase critical

- "Frequency"-encode gradient ($G_x$) turns on during
  during echo causes precession rates
to immediately vary with $x$-position

  \[ G_x \rightarrow t \rightarrow \]

  $\uparrow B_z$ in $x$-direction

  \[ G_x \text{ levels } (= \text{slope}) \]

  \[ x \]

  \[ t \]

  at beginning of gradient on, the phase of
  signal coming from each $x$-position is the same
  \[ \text{summed phase angle is what we measure} \]

- early after gradient on, phase advances (because
  of faster precession frequency) arise with greatest
  phase advance at largest $x$-position

  \[ \text{Single} \]

  \[ \text{time point} \]

  \[ \text{Early} \]

  \[ t \rightarrow \]

  \[ 360^\circ \]

  \[ \phi \]

  \[ 0^\circ \]

  \[ x \rightarrow \]

  \[ \Rightarrow \text{one cycle of} \]

  \[ \text{spatial frequency} \]

  \[ \epsilon \text{ phase angle} \]

  \[ (= \text{low spatial freq}) \]

- later during gradient on, phase advances cause
  multiple wraparounds of phase angle across space

  \[ \text{Single} \]

  \[ \text{time point} \]

  \[ \text{Later} \]

  \[ t \rightarrow \]

  \[ 360^\circ \]

  \[ \phi \]

  \[ 0^\circ \]

  \[ x \rightarrow \]

  \[ \Rightarrow \text{multiple cycles of} \]

  \[ \text{spatial frequency} \]

  \[ \epsilon \text{ phase angle} \]

  \[ (= \text{hi spatial freq}) \]

- in practice, the lowest spatial frequency ($\epsilon 0$)
  occurs in the middle of the gradient on time
  because the phase is "rounded" negatively by
  a preparatory gradient (to the highest negative
  spatial frequency) before data collection occurs

  \[ \text{Individual} \]

  \[ \text{RF data} \]

  \[ \text{Samples} \] (after

  \[ \text{demodulation} \]

  \[ G_x \rightarrow t \rightarrow \]

  \[ \text{Spatial frequency} \]

  \[ \phi = 0 \]

  \[ \phi = \text{max neg} \]

  \[ \phi = 0 \]

  \[ \phi = \text{max positive} \]
FREQUENCY ENCODING (3)

Standard Fourier transform:

\[ H(\nu) = \int_{t=-\infty}^{t=\infty} h(t) e^{-i 2\pi \nu t} dt \]

"\( k \)" is often used instead of "\( \nu \)" for the frequency variable.

Imaging equation:

\[ S(\nu) = \int_{x=-\infty}^{x=\infty} I(x) e^{-i 2\pi \nu x} dx \]

One data point (i.e., one spatial freq) during readout (2 components).

Get this single readout point by

- Summing signal across x-position
- Recording sum even though variable is \( \nu \), it represents one time period during readout

Signal strength at one x-position

- Brightness of image point
- Spin density (spectral density)

Oscillations come from readout phase wrapping, where \( f \) is single spatial freq (e.g., 5) and \( x \) goes across object

\[ f = G_x / (t - TE) \]

Don't confuse with instantaneously changed precession freqs which are constant across entire readout time (for each \( x \) position)

To make image, do inverse Fourier transform of recorded signal \( S(\nu) \).
Alternate derivation (incl. effects of $G_x$) Signal EQ

- Oscillators at $w = fB$ at each position (just $x$ for now)

$$S(t) = M(x) e^{-i\phi(x)} dx$$

- By definition, freq, $w$ is rate of change of phase, $\phi$

$$\frac{d\phi(x,t)}{dt} = w(x,t) = fB(x,t)$$ and integrating $\phi(x,t) = \int_0^t w(x,t) dt = f\int_0^t B(x,t) dt$

- Assuming phase initially 0, $B$ affected by gradients

$$B(x,t) = B_0 + G_x(t) x$$

$$\phi(x,t) = \gamma \int_0^t B_0 dt + \left[ f \int_0^t G_x(t) dt \right] x$$

$$= \omega_0 t + 2\pi k_x(t) x$$

$k$ is time integral of gradient waveform

- Demodulation removes the $B_0$-caused carrier frequency $e^{-i\omega_0 t}$ from the first equation

$$S(t) = \int_x M(x) e^{-i 2\pi k_x(t) x} dx$$

Amplitude of each oscillator, gradient-controlled phase
- Turn on gradient after excitation but before readout

- Different levels of \( G_y \)

- Higher levels of \( G_y \) (slope of \( B_z \) in \( y \)-direction!)
  \[ \text{\( \Rightarrow \text{higher spatial freq. (more phase wraps) in \( y \)-direction} \) \]

- Phase wraps persist after phase-encode gradient off

- Read-out gradient (\( G_x \)) phase wraps then add to phase-encode phase

2D Imaging Equation

\[
S(k_x, k_y) = \iint I(x, y) e^{-2\pi i (k_x x + k_y y)} \, dx \, dy
\]

- Signal recorded at single time point
- Complex signal (from phase-sensitive detection)
- Done by RF coil
- Scalar (what we try to reconstruct)
- Phase angle (of scalar magnetization!) in rotating frame, set by gradients

- Ignoring relaxation, spatial frequency \( k_x \) and \( k_y \) have no "inertia" — they stay wherever the gradients last left them.
3-D IMAGING - two phase-encode gradients

- Use z-gradient for 2nd phase-encoding instead of slice selection

- Excitation of whole slab (slice-select is whole brain)

- Simple spin echo example (in real life, usu. done with echo trains [FSE] or small flip angle to allow short TR [SPGR])

\[ S(k_x, k_y, k_z) = \text{signal recorded at single point of readout} \]

\[ I(x, y, z) = e^{-2\pi i (k_x x + k_y y + k_z z)} \]

- i.e., freq-encode phase, first phase-encode phase, and second phase-encode phase all just add (= 3D rotation of phase stripes)

- S/N much better than 2-D because each entire excited volume contributes to signal from each pulse instead of just slice

\[ \text{phase stripes created throughout volume vs. slice:} \]

N.B., this ignores relaxation effects for now
PHASE & FREQ, 2D & 3D

Since the phase-encode gradient and the freq encode gradient both affect phase, the result is a rotation of phase "stripes" when the two add.

N.B.: Stripes have sharp edges from phase wrap (not sinusoid since \( \phi \) from 2-comp quadrature!)

Stripes here represent complex value

Phase of whole image summed to one (complex) number by RF coils

successive read out steps:

1. Small phase encode \( G_y \)
2. Large phase encode \( G_y \)
3. Large phase encode \( G_z \) and \( G_x \) starts rotated in y-z plane

- 3D phase encode w/ \( G_y \) and \( G_z \) starts rotated in y-z plane

Example: after y-gradient, spins at a point might be 2 cyc ahead while after x-gradient spins at same pt 8 cyc ahead, but counting wraps in y-direction still only 2 ahead
Gradients move k-space location of data point

- k-space (spatial-frequency space) location is set by integral of gradient over time up to recording point:

\[ k = \int_0^t G(t) \, dt \]

spatial freq recorded at t = record time

- all of the following gradients end up at the same point in k-space:

- Frequency-encode (FID)

- Frequency-encode gradient echo

- Frequency-encode spin-echo (plus gradient echo !!)

- Phase-encode then frequency encode gradient echo

- N.B. 180° waves to conjugate point
\[
S(k_x, k_y) = \sqrt{\sum_{y} \sum_{x} I(x, y) e^{-i2\pi (k_x x + k_y y)} dx \, dy}
\]

\[
I(x, y) = \sqrt{\sum_{k_y} \sum_{k_x} S(k_x, k_y) e^{i2\pi (x k_x + y k_y)} dk_x \, dk_y}
\]

\[
S(k_x, k_y) = \frac{A \phi}{n^2} e^{i\phi}
\]

Adding exponents same as multiplying two \(e^{i2\pi k x}\)'s

Same as two sequential 1D FFTs (actual code)

In practice, finite number of samples, \(N\) and \(M\), are collected

\[
I(x, y) = \sum_{m=-M/2}^{M/2-1} \left[ \sum_{n=-N/2}^{N/2-1} S(n, m) e^{i2\pi n \Delta k_x \Delta k_x} e^{i2\pi m \Delta k_y y \Delta k_y} \right]
\]
**SAMPLING**

aliasing, FOV

- must consider effects sampling
  - limited points in k-space
  - limited in range of frequencies sampled \((k_{\text{min}} \rightarrow k_{\text{max}})\)
  - limited in rate of sampling \((\Delta k)\)

- N.B. aliasing less familiar when result of limited frequency domain sampling than limited space or time domain sampling

- correct reconstruction
  - as above w/ blurring, ringing

- as above w/ blurring, ringing
  - finite frequency range
  - finite spacing of samples

- aliasing occurs in spatial domain
  - replicas overlap, causing wraparound
  - finite frequency range
  - too-wide spacing of samples

thus, finer sampling of same range of spatial freqs increases FOV
UNDER/OVER SAMPLE

\[ FOV_x = \frac{1}{\Delta k_x} \]

\[ \delta_x = \frac{FOV_x}{N} = \frac{1}{N \Delta k_x} \]

FOV (distance to repeat) is reciprocal of spatial frequency sampling interval

Pixel size is FOV divided by K-space sample count

3 more examples (not incl. less samples to same spat. freq [bottom last page])

Basic Image

Same num samp. to 2X spat. freq. (i.e. gradients stronger or time ON longer)

2X num. samples to same spat. freq. (i.e. gradients weaker or time ON shorter)

2X number samples to 2X spat. freq. (i.e. gradients stronger or time ON longer)

Spatial freq. K-space

N=10
\[ k_x = 5 \]
\[ \Delta k_x = 1 \]
\[ FOV = 1 \]
\[ \delta_x = 0.1 \]

N=10
\[ k_y = 10 \]
\[ \Delta k_y = 2 \]
\[ FOV = 2 \]
\[ \delta_x = 0.05 \]

N=20
\[ k_y = 5 \]
\[ \Delta k_y = 0.5 \]
\[ FOV = 2 \]
\[ \delta_x = 0.05 \]

Space Pix

- basic image
- square pix

- x-pix half width
- replicas intrude
  [scanning makes square image "wrap" occurs]

- square pix
- twice \( x \)-pix count so FOV=2X
  [scanning crops to square replicas move out]

- x-pix half width
- twice \( x \)-pix count
  [same FOV]

- this is decrease pixel size w/o change FOV
Fourier Transform Solution to Replicas

1. image/brain space
2. sampled data spatial frequency

- Convolve
- Multiply

Useful FTs

- Rect
  \[ \text{Rect}(\frac{x}{w}) \xrightarrow{\mathcal{F}} W \cdot \text{sinc}(\pi Wk) \]

- Gaussian (special case)
  \[ e^{-\pi x^2} \xrightarrow{\mathcal{F}} e^{-\pi k^2} \]

- Gaussian (adj width)
  \[ e^{-\alpha x^2} \xrightarrow{\mathcal{F}} \frac{\sqrt{\pi}}{\sqrt{\alpha}} e^{-\frac{\pi k^2}{\alpha}} \]

- Comb
  \[ \sum_{n=-\infty}^{\infty} \delta(x - \frac{n}{\Delta k}) \xrightarrow{\mathcal{F}} \Delta k \sum_{p=-\infty}^{\infty} \delta(k - p\Delta k) \]

Limit approach to Fourier transform of comb

\[ \text{FOV} = \frac{1}{\Delta k} \]
\[ \Delta k = \frac{1}{\text{FOV}} \]
Point-Spread Function

\[ \hat{I}(x) = \Delta k \sum_{n} S(n\Delta k) e^{i 2\pi n \Delta k x} \]

- Set true image to \( S \)-function, then measured signal is:
  \[ S(n\Delta k) = 1 \]
- Substitute into recon to get PSF:
  \[ h(x) = \Delta k \sum_{n} e^{i 2\pi n \Delta k x} \]
- Simplify
  \[ h(x) = \Delta k \cdot \frac{\sin(\pi N \Delta k x)}{\sin(\pi \Delta k x)} \Rightarrow \text{periodic} \]
- That is, image is reconstructed from a sum of sinc's, because the FT of a boxcar pixel in \( k \)-space is an image sinc.

- How PSF modifies ideal (infinite \( k \)) image
  - Convolve
  - FT
  - FT (under rect) narrower sinc
  - Multiply
  - Spatial freq. data
  - Acquisition window (truncates high spatial)
GENERAL LINEAR INVERSE RECON FOR MRI

\[ S(k_x) = \int I(x) e^{-i2\pi k_x x} dx \]

Signal eq. \( \rightarrow \) fwd problem

\[ I(x) = \sum_{k_x} S(k_x) e^{-i2\pi k_x x} \]

Recon eq. \( \rightarrow \) inv. problem

\[ \hat{s} = \hat{F} \hat{i} \]

\[ s = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} F_i \end{bmatrix} \]

Linear "forward solution"

Matrix & vectors have complex entries

Can build in any measurable priors

\[ F_{x,y,t} = g(x,y) e^{-i\Phi(x,y)} \left( e^{-\left(\frac{nT \pm m\Delta T + TE}{T_2}\right)} - i\frac{\gamma B (x,y, T_2 \pm \Delta T)}{\Delta y} \right) \]

Cal gain at this location

Cal phase

T2 decay

\( \Phi \) error

Freq. + phase

(complex)

Multi-cal:

\[ \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} k_x \\ k_y \end{bmatrix} \begin{bmatrix} F_{\text{coil 1}} \\ F_{\text{coil 2}} \end{bmatrix} \]

Naturally incorporates undistorted field map

different sensitivity function for each coil
contains additional info about some loc.

But, need reference scan, lo-res OK

(need phase corrections for each coil)

\[ i = F^+ s \]

over-determined

More Precise inverse

\[ F^+ = (F^T F)^{-1} F^T \]

More Precise inverse

\[ F^+ = F^T (F F^T)^{-1} \]

Slice-by-slice

\[ i = \left[(F^T)^{-1} F^T\right] s \]

Assume slice select swamps other
Fast Spin Echo (FSE)

One 90° pulse followed by multiple 180° pulses (e.g., 8) each with a different phase-encode gradient.

Each phase "winder" is "unwound" because left over phase would be re-focused away by 180° (i.e., EPI where it persists between blips).

The effective TE is the TE when center of k-space is collected (largest effect on contrast, largest echo).

Each subsequent echo has more T2 decay: $E_n = e^{-nTE/T2}$, $n = 1, 2, ..., M$.

By arranging to collect $k_y=0$ early, PD-weighted instead of T2-weighted.

Possible to correct different T2-weighting of echoes by estimating T2 curve from $G_y=0$ echo train.

3DFSE — like 2D except wind/unwind added to thickness slice select (with markers on 180°) $G_z$ (phase 2) $G_y$ (phase 1) $G_x$ (field).
**MULTI-SLAB 3DFSE, PROBLEMS**

- RF$_{in}$
- G$_z$
- Gy
- G$_x$
- RF$_{out}$

- Echo train e.g., 20 etc to fill 3D k-space

- G$_z$ is "partition"
- Gy is "phase encode"
- G$_x$ readout needs no pre-wind since 180$^\circ$ does it.

- TE$_{eff}$ is time from 90$^\circ$ to echo thru center of k-space

- Echoes die out quickly by $e^{-t/12}$
- Since echoes after 90$^\circ$ limited to <30, can't fill 3-D k-space in a reasonable time

- SAR constraint $\text{SAR} \propto B_0^2 B_1^2 \Delta f$

- 180$^\circ$ pulses deposit 4-6x power of 90$^\circ$

- "multi-slab" is halfway between slices and single-slab

- Problem at slice boundaries — esp. movement

- Multislab requires slice selective RF pulses — longer than non-selective 'hard' pulses

- Rough approx. for non-adiabatic standard pulses

- Single slices
- Multi-slab
- Single slab

- 4 ms RO

- Hard to get under 8 msec inter-echo spacing

- Limits speed of covering k-space
**SINGLE-SLAB 3D FSE**

- Regular FSE (180° pulse train)
- Sub 180° pulses cause each successive pulse to also generate a stimulated echo (STE)
- This "storage" in Z-axis preserves magnetization for longer time
- Smaller flip angles allow much longer echo trains
- Enough to collect whole plane of 3-D k-space
- Different than hyper-echoes (not symmetric)
- Contrast must consider STE

\[
SE = \sin \alpha, \sin^2 \frac{\alpha}{2} e^{-\frac{2\pi}{T_2}} \quad \text{STE} = \frac{1}{2} \sin \alpha \sin^2 \frac{\alpha}{2} e^{-\frac{2\pi}{T_1}} e^{-\frac{2\pi}{T_2}}
\]

- Single-slab 3D FSE pulse seq,

Variable flip angle (<1 msec)

Hard (non-selective) pulse not 180°

\[
\text{Echo num} \quad \text{Echo train num (partition num)}
\]

\[
\text{Long TE eff (T2)} \quad \text{Short TE eff (T1)}
\]

NB: time to count k-space is \(0.5X\) apparent contrast time b/c of "storage"
(e.g. TE eff = 585 ms looks like FSE TE = 140 ms)
FAST GRADIENT ECHO

- small tip so TR can be greatly reduced (e.g. 10 msec, less than T2)
- 'leakage' undelayed transverse magnetization \[ \text{"unwound" and re-used \"spooled\" before next shot} \]

RF \[ A^\alpha \]
select (balanced)
\[ G_2 \]
phase unwind
\[ G_y \]
makes GE\[ \rightarrow \text{read} \]
unwind read
\[ G_x \]
\[ \text{Signal} \]
TR - as small as 10 msec

STEADY-STATE COHERENT (GRASS, FISP)
- unwind phase from phase-encode \( M_z \)
  before next pulse (here because \( TR < TE \))
- unwind read gradient, too
\[ S = k \sin x \left[ \frac{1}{1 + \cos x + (1 - \cos x) T_1/T_2} \right] e^{-T_2/2} \]
  \( T_2/2 \text{- weighted contrast (bright CSF)} \)
  \( \text{brain 0.1, fat 0.3, CSF 0.7} \)

STEADY-STATE SPOILED (SPGR, FLASH)
- spoil with random gradient (but this still allows some \( \alpha \) refocusing)
- spoil with gradient plus incremented phase of RF pulses (RF spoiling)
- good gray-white contrast (T1-weighted)

(Shown as 3D sequence - possible with ones above, too)

NON-STEADY STATE, MAGNETIZATION-PREP

RF \[ 180^\circ \]
\[ \alpha \]
\[ G_2 \]
\[ G_y \]
\[ G_x \]
\[ \text{TR \times 10 msec} \]

- preparation pulse \( \rightarrow \text{strong T1-weighting} \)
- contrast varies in spatial - freq-dependent way

MP-RAGE
- longitudinal mag. not affect much by low angle pulses
- record \( k_y = 0 \) here
QUANTITATIVE T1 — INTRO, METHODS

Motivation

- Image values are arbitrary/relative (cliff seg's, manufacturers)
- Uncorrected coil fall-off (receive inhomogeneity) can result in 2-3x differences in voxel brightness
- Uncorrected variation in local B1 field can cause contrast variation
  - At 3T, B1 can vary by 25% across the brain
  - This can invert contrast in a fast gradient echo

Pre-scan normalise

- Collect lo-res GE image, receive w/ body coil (no coil fall-off)
- Set params to get low GM/WM contrast
- Collect data scan (e.g. MPRAGE) w/ surface coils, strong GM/WM
- Use ratio between scans to generate smooth correction field

T1 divided by T2

- MPRAGE → strong T1-Contrast
- SPACE → T2-weighted (no T1 weighting)
- T1/T2 removes coil fall-off
- Problems: distortion different in GE (MPRAGE) and SE (SPACE)
- Noise in regions of low signal

MP2RAGE

RF 180°

Gz / / / / / / / / center of k-space
Gxy / / / / / / / / copy 2
Gz / / / / / / / /

TI1 → TI2

- N.B. SSFP-like in partition, phase-encode directions
- Convert to -0.5 to 0.5 image: \( S = \text{real} \left( \frac{S_{TI1} \times S_{TI2}}{||S_{TI1}||^2 + ||S_{TI2}||^2} \right) \)
- Calc. PD & T1 from above (cf. 2 flip angles)
QUANTITATIVE T1 - HELMS 2-FIIP ANGLE METHOD

- Start w/ gradient echo signal e.g., dropping T2 decay: $e^{-TR/T2}$
- Simplify/linearize estimate
  \[ TR \ll T1 \]
  Linear approximation of exponentials
  Taylor expansion simplification of $\sin$, $\cos$, drop small terms

$$ S_{\text{est}} = A \cdot \sin x \cdot \frac{1 - e^{-TR/T2}}{1 - \cos x \cdot e^{-TR/T2}} $$

Ernst eq.
  \[ (\text{max: } \cos x_E = e^{-TR/T2}) \]
  "Ernst angle"
  \[ x_E = \cos^{-1}(e^{-TR/T2}) \]

Helms et al. (2006)

$$ S \approx A \cdot x \cdot \frac{TR/T1}{x^2/2 + TR/T1} $$

- Solve for TD and $A$ (proton-density) given signals from 2 diff flip angles

\[ T1_{\text{est.}} = 2TR \cdot \frac{S_1/x_1 - S_2/x_2}{S_2x_2 - S_1x_1} \]
\[ A_{\text{est.}} = S_1S_2 (x_2/x_1 - x_1/x_2) \]
$$ S_2x_2 - S_1x_1 $$

- Problem: flip angle varies a lot at 3T (e.g., 25%) from nominal/requested (e.g., flip series)

2FIIP angles

- acq.: spin-echo and stimulated echo (EVI)

$$ S = k \cdot \sin^3 x \cdot e^{-TE/T2} $$
$$ S_{\text{STE}} = k/2 \cdot \sin^2 x \cdot \sin 2\alpha \cdot e^{-TM/T1} $$
$$ \alpha = \cos^{-1}\left(\frac{S_{\text{STE}} \cdot e^{-TM/T1}}{S_{\text{SE}}} \right) $$

T2* - add EPI-like echo train to each FLASH excit.
Echo Planar Imaging (EPI) (another fast gradient echo)

- Single shot EPI collects all k-space lines (e.g., 64) after a 90° RF pulse using a train of gradient echoes.

- Since there is only one RF pulse per slice, spins never get reset to all-the-same (= zero freq, center of k-space).

- Therefore, the recording point (Δt) in k-space (= spin phase stripe pattern) stays wherever the x and y gradients last left it.

- That explains why successive y phase-encode steps are achieved without changing the size of the G_y "blips".

- Echoes are T2*-weighted (gradient echo).

- Contrast mainly determined by echoes near center of k-space, which are only recorded after about 32 echoes.
SPIN ECHO EPI

why SE-BOLD may be selective for capillary bed

- Standard EPI is a gradient echo method, which results in T2*-weighting

- Deoxyhemoglobin is paramagnetic, which reduces signal in a T2*-weighted image due to greater dephasing

- The excess of deoxyhemoglobin (probably the result of the need to drive O2 into tissue, which requires more O2 in the blood than is actually used) leads to the positive BOLD effect

- Spin echo corrects (cancels) static T2* (T2') dephasing, incl. deoxy

- If all spins stayed in the same position, spin echo would eliminate the BOLD effect by eliminating dephasing

- Diffusion exposes spins to different fields (reducing gradient echo dephasing)

- Magnetic field gradients produced by large vessels are smoother across space than those produced by small vessels

- For TE \approx 100 \text{ ms}, spins diffuse 10's of \mu m, which is larger than diameter of small capillaries, meaning that spins will likely experience different fields over time

- Therefore, spin echo will be less successful at canceling BOLD effect near small vessels (BOLD effect will be reduced near large vessels where diffusion is less likely to expose spin to different fields)

- This argument only works for extravascular spins — intravascular signal in BOLD is large (despite being only 4% by volume) because large gradient produced around red blood cells

- Measure intra/extraw ith bipolar pulse which kills signal in faster moving blood in moderate and larger vessels

Over half of SE-BOLD at 1.5T is venous...
SPIN ECHO EPI

- EPI is a multi-gradient echo pulse sequence.

- "spin-echo EPI" uses a 180° pulse to add a single spin echo to the contrast-controlling gradient echo through the center of k-space.

- "asymmetric spin-echo EPI" arranges for the spin echo to occur 1 msec before the gradient echo, which gives more T2*-weighting (for ky=0 echo).

- The 180° pulse rephasing reduces the T2* signal, which is why the partially replaced asymmetric spin echo has been more commonly used.

- At higher fields, spin echo EPI is more promising:
  - Signal to noise is higher so we can take spin echo hit
  - Contribution from venous blood is reduced, since blood T2 is so short, we can let it decay away before recording.
- **Coil fall-off** intuitively contains info about location if same brain location imaged by different coils w/ diff. fall-offs

  > but what does this look like in k-space?

- Slow variation in RF-field fall-off (e.g., 1-4 cyc/FOV) causes a blur in acquired data in k-space  
  > (N.B. not addition!)

- To see this, consider **multiplication** by coil fall-off function in image space, which equals **convolution** (w/ FT of that function) in k-space - at all spatial frequencies!!

- Simple example w/ "brain" consisting of one spatial freq. in image domain

  
  ![Image](image.png)

- N.B. inverse FT of k-space data "smeared" in spatial freq. space is sharp image w/ fall-off (not blurred img.)

- "Smeared" means normally orthogonal spatial freq's "leak" to adj. freqs.

  $$\text{GRAPPA} - \text{construct k-space "kernel" to fill in missing k-space lines by training on fully-sampled data from near k-space center}$$

  $$\text{SENSE} - \text{general linear inverse approach}$$

- N.B.: neither would work unless normally orthogonal spatial freqs. blurred!
Multiband, EVI, Spiral

- Excite multiple slices at once
- Function of $G_2$ blips is to shift slices in $G_y$ direction
- This occurs because for given slice, a phase pedestal is added to the entire slice
  \[ \Rightarrow \text{this } "\text{Fourier Shift Theorem}" \]
  \[ \Rightarrow \text{[N.B.: different than } B_0 \text{ defect-induced incremented phase errors]} \]
- Problem w/ all up $G_2$ blips $\Rightarrow$ phase error builds up

*trick*#1
- Start w/2 slices, one at $z=0$, other above
  \[ \Rightarrow \text{if } \pi \text{ (180°) phase shift used, blip up/down same! (no effect at } z=0) \]
  \[ \Rightarrow \text{i.e., move top or bottom replica} \]

*trick*#2
- For multiple slices not all at $z=0$, phase no longer same for even/odd
  \[ \Rightarrow \text{but can add phase to equilibrate to k-space before recon.} \]

*trick*#3
- For more than 2 slices:
  \[ \text{1st even odd even odd etc.} \]
MULTI BAND/BLIPPED CSPI (cont.)

- relation between leave-one-out aliasing and nominally fully-sampled SMS

- leave alternate lines out wraps image
- SENSE/GRAPPA to fix b/c coil view swears K-space data
- nominally, w/ SMS we record every line of K-space
- but equivalent to leave alternate out b/c our multi-slice FOV was not big enough

- slice-GRAPPA

  - reg GRAPPA -> recon missing lines
  - slice GRAPPA -> recon multiple K-spaces

i.e. not for each overlapped slices by training on fully-sampled data at beginning of scan

- inter-slice "leakage block"

  - when training GRAPPA kernel on fully-sampled data, also minimize inter-slice leakage (split-slice-GRAPPA)

  - can also do regular GRAPPA on top of this
  - reason: for diffusion, loss in S/N from undersample cancelled by shorter TE readout

  - gain from reduced more distortion from shorter readout
ECHO-VOLUME IMAGING EVI

- Multi-shot (like FLASH) but acquiring one plane of 3-D k-space per shot (can do spin echo, too)

- Main issue is movement artifact since data assembled from many shots over several secs

- Breathing-induced BP problems in different partitions may cause blur

- Entire k-space must be filled before 3D image is reconstructed

- Since entire volume is excited each shot, potentially higher S/N

- Must use smaller flip angle to avoid killing M_L since entire volume excited every partition (e.g. every 30 msec)
SPIRAL IMAGING

- By using smoothly changing gradients (sinusoids), less gradient power required than w/trapezoids (less eddy currents).

- Earlier EPI hardware like this: sinusoidal gradient waveform from resonant circuit w/non-uniform sampling to get constant $\Delta k_x$.

- Sinusoids in both $G_x$ and $G_y$ give spiral $k$-space trajectory.

RF

$G_x$

$G_y$

$G_x$

$G_y$

Sig [Sample continuously]

- Constant angular velocity goes too fast at large $k_x$, $k_y$.
- Constant linear velocity better but impractical near $k_x=0, k_y=0$.
- Compromise: start constant angular, end constant linear.

**Constant angular velocity**

$$w(t) = w_0 t$$

$$k(t) = A t e^{i w_0 t}$$

$$G(t) = \frac{1}{A} \frac{d}{dt} k(t)$$

$$= A e^{i w_0 t} + i A w_0 e^{i w_0 t}$$

$$G_x(t) = A \cos w_0 t - A w_0 \sin w_0 t$$

$$G_y(t) = A \sin w_0 t + A w_0 \cos w_0 t$$

**Constant linear velocity**

$$w(t) = \frac{A}{20} T E$$

$$k(t) = A T E e^{i w_0 T E}$$

$$G(t) = \frac{1}{A T E} \frac{d}{dt} k(t)$$

$$= \frac{A}{20} e^{i w_0 T E} + \frac{A}{2} \frac{w_0 e^{i w_0 T E}}{20}$$

$$G_x(t) = \frac{A}{20} \cos w_0 T E + \frac{A}{2} w_0 \cos w_0 T E$$

$$G_y(t) = \frac{A}{20} \sin w_0 T E + \frac{A}{2} w_0 \sin w_0 T E$$

$G_x$

$G_y$
**SPIRAL 3D IR FSE** (from Eric Wong)

- **3D**: block select vs. slice select
- **FSE**: multiple echoes from one 90°
- **spiral**: multiple spirals vs. multiple lines
- **interleaved spirals** (like FSE interleaves)
- **true IR (vs. MPRAGE)**: all echoes after 90° derive from mag w/ same T1 contrast (vs. non-steady-state)
- possible to preserve sign
- high, uniform contrast, but lots of waiting (T2), high BW

180° (prep1) TR=700 msec  
180°  
180°  
180° (prep2)

**RF**  
**Gz**  
**Gy**  
**Gx**  
**Sig.**

**Loop order**

3D k-space  
("stack of spirals")
PHASE ERRORS & ECHO-CENTERING ERRORS

anything that causes a deviation of the \( B_z \) field strength from the expected value \((B_{o,z} + G_{x,z} x + G_{y,z} y + G_{z,z} z)\) changes precision frequency and therefore, expected phase angle

- incorrect phase of spatial frequency stripes results in a shift in space in the magnitude image after reconstruction

- first defense: freq prescan
- refine w/shimming and \( B_0 \)-mapping/phase unwrapping before reconstruction

- if realignment of all spins \((k_x = k_y = 0)\) doesn't occur at the middle of read gradient, echo is shifted
- since echo is in spatial frequency domain, this is frequency shift

- spatial frequency shift results in wrapping in phase image after reconstruction

\[
I(x - x_0) = \int e^{-i2\pi k_x x} S(k_x) e^{i2\pi k_x x} dk_x
\]

N.B.: this is a "pedestal" of phase, not a gradient

- Fourier freq. shift theorem
- freq. shift in freq. domain causes phase shift in spatial

magnitude noise

\[
e^{i2\pi k_x X} I(x) = \int S(k_x - k_0) e^{i2\pi k_x x} dk_x
\]
FAST SCAN ARTIFACTS

EPI vs. Spiral

Brain-induced field defects lead to phase errors

**EPI**
- $G_x$ readout gradient strong $\rightarrow$ field defects smaller percentage less deformation of $k_x$ (vertical stripe components)
- Gy "blips" are weak and total Gy readout time much longer (5 times) than standard readout (50 ms vs. 10 ms)
- An extra gradient in the x-direction, for example, maps and unmaps phase as a function of x-position
- But $G_x$ big, so effect on freq.-encode direction is much less than on phase-encode direction, where error accumulates

**Spiral**
- With center-out spirals phase errors accumulate in a radial direction
- Thus, higher spatial frequencies have more error (= more shearing)
- For spurious x-direction gradient as above, there is a radial blurring, rather than a vertical shift because higher frequency phase stripes misaligned relative to low spatial freq.

- For defects with more complex contents in the y-direction (than linear, as above) the vertical shifts (in EPI) will vary with y-position, and may result in signals from different y-positions being reconstructed on top of each other
- Localized BF defects often arise from air pockets embedded in tissue
  - air in middle/outer ear → indentation in inferior temporal lobe
  - air under olfactory epithelium → orbitofrontal dy, ant, thal. compression

- Collect one data (k-space) point

- Localized BF defect

- Complex multiply
  - = correlate sin/cos with brain

- Brain structure sampled with distorted stripes

- One complex number

- Reconstruction from distorted data points

- Undistorted stripes used by inverse FFT

- Same defect makes leftward dent in vertical phase stripes

- Spatial information can be lost when continuous changes in phase are flattened by BF defect
  - $\Delta \phi$

- Shifts can pile multiple pixels on top of each other into one bright pixel

- Local estimates of $\Delta \phi$ needed to correct images
  1. Fieldmap method: Shift each image pixel proportional to $\Delta \phi$ from TE slope
  2. Point-spread-function: Extra phase encode to estimate PSF (should be $\delta$-function) deconvolve distorted image in phase encode direction

N.B.: Image shift only occurs if shift coil k-space sampled w/ successively later echoes times (see next page)
LOCALIZED Bφ DEFECT, EFFECT ON RECON

- when local Bφ defect disturbs image space phase stripes during signal acquisition, estimates of local spatial freq. are affected (compressed stripes = higher spatial freq.)

- if each successive ky line recorded with same echo time (e.g., w/ single line phase encoding) this will correspond to constant spat. freq. offset in k-space

- a k-space freq. offset only results in image space phase shift (Fourier freq. shift theorem), which is invisible in amplitude image (cf. echo cont. error)

- however, with w/EPI, static Bφ defect causes more and more local displacement of image phase stripes for each additional ky line

  - that is, later lines have greater spat. freq. offset
  - effectively stretches k-space in ky direction
  - same num samples to higher spatial freq. shrinks FOV (squished voxels — see FOV page)

- when image is reconstructed, region with local Bφ defect shifted oppositely

- Thus, local shift effect due to combination of 3 things:
  1) static local ΔBφ defect
  2) successive increases in phase error for successive spat. freq. measurements during long EPI readout
  3) small size of ky phase encode blips relative to Bφ defect (much less of this effect in freq. encode direction)

- respiration (which affect Bφ) in 3D FLASH might cause similar effect within k2 partition (if successive spatial freqs.)
GRADIENT NON-LINEARITIES

- Ideally, the $G_x$, $G_y$, and $G_z$ gradient coils attempt to impress a linear variation onto the $z$-component of the $B$ field — $B_z$ — in the $x$, $y$, and $z$-directions.

- In practice, gradient coils are non-linear (esp. printed-circuit-like).

- Non-linearities are worse in smaller coils, but also in higher performance coils, designed for post-processing correction of distortions.

- Non-linearities result in phase errors, which result in 3-D image distortion.

  - A non-linear slice-select gradient will excite a curved slice.
  - Non-linear phase and frequency encode gradients will distort in-plane features.

- Some scanners correct these differently for 3-D scans (all directions), 2-D scans (just in-plane), and EPI scans (no corrections!).

- This can result in errors approaching 1 cm in funct-struct overlays.

- Different coil designs have different directions of distortion (!).

- The assumption of non-Maxwellian gradients results in additional phase errors.

- These can also be corrected since the $B_x$ and $B_y$ components are known.

- These effects do not build up over time in phase-encode direction since they only occur when gradient are turned on.

- Fourier shift theorem

- These distortions are predictable and can be corrected.

- That is, the assumption that gradients cause no field in the $B_x$ and $B_y$ direction.
SHIMMING AND \( B_0 \)-MAPPING

- Passive iron shims inserted to flatten \( B_0 \) field

- Additional coils (usu. in the gradient coils) can be statically energized in an attempt to flatten the \( B_0 \) field (a few ppm good)

- Primary use is to compensate for defects in flatness present without a sample in the magnet (geometric imperfections, impurities in metal, etc.) (= several hundred ppm)

  \[
  \text{linear shim coils impose gradients in } x, y, \text{ and } z \\
  \text{higher order shims impose higher order spherical harmonic field components (e.g. } z^2) \\
  \]

- Secondary use is to compensate for inhomogeneities caused by introducing the sample into the \( B_0 \) field

- Local resonance offsets caused by \( B_0 \) defects estimated from images
  - \( \Rightarrow \text{e.g., sample phase at multiple echo times} \)

- Fit defective field using combination of fields generated by shim coils, then add these corrections to base shim currents
  - \( \Rightarrow \text{this only corrects spatially gradual field defects} \)
  - \( \Rightarrow \text{local defects due to air in sinuses much higher order than shims} \)

- After shimming, field map measured again

- Image voxel displacements calculated from resonance offset map are used to un warp the reconstructed magnitude image

- For EPI images, assume displacements all in phase-encode direction (since freq encode gradient is strong relative to defects)
**NAVIGATOR ECHOES**

- **1D navigator**
  - **B0 drift problem**
    - Slow up/down drifts in B0 continuously occur
    - A pedestal in B0 is pedestal in phase (not gradient)
      which causes spatial shift (Fourier shift theorem)
    - In EPI, mainly affects phase-encode dir b/c small slip
     .imag readout
    - Result is successive volumes drift in phase encode dir

- **Gradient balance problem**
  - Unequal L/R readout gradients cause L/R shift in position of even/odd lines in k-space
  - Causing N/2 (Nyquist) ghosting → another phase error

- **3D navigator**: collect 3D sphere in k-space
  - Rotation of object → rotation of k-space amplitude pattern
  - Translation of object → phase shift of k-space phase (Fourier shift)
  - Sample at sufficient radius to pick up high spatial freq features
  - N.B.: excite whole volume
  - Do N,S hemispheres separately (less T2*, cancel EPI-like error accumulation)

  Walsh et al. (2002) MEM

 Ẹọ ọtọ ọtọ → equator → up, equator → down

- RF
  - 90°
  - Crush

- \[ z(n) = \frac{2n - N - 1}{N} \]
- \[ y(n) = \cos(TN\pi \sin^{-1}z(n))(1-z(n)) \]
- \[ x(n) = \sin(TN\pi \sin^{-1}z(n))(1-z^2(n)) \]
- (skip poles — slew rate too high)

- Can be used for prospective motion correction (rotate, translate w/ gradients)
- Better estimate, because of speed, than full TR of EPI images (27 ms vs. 2.4 sec)
- May need to smooth rot, trans estimates across time (e.g. Kalman filter)
RF FIELD INHOMOGENEITIES

- receive coil inhomogeneities alter the amplitude of the received signal, altering the reconstructed proton density in a spatially varying way
  - variations can be used (cf. GRAPPA, SENSE) and/or corrected

- transmit coil inhomogeneities affect the flip angle in a spatially varying way (can affect contrast: FLASH)
  - potentially worse (why local transmit is still in progress)
  - usu. fixed by using a large transmit coil (e.g. body coil)

- RF penetration at higher fields (\(\leq\) higher RF frequencies)
  is less uniform:
  1) decreased RF wavelength (closer to size of head) at higher freq.
  2) increased permittivity (\(\varepsilon\)) and conductivity (\(\sigma\)) at higher field

- 2nd advantage of the falloff in signal recorded with a small, receive-only RF coil is better signal-to-noise (less noise received from other parts of brain)

- different sensitivity functions from different coils can be used to scan less lines in k-space (GRAPPA/SENSE/SPACE-RIP)

  - record lo-res volume (b/c coil falls off is smooth) through both body coil and small coil(s)
  - divide small coil body coil at each voxel to determine receive field
  - use receive field to normalize main image(s)

  [see also: qT1, MP2RAGE, T1/T2]
**Diffusion - Weighted Imaging**

- **Simple Diffusion Weighting**
  - RF 90° \( T \) (or Δ)
  - Select \( G_2 \)
  - \( G_y \) (not a practical sequence)
  - Prepare \( \rightarrow \) Readout

- **Apparent Diffusion Coefficient**
  - Calculate from \( b=0 \) image and at least 6 \( b \)-“large” (e.g., 1000) images.
  - Two useful scalar values from 3 eigenvalues:
    - \( A_D = MD = (\lambda_1 + \lambda_2 + \lambda_3)/3 \)
    - Fractional Anisotropy \( FA = \sqrt{\frac{(\lambda_1 - \lambda_2)^2 + (\lambda_2 - \lambda_3)^2 + (\lambda_3 - \lambda_1)^2}{2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)}} \)

- **Anisotropic Diffusion** (Gaussian)
  - Measure \( D \) along multiple axes.
  - Have to measure tensor, not scalar.
  - Even for determining one primary direction.

- **Diffusion Surface** (non-Gaussian)
  - Need to measure diffusion in many directions (>6) to properly characterize even 2 main directions.

- **Length Scale by Multiple b-values**
  - Fit line to semi-log signal as function of \( b \).
  - If not straight line: multi-exponential, e.g., hi ADC/fast/extra vs. lo ADC/slow/intercellular.

- Assume Gaussian diffusion
  - Spins acquire phase during first \( \Delta t \)
  - If spins diffuse (move) along gradient by time \( T \), signal is lost because negative \( \Delta t \) doesn't re-phase.

- Attenuation:
  \[ A(D) = \frac{S_0}{S_b} = e^{-bD} \]

  where:
  \[ b = \gamma^2 G^2 \Delta t^2 \left( T - \Delta t/3 \right) \]

- Long \( T \), \( T \) can give spurious T2-weighting.
- Can use stimulated echo to get long T w/o less T2-weighting.

**1)** Anisotropic Diffusion (Gaussian)

- Measure \( D \) along multiple axes.
- Have to measure tensor, not scalar.
- Even for determining one primary direction.

- Isotropic: \( \lambda_1 = \lambda_2 = \lambda_3 \)
- Since \( D \) is symmetric, need minimum of 6 different measurement directions.

**2)** Length Scale by Multiple b-values

- Fit line to semi-log signal as function of \( b \).
- If not straight line: multi-exponential, e.g., hi ADC/fast/extra vs. lo ADC/slow/intercellular.

- Diffusion Tensor Measurement Direction

**Diffusion Surface (non-Gaussian)**

- Need to measure diffusion in many directions (>6) to properly characterize even 2 main directions.

- Scalar diffusion
- Diffusion tensor measurement direction

**Tract Tracing**

1. Markov process
2. Crossing fibers
3. "Freeway ramp" prob
4. Sharp turns into gyri

**Voxels vs. Tract Tracing**

(e.g., assume \( A_1 = A_2 = 0.5 \))

- D=0: both D=0
- D=3: both D=3

- Log signal (Mulkern, 1999)
- E=BD^2 (good)
- Curve
**PRACTICAL DIFFUSION-WEIGHTED PULSE SEQ**

- spin-echo 'Stejskal-Tanner' EPI seq. (standard on scanners)

  - allows longer TE

    - 90°
    - 180° flips $M_z$ so rephase gradient, same sign as dephase

    - [expanded in time for clarity]

    - $\text{TE}_{\text{eff}}$
    - $\text{TE}_{\text{eff}}/2$

  - eddy-currents are long time-constant currents in metal of scanner that distort B field → spatial image distortion

  - "doubly-refocused" spin echo sequence (DSE) can cancel the effects of eddy current (w/ partic. time constants)

    - (also, keep crushers orthogonal to diffusion-encoding gradients)

    - $Y_{\text{TRSE}} = 0 = Y_1 - Y_2 - Y_3 + Y_4$

    - Nagy et al., (2014) MRM

    - Phase dispersion (6 echo)

    - $\text{TE}_{\text{eff}}$

    - Twice refocused Spin-echo (for center k-space)
**Perfusion - Arterial Spin Label**

- **Basic idea:**
  - Tag blood below area of interest, collect control & tagged image, assume directional input flow!
  - $\text{tag is } 180^\circ \text{ pulse}$
  - Sign not problem when delay long enough (see below)

  - Continuous ASL (CASL) — continuously tag a plane, greatest on, blood gets adiabatically inverted as it passes through location w/corresponding resonant tag.
  - Pulsed ASL (PASL) — e.g., EPISTAR, FAIR, PICORE, QUIPPS II
    - Tag block of tissue below slice(s)

- Small diff. between control and tag (~1%)
  - Requires accurate balancing of control & tag images, control mag. transfer
  - Transit delays $\rightarrow$ biggest confounding factor
    - Relaxation rate diff.
    - Venous clearance (vs. microphones, which get stuck!)

- Solutions for quantitative
  - Invert delay so all spins arrive into low-velocity capillaries
  - Kill end of tag to reduce spatial variation of tag

- **QUIPPS II** — Quantitative Perfusion
  1. Pre-saturate spins in target slices
  2. Tag — $180^\circ$ pulse below slices
  3. Control — $180^\circ$ pulse above slices (to control off-resonance)
  4. Saturate tagged block to end tag ($T_1$)
  5. Both tag and control
  6. Can use train of thin slices pulses at top of tag band
  7. EPI or spiral images of target slices ($T_2$)
  8. Image most distal slice first to cancel delays
  9. Fast between slice so imaging excitations don't get interpreted as flow

\[
\Delta M \approx \text{flow} \times \left[ 2M_0 \cdot TI, e^{-T_2/T_1} \right]
\]

- $\Delta M$ extract flow and BOLD
  - Adjacent substrata minimize movement artifact

1. Alternate tag and control, GRE TE = 30 ms
2. Dual echo spiral
  - $k=0$ early $\Rightarrow$ hi S/N flow
  - TE = 30 ms $\Rightarrow$ BOLD
PERFUSION – pCASL

- original CASL (continuous arterial spin labeling) requires
  RF on continuously to adiabatically invert blood flow
  through one plane
  → can only image one slice (due to dephasing from gradient)
  → hard to keep RF on continuously on modern scanner (esp. BC)
  → can use special purpose RF transmit (separate xmt channel)

A) original CASL

<table>
<thead>
<tr>
<th>RF</th>
<th>G2</th>
</tr>
</thead>
<tbody>
<tr>
<td>image formation module (&quot;readout&quot;)</td>
<td>multiple possibilities</td>
</tr>
</tbody>
</table>

B) pCASL – pseudo-continuous arterial spin labeling  
Dai, Alsop (2008)

<table>
<thead>
<tr>
<th>RF</th>
<th>G2</th>
</tr>
</thead>
<tbody>
<tr>
<td>begin readout</td>
<td></td>
</tr>
</tbody>
</table>
- problem: multiple pulsers create aliased slice planes
- use: convolution of 2 function multiplying their FTs
- aliased labeling planes at: β = w / Δt in frequency space, modulated by broad sinc(πβΔt)
- use Hamming or hyperbolic secant to reduce replicas

C) pCASL w/ shaped gradients

<table>
<thead>
<tr>
<th>RF</th>
<th>G2</th>
</tr>
</thead>
</table>
| (control called a "transposed" pulse) | tag pulses have phase offset respecting gradient
| — tag pulses have phase offset respecting gradient | control identical except every other has π phase
| Readout | FLASH |
| — control identical except every other has π phase | EPI |
| — control identical except every other has π phase | SMS |
| SMS | Stack spirals 3D |
| 0.8 ms ac | no net flip |
**Off Resonance Excitation**

- **Main Idea:** Examine evolution of $\vec{M}$ vector in rotating coordinate system set to "off-resonance" $\vec{B}_1$ field freq ($\omega_f$), not Larmor freq $\omega_0$.

- Normally, if rotating coord syst freq set to Larmor freq ($\omega_f = \omega_0$), an actually precessing $\vec{M}$ will be stationary (ignoring decay) $\Rightarrow$ implies effective $B_z = 0$ in rotating.

- Now, move $\vec{M}$ to rotating coord syst at $\vec{B}_1$ freq lower than $\omega_0$ (assume $\vec{B}_1 = 0 = \omega_f$): Existing $\vec{M}$ will now appear to precess around $z$-axis.

  \[ \omega_0 = \omega_f \]

  \[ \Delta \omega_0 = \omega_0 - \omega_f \]

  - freq of precession in rotating coord syst
  - Larmor rotation freq $\vec{B}_1$
  - "incorrectly set rotating coord syst freq"

- Thus, viewing $\vec{M}$ vector in off-resonance rotating coord syst makes it look like additional $\vec{B}_z$ field is causing "extra" precession.

- "Extra" $\vec{B}_z$ component is proportional to $\Delta \omega_0$, offset $\Rightarrow$ can be pos or neg: rot coord too low $\Rightarrow$ pos $\vec{B}_z$; too high $\Rightarrow$ neg $\vec{B}_z$.

- Extra $\vec{B}_z$ adds to $\vec{B}_1$ resulting in slow precession around tipped axis: $\vec{B}_{eff}$ (effective).

- Extra $\vec{B}_z$ from any gradient $\Rightarrow$ same effect on $\Delta \omega_0$ (changes $\omega_0$ instead of changing $\omega_f$).

  \[ \vec{B}_{eff} = \left( \Delta \frac{\omega_0}{Y} \right) \hat{z} + \vec{B}_z \hat{z} + B_1 i \]

  - Effective $\vec{B}$ in rotating frame set to $\vec{B}_1$ freq.
  - Apparent "extra" $\vec{B}_z$ from Larmor-$\vec{B}_1$ freq mismatch (pos or neg).
  - Extra $\vec{B}$ from optional gradient (pos or neg).

- **Off Res., Opposite Way**

  \[ \vec{B}_2 = \left( \Delta \frac{\omega_0}{Y} \right) \hat{z} \]

  - Can cause 180° flip.

- **ADIABATIC RF PULSE**

  - Flaw-driven CASL tag

  - RF: sweep freq ($\omega_0$); constant freq ($\omega_0$);
  - RF: const freq, $\omega_0$: sweeps because spins flow along gradient direction.

- "adiabatic RF pulse" $\approx$ flaw-driven CASL tag.
**SPECTROSCOPY + IMAGE**

- **chemical shift**: small displacement resonant freqs due to variable shielding of target nucleus (e.g. \(^1\text{H}\)) by surrounding electron orbitals.

- **e.g., acetic acid**:
  - Oxygen attracts electron so less shielding of target nucleus.
  - H: 3 of these H's (more shielded).
  - 1 of these H's (less shielded).

- **how we get chemical shift spectrum**:
  - Larmor oscillations are multiplied (PSD) by center freq to obtain \(\Delta f\) (not \(\text{MHz} \times \text{high freq}\)).

- **data before FT is a series of time-domain samples of the mix of shifted-freq offsets**.

- **FT turns data into "shift spectrum"**.

---

**MRI**

- **Spatial** → spatial object freq samples (like time domain signal).

---

**Pulse Sequence**

- Since we are already using phase (freq) encoding for space, we need an "extra dimension" with all gradients OFF!

- Use spin-echo to undo built-up chemical shifts, then record chemical-NMR-like signal.

- And FT-it like chemists do!
PRESS, MEGA-PRESS

- Usual single voxel by using 3 orthogonal slice select (interleaves gradients & more excitations to get multiple voxels)

**PRESS** - 3 orthogonal slice select

**MEGA-PRESS** - add "editing" RFs to suppress solvent (water)

FT to get shift spectrum
**Phase-encoded Stimulus & Analysis**

**Map Polar Angle**

**Map Frequency**

**Map Eccentricity**

**Map Prox/Distal Axis, Road Maps**

Periodic stimuli (phase-encoded) - e.g., 8 cycles at 64 sec/cycle

**Calculate Significance**

- Ratio between amplitude at stimulus frequency (= signal) and average of amplitudes at other frequencies (= noise)
- Ignore harmonics, 0 freq (= movement)

**Smooth**

- Vector average of complex significance (A, φ) with that at nearest neighbor surface points

**Display**

- Plot phase using hue and saturation to indicate significance

**Delay Correction**

- Record responses to opposite directions of stimulus (CCW/LCW, in/out, up/down)
- Vector average after reversing angle of one
  - Penalizes inconsistent more than just avg of angles

Typically 0.5 - 5% amplitude

Strongly periodically activated single voxel time course

Remove constant (avg) and linear trend

Real

Imaginary

FFT, convert to A, φ

A

φ

b_{eq} = total TR's / 2

Reversed CCW

Vector average

CCW significance (complex)
**CONVOLUTION**

\[ h(x) = f(x) \ast g(x) = \int_{-\infty}^{\infty} f(z) \cdot g(x-z) \, dz \]

- definition of Convolution \( f \ast g(x) \)
- commutative

**Why reverse makes sense**

Centered at \( x \), place kernel at \( x \)
- reverse kernel
- current \( x \)
- impulse response function (HDR)

**Intuitive non-reversed view of convolution output**

Expt. start here!

N.B. same as convolution except no reversal
- impulse occurred a while ago
- small effect
- impulse occurred recently
- large effect

N.B. auto-corr same, except no reversed and use same function for both \( f, g \)

How to calculate convolution output for this time point (only 3 terms in sum, all others zero)
**General Linear Model (GLM)**

\[
\tilde{y} = X \tilde{h} + S \tilde{b} + \tilde{n}
\]

data = design \cdot HDR + drifts \cdot weights + noise

- **Goal:** solve for the hemodynamic response function, \( \tilde{h} \)

\[
\begin{bmatrix}
\text{cond1 occurs} \\
\text{cond2 occurs} \\
\text{cond1 re-occurs}
\end{bmatrix}
\]

- **Maximum likelihood estimate**

(Liu et al. 2001 Neuroimage)

1) assume white noise, solve for \( \tilde{h} \)

2) \( \hat{h} = (X^T P_s^\perp X)^{-1} X^T P_s^\perp y \) where \( P_s^\perp = I - S(S^T S)^{-1} S^T \)

3) significance (how to construct F-ratio)

\[
F = \frac{N - K - L}{k} \left[ \frac{y^T (P_{xs} - P_s) y}{y^T (I - P_{xs}) y} \right]
\]

- \( P_{xs} \) - projects data on event + nuisance subspace
- \( P_s \) - projects data onto nuisance subspace
GEOMETRIC INTERPRETATION OF GENERAL LINEAR MODEL

- with no nuisance functions (S), we could look at orthogonal projection of data onto experimental design and compare that to error to determine significance.

\[ \hat{y} = X\hat{h} + \hat{e} \]
\[ X\hat{h} = P_x \hat{y} \]

Projection matrix, \( P_x \), operates on \( \hat{y} \) to give projection of data into experiment space, \( X \)

- when nuisance functions, \( S \), are considered, problem: \( S \) may not be orthogonal to \( X \)

→ for example: linear trend not orthogonal to std. block design.

[remember: "orthogonal" means dot prod. = 0 corr = 0]

Data orthogonal to nuisance

Error (e) not explained by exp design and nuisance (F denoted)

\[ [(I - P_x)S]y \]

How much more of data you can explain by adding experimental design (F numerator)

\[ (P_x - P_s)S \]

Same as projection onto reference only in special case where \( S \perp X \)
WHY USE SURFACES?

- raw MRI data is a 2D flat slice or a 3D volume $ightarrow I(x,y)$ or $I(x,y,z)$
  but...

1) the neocortex (and cerebellar cortex) are thin, folded 2D sheets
   - cortex starts as smooth "balloon" $ightarrow$ 
   - major sulci, temporal lobe form $ightarrow$
   - great size increase, "crinkles" form

2) neocortex contains many topological maps along its surface
   - retinotopy
   - tonotopy
   - somatotopy
   - musculotopy
   - plus higher level maps $ightarrow \approx \frac{2}{3}$ of its area

3) surface displays allow seeing (almost) all of data at once
   - only 30% exposed
   - everything visible
   - [flattened]

4) differences in major sulci make 3D-based alignment difficult
   - e.g. STS, monkey-like IPS vs. postcentral
   - requires mapping
   - extremely anisotropic def.

5) idiosyncratic sulcal crinkles

   - these introduce additional noise into alignment in 3D
   - exact position of crinkles unlikely to have functional implications (the 3D align might respect them)
1) MNI auto-Talairach → generates 4x4 matrix
   - make average brain target (blurry)
   - blur target (further), blur single brain (a lot), gradient descent on xcorr
   - repeat with less blurring of avg target and current brain
   - problems: variable neck cut-off
     - but much better than standard! < fit to bounding box

2) Intensity Normalization (output: "T1")
   - histogram of pixel values in 10 mm thick T1R slices
   - smooth histogram
   - peak and to get initial estimate of white matter
   - discard outlier peaks across slices
   - fit splines to peaks across slices
   - interpolated scaling factor 1 to T1R
   - scale each pixel so WM peak is 110
   - refine estimate to interpolate in 3D
   - find points in 5x5x5 within 10% of WM, get new scale for them
   - build Voronoi to interpolate scales on above soul bubble smooth Voronoi boundaries (3 iterations)
   - re-scale each voxel

3) Skull Stripping (output: "brain")
   - "shrink-wrap" algorithm
   - start with ellipsoidal template
   - minimize brain penetration and curvature
   - curvature: spring force
     (from center-to-neighbor vect sum)
   - brain penetration
     apply force along surface normal that prevents surface from entering gray matter

Talairach, Normalize, Strip Skull
SEGMENTATION & SURFACE RECON

- Implementing a "force" is like directly constructing the operator that minimizes something (without first defining the "something")
- More formally, we would define cost function, then take its derivative (gradient) to minimize it

Shrinkwrap update e.g. (skull strip, original Dale & Sereno surface refinement)

\[
\mathbf{r}_{\text{center}}(t+1) = \mathbf{r}_{\text{center}}(t) + F_{\text{smooth}}(t) + F_{\text{MRI}}(t)
\]

for one vertex previous local quantities

\[
F_{\text{smooth}} = \lambda_{\text{tang}} \sum_{\text{neigh}} (\mathbf{I} - \mathbf{n}_{\text{center}} \mathbf{n}_{\text{center}}^T) \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}})
\]

vector to neighbor vertex

\[
+ \lambda_{\text{normal}} \sum_{\text{neigh}} (\mathbf{n}_{\text{center}} \mathbf{n}_{\text{center}}^T) \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}}) \cdot \frac{1}{\text{#vertices}} \sum_{\text{neighbors}} (\mathbf{n}_{\text{center}} \mathbf{n}_{\text{center}}^T) \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}})
\]

average normal component

\[
F_{\text{MRI}} = \lambda_{\text{MRI}} \mathbf{n}_{\text{center}} \sum_{d} \max \left[ 0, \tanh \left( I(\mathbf{r}_{\text{center}} - d \mathbf{n}_{\text{center}}) - I_{\text{threshold}} \right) \right]
\]

d sample points into brain along the direction of normal

\[
\text{max force at } 1.0 \text{ (max product = } 1.0) \text{ if any are zero}
\]

Outside (dark)

Snapshot of surface and "core sample" from one vertex

GM, ideal skull strip

etc.
4) Non-isotropic filtering (output: "wet") — "floss" and "pimple"
- Preliminary hard thresholds: output
- Find ambiguous/boundary voxels
  \[ \text{avg brightness} \]
  \[ \geq 20\% \text{ of more of } 26 \text{ immediate neighbors different} \]
- Find plane of least variance
  \[ \text{for each direction (from icosahedral supertessellation)} \]
  \[ \text{consider } 5 \times 5 \times 5 \text{ volume around 1 voxel} \]
  \[ \text{find plane of least variance in this hemisphere} \]
  \[ \text{median filter w/ hysteresis (1 voxel thick)} \]
  \[ \Rightarrow \text{if } 60\% \text{ of within-slab differ, reverse classification} \]
  \[ \Rightarrow \text{"flosses" sulci without blurring} \]

5) Find cutting planes
- Midbrain
- Callosum, to separate hemispheres (SAH)
- Midbrain, to avoid fill into cerebellum (Talairach to start, fill WM in SAH or TOR till min area)

6) Region-growing to define connected parts (output: "filled")
- Inside-out, outside-in, inside-out — for each hemisphere
- Up/down cycles within each plane
- Plane-by-plane
- "Wormhole filter" \((3 \times 3 \times 3 = \text{center } + 26)\)
  \[ \Rightarrow \text{fill (unfilled) voxel if } 66\% \text{ neighbors differ} \]
  \[ \Rightarrow \text{eliminates structures within, 1-D structure} \]
7) **Surface Tessellation** (output: rh.orig, lh.orig)

- Variable num neighbors possible!
- Quads to triangles

- Find filled voxels bordering unfilled
- Make ordered list of neighboring vertices
  \[ \rightarrow \text{so cross-products oriented properly} \]

- Long list of values associated with each numbered vertex
  
  *Position (orig, morphed)*
  *Area (orig, morphed)*
  *Curvature (intrinsic, Gaussian)*
  *"Sulcushness" (summed 1 movement during unfolding)*
  *Cortical thickness*
  *fMRI data* \[ \leq \]
  *EEG/MEG dipole strength*

- Separate fMRI data set must be aligned, sampled

  - fMRI voxels larger
  - Sample at each surface vertex
    - Nearest-neighbor "soap bubble" smoothing to interpolate data onto hi-res mesh

- Some quantities only well-defined on surface
  
  \[ \rightarrow \text{gradient of magnitude of cortical map measure (e.g., eccentricity)} \]
SEGMENTATION & SURFACE RECON

Smooth, inflate, final surfaces

- smoothing/inflation/WM/pial done as derivative of energy functional

\[ J = J_{\text{tangential}} + \lambda_{\text{normal}} J_{\text{normal}} + \lambda_{\text{image}} J_{\text{image}} \]

- total scalar error to minimize
- scalar tangential error (fixed by redistributing vertices)
- small curvature error (fixed by reducing curvature)
- scalar image error (fixed by moving toward target image value)

\[ J_{\text{normal}} = \frac{1}{2 \# \text{vert}} \sum_{\text{centers}} \sum_{\text{neighbors}} \left[ \mathbf{n}_C \cdot (\mathbf{r}_N - \mathbf{r}_C) \right]^2 \]

- across all vertices, curvature error
- \( \frac{1}{2} \) so no coefficient on derivative
- across all vertices of one vertex
- vector from current center to one neighbor
- position vector diff.

\[ J_{\text{tangential}} = \frac{1}{2 \# \text{vert}} \sum_{\text{centers}} \sum_{\text{neighbors}} \left[ \mathbf{t}_C^x \cdot (\mathbf{r}_N - \mathbf{r}_C) \right]^2 + \left[ \mathbf{t}_C^y \cdot (\mathbf{r}_N - \mathbf{r}_C) \right]^2 \]

- "squishing" of mesh
- \( \mathbf{t}_C^x \) direction in tangent plane
- \( \mathbf{t}_C^y \) direction in tangent plane

\[ J_{\text{image}} = \frac{1}{2 \# \text{vert}} \sum_{\text{centers}} \left[ I_{\text{tag}} \left( \mathbf{r}_C \right) - I_{\text{tag}} \left( \mathbf{r}_N \right) \right]^2 \]

- image error
- \( I_{\text{tag}} \) for WM:
  - mean of voxels labeled WM in 5 mm neighborhood
- \( I_{\text{tag}} \) for pia:
  - global + small num for C.S.F.-like

- take directional derivative of energy functional (to find steepest uphill)
- move each vertex in the opposite (negative) direction w/ self-interest test

\[ \frac{\partial J}{\partial \mathbf{r}_C} = \lambda_{\text{image}} \left[ I_{\text{tag}} \left( \mathbf{r}_C \right) - I_{\text{tag}} \left( \mathbf{r}_N \right) \right] \left( \nabla I \left( \mathbf{r}_C \right) \right) + \sum_{\text{neighbors}} \lambda_{\text{normal}} \left[ \mathbf{n}_C \cdot (\mathbf{r}_N - \mathbf{r}_C) \right] \left( \mathbf{n}_C \right) \]

- go in the opposite direction (vector of largest scalar error for each vertex movement)
- \( \nabla I \) gone b/c const
- scalar
- vector — calculate gradient on image (first blur w/ Gaussian)

\[ \frac{\partial J}{\partial \mathbf{r}_C^x} = \lambda_{\text{tangential}} \left[ \mathbf{t}_C^x \cdot (\mathbf{r}_N - \mathbf{r}_C) \right] \left( \mathbf{t}_C^x \right) + \left[ \mathbf{t}_C^y \cdot (\mathbf{r}_N - \mathbf{r}_C) \right] \left( \mathbf{t}_C^y \right) \]

- \( \mathbf{t}_C^x \) — component of tangential
- \( \mathbf{t}_C^y \) — component

N.B.: eq. 9 in Dola, Fischl & Steward different — and incorrect!
**SULCUS-BASED CROSS-SUB. ALIGN**

- use summed perpendicular vertex move during inflation as vtx measure of "sulcus-ness"
- add term to error function, \( J \): "sulcus-ness" error
  \[
  J_{\text{sulc}} = \frac{1}{2 \# \text{vtx}} \sum_{\text{vtx}} \left[ S_{\text{subj}} - S_{\text{avg}} (\text{recenter}) \right]^2
  \]
- find neg of steepest uphill direction of change in sulcus-ness of target
- take derivative
- \( \frac{\partial J_{\text{sulc}}}{\partial S_{\text{recenter}}} = \lambda_{\text{sulc}} \left[ S_{\text{subj}} - S_{\text{avg}} (\text{recenter}) \right] (-\nabla S_{\text{avg}} (\text{recenter})) \)
- icosahedron (5-fold symmetry)
- 7th triangular sub- tessellation
- morph to one brain
- make avg. tary re morph to avg. tary
- sulcus-ness of movable subj vtx

- smooth wm
- inflated
- sphere
- registered sphere

- each sub's native surf has diff # vertices
- interpolate values (coords, surface measures, stats) to each icosahedral vertex from neighboring vertices of native mesh (dashed lines)
- average surface made from folded/inflated avg coords
  - folded: loses area from sulcal wrinkles (for average "inflated")
  - inflated: retains orig area, correct sulc/gyrs ratio ("inflated-avg")
- can use sampled-to-icos individual subj coords to draw icosahedral surface in shape of indiv. brain

\( \text{N.B.: morph will have changed local vertex density compared to more uniform native mesh (use native for sing. subj.)} \)
**SOURCE OF EEG/MEG**

**PSPs**
- anisotropic cables + aligned spatially + coherent/biased stim
- one end
- no distant signal from axon spike too close

**Isotropic**
- "closed field" (invisible at distance)

**N.B.:** spikes only detected by 15μm microelectrode in gray matter!

**Head**
1) Local dipole
2) EEG through skull, skin
3) Sweating because skull 1/50 conductivity of brain

**MEG**
- Radial dipoles lost
- Tangential dipole generates Gabor-like scalp distribution of B field
**INTRACORTICAL CIRCUITS & ORIGIN OF EEG**

**Cell types**
- **Excitatory (spiny)**
  - pyramidal
  - spiny stellate (e.g., V1 layer 4C)
- **Inhibitory (smooth)**
  - basket
  - double bouquet
  - chandelier
  - clutch

**Circuits**
- huge complexity
- first principal components: input → layer 4 → layer 2/3 → feedforward → layer 5/6 → output feedback

- microelectrode recording (e.g., 10 µm tip)
  - high pass → spikes
  - low pass → local field potentials
- spikes only recordable in gray matter
- white matter spikes only recordable with pipette w/ very fine tip b/c inward & outward currents so spatially close in axon/spike (> 1 µm)

**Intra/Inter cortical connections cartoon**

"lower" (e.g., V1)
- 2/3 feedforward
- 4 input
- 5 motor output
- 6 feedback

"higher" (e.g., V2)
- 2/3
- 4
- 5
- 6

ascending input (e.g., dLGN) → output to sup. collicl, feedback avoids layer 4 → motor striatum → to higher areas
GRADIENT, DIVERGENCE, CURL

**Gradient** ($\nabla$) (generalized deriv)

$$\nabla s(\mathbf{r}) = \frac{\partial s(\mathbf{r})}{\partial x} \hat{i} + \frac{\partial s(\mathbf{r})}{\partial y} \hat{j} + \frac{\partial s(\mathbf{r})}{\partial z} \hat{k}$$

- turns scalar function defined at each $x, y, z$ point, $\mathbf{r}$ into vector field
- unit vector in $x$ dir
- $s$ in $x$ change of $s$ in $x$ dir at point $\mathbf{r}$

**Divergence** ($\nabla \cdot$) (deriu: "dot prod")

$$\nabla \cdot \mathbf{v}(\mathbf{r}) = \frac{\partial v_x(\mathbf{r})}{\partial x} + \frac{\partial v_y(\mathbf{r})}{\partial y} + \frac{\partial v_z(\mathbf{r})}{\partial z}$$

- turns vector field into scalar field
- $v_x$ change of just $x$ component of $\mathbf{v}$ in $x$-dir at point $\mathbf{r}$

**Curl** ($\nabla \times$) (deriu: "cross product")

$$\nabla \times \mathbf{v}(\mathbf{r}) = \left( \frac{\partial v_z(\mathbf{r})}{\partial y} - \frac{\partial v_y(\mathbf{r})}{\partial z} \right) \hat{i} + \left( \frac{\partial v_x(\mathbf{r})}{\partial z} - \frac{\partial v_z(\mathbf{r})}{\partial x} \right) \hat{j} + \left( \frac{\partial v_y(\mathbf{r})}{\partial x} - \frac{\partial v_x(\mathbf{r})}{\partial y} \right) \hat{k}$$

- turns vector field into cross vector field
- $v_z$ change of just $z$ component of $\mathbf{v}$ in $y$-dir at point $\mathbf{r}$

**Vector identities**

- $\nabla \times \nabla s = 0$ curl of the gradient of any scalar field is zero
- $\nabla \cdot (\nabla \mathbf{A}) = 0$ divergence of the curl of any vector field is zero
- $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$
**Potential** ($\Phi$), Electric Field ($\nabla \Phi$) & CSD. ($\nabla \cdot (\nabla \Phi) = \nabla^2 \Phi$)

Low-frequency field approximation
- Electric fields uncoupled from magnetic (vs. electromagnetic radiation)
  - Pre-Maxwellian approx. (EEG freq's $\ll 1$ MHz)
- Calculate electric fields as if magnetic fields don't exist
- Calculate magnetic fields strictly from distribution of currents
- Ignore capacitative effects, too

**Scalar potential**, $\Phi$ (what we measure with electrode)

\[
\vec{E} = -\nabla \Phi - \frac{\partial \vec{A}}{\partial t} = - \left[ \frac{\partial \Phi}{\partial x} \hat{i} + \frac{\partial \Phi}{\partial y} \hat{j} + \frac{\partial \Phi}{\partial z} \hat{k} \right]
\]

Gradient of scalar field

1. $\vec{E}$ defined as force (vector) acting on unit charge at a given point in space (as result of arbitrary distribution of other charges)
2. Current density, $\vec{j}$ (not curr. source dens., it is) is proportional to $\vec{E}$ (Ohm's law) $\sigma$ is conductivity
3. Two defs of $\vec{A}$: $\vec{A} = \frac{\lambda_0}{4\pi} \int \frac{\vec{j}}{r^2} \, d\text{vol}$
   - Dist-weighted sum of directional currents

CSD is Laplacian of $\Phi$ ($= \text{div} \vec{E}$)

\[
\nabla \cdot (-\nabla \Phi) = \text{Scalar field} = \left[ \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} \right] = -\nabla^2 \Phi
\]

3D CSD gold standard (rat BAER paper)

\[
\Phi \text{ data } \nabla \rightarrow \nabla \Phi \rightarrow \nabla \cdot (-\nabla \Phi)
\]
Scalar field source/sink movie as function of $z$
1D AND 2D CURRENT SOURCE DENSITY EXPERIMENT.

1D CSD
- Raw, event-related signal relative to ground, $\frac{d}{dt}$ (e.g., skull)
- $\mu V$
- High pass
- Low pass
- Spikes (upside down of extracellular)
- LFP (local field potential)
- Both types of data can be recorded from same electrode
- $\nabla^2$ means find spatial (i.e., 1D depth) curvature of potential
- Discrete approx: center $= \frac{\text{above} + \text{below}}{2}$
- N.B. in example above, even though all 3 potentials are positive, smaller value of center point implies sink!
- Rationale: all electrodes record along same surface so assume depth profiles are constant

2D CSD
- 2D array of electrodes on pial surface or on scalp
- $\nabla^2 \Phi$
- For scalp recordings, sources and sinks are at the scalp (not a depth loss method unless done in 3D)
- Can tell curvature from sign of potential:
  - Concave $\rightarrow$ sink
  - Convex $\rightarrow$ source
INTRACORTICAL C.S.D.

- e.g. click evoked rat A-I
  (Sukou & Barth, 1998)

\[ \text{CSD - 10} \]

- phase-locked CSD \( \rho \)
  gamma shifts w/each cycle

\[ \text{Source} \]
\[ \text{Sink} \]

\[ \text{P1} \]
\[ \text{N1} \]
\[ \text{P2} \]
MAXWELL EQUATIONS

\[ \nabla \cdot (\sigma \nabla \Phi) = \nabla \cdot \mathbf{J} \]

Ohm's law: \( \sigma \nabla \Phi = \mathbf{J} \)

\[ \nabla \times \mathbf{B} = \mu_0 (\mathbf{J} - \sigma \nabla \Phi) \]

Curl magnetic field.

Impressed currents:
- currents due to ionic flow that "appear out of nowhere" (neural batteries)

N.B. these are all defined at a (every) point in space

\[ \nabla \cdot \mathbf{B} = 0 \]

Divergence magnetic field vectors at each point

\[ \nabla \times \mathbf{E} = \mathbf{J} \]

Curl electric field.

\[ \nabla \Phi \rightarrow \mathbf{B}, \mathbf{E} \]

Potential \( \Phi \) and magnetic fields \( \mathbf{B} \) produced by a weighted sum of two current source distribution are equal to weighted sum of fields produced by each current source distribution by itself.

- propagation of potentials, magnetic fields instantaneous (no capacitance)
- simultaneous Eq. to solve if \( \mathbf{J} \) are sources, \( \Phi, \mathbf{B} \) are data
- linear
WHY WE CAN IGNORE MAGNETIC INDUCTION

\[ \vec{E} = -\nabla \Phi - \frac{\partial \vec{A}}{\partial t} \quad \text{(from Nunez, 1981)} \]

- Electric field
- Field component due to charge distribution
- Field component due to coupling between electric & magnetic fields

\[ \vec{B} = \nabla \times \vec{A} \]

- "vector potential"

\[ \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \]

- Magnetic field (in given medium, induced)
- Currents (in given medium)
- Time-varying electric fields (in given medium)

\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]

- Take \( \nabla \times \) of both sides
- Use \( \vec{B} = \mu \vec{H} \)
- Substitute this into LHS

\[ \nabla \times \nabla \times \vec{E} = -\mu \frac{\partial}{\partial t} \left( \vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \]

- Linear in conductivity and dielectric, too, and fields periodic with

\[ \nabla \times \nabla \times \vec{E} = -2\pi f \mu (\sigma + 2\pi f \varepsilon) \vec{E} \]

To neglect:

\[ \frac{2\pi f \mu (\sigma + 2\pi f \varepsilon) |\vec{E}|}{|\nabla \times \nabla \times \vec{E}|} \ll 1 \]

1) \( |\nabla \times \nabla \times \vec{E}| \approx |\vec{E}|/L^2 \) where \( L \) is dist over which \( \vec{E} \) varies significantly
2) \( \mu \) is tissue similar to empty space
3) Assume conservatively (large) \( \sigma \), dielectric unit, and EEG freq.

- Number is about \( 10^{-6} \rightarrow \) small
**MONPOLE, DIPOLE FORWARD SOL’N**

\[ \Phi_1 = \frac{s}{4\pi\sigma r} \]

Potential recorded for source monopole

\[ \Phi_2 = \frac{s}{4\pi\sigma} \left( \frac{1}{r_1} - \frac{1}{r_2} \right) \]

Potential recorded for source-sink pair ("near field")

**Scalar**

\[ \Phi_2 \approx \left( \frac{1}{4\pi\sigma} \right) \frac{s d \cdot \hat{r}}{r^3}, \quad r \gg d \]

Approximations for "far enough away" measurements (subtracting two \(1/r\) gives inverse square)

**Vector**

\[ \mathbf{B}_2 \approx \left( \frac{\mu_0}{4\pi} \right) \frac{s d \times \hat{r}}{r^3}, \quad r \gg d \]

\( \gg 3.4x \)

N.B.: both assume inside infinite isotropic conductor

**Linear superposition with fixed electrodes and sensors**

\[ \Phi_i(t) = e_i \cdot s_i(t) \]

\[ \mathbf{b}_i(t) = \mathbf{m}_i \cdot s_i(t) \]

\[ \mathbf{B}_i(t) = \sum_j \mathbf{m}_j \cdot s_j(t) \]

\[ \mathbf{X}(t) = \sum_j \mathbf{G}_j \cdot s_j(t) \]
Forward Solution

- well-posed (one answer)
- linear: \( b(A) + b(B) = b(A+B) \)
- approximations due to unknown electrical properties of head

- 3-shell spherical analytic

- skull voxel conductivity of brain
- "smearing" (cf. cable theory)

- 3-shell boundary element

- arbitrary shape
- homogeneous conductivity

solution = infinite homogeneous + \( \sum \) factors

- finite element
  - most general
  - computational intensive w/ small grid
  - many unknown parameters to estimate
**Forward Solution**

\[ \mathbf{V} = \sum_j \mathbf{E}_{ij} \mathbf{s}_j + \mathbf{n} \]

- **Matrix Form**: 
  \[ \begin{bmatrix} \mathbf{v} \\ \mathbf{n} \end{bmatrix} = \begin{bmatrix} \mathbf{E} \\ \mathbf{S} \\ \mathbf{n} \end{bmatrix} \]
  - **E** fixed across time
  - **v, S, n** vary

**Lower case bold → vector**
**Upper case bold → matrix**

- Electric recordings
- Magnetic recordings

\[ \hat{\mathbf{x}} = \mathbf{A} \hat{\mathbf{s}} + \hat{\mathbf{n}} \]

**Note**: only one current source for each column in the \( \mathbf{E} + \mathbf{B} \) matrix!
WHY LOCALIZE?

- most of ERP literature based (indeed) on temporal "components"
  - underlying local cortical generators (from micro electrode LFP, CSD)
    - extended in time (400 msec), visible from every scalp electrode
    - multiphasic in every cortical area
    - temporally non-static depending on stimulus
    - e.g. simple contrast, brightness diffs can modulate retinal delay by 50 mSec!
  - thus, any "component" consists of sum of activity from multiple cortical areas at different hierarchical levels
  - stimulus manipulations will change temporal overlap
    - may cause "component" peak to disappear without changing cortical areas being activated
  - verified by intra cortical LFP/CSD (Schroeder et al., 1998)

  ![Graph of LFPs from approx. layer 4 in cortex ("input layer")]

- macaque monkey intracortical data
  - these areas span the visual system from bottom to top, accounting for roughly 50% of the entire macaque monkey cortex
  - by contrast, the spatial signature of the signal from one cortical area is static — a better area-based "component"

- temporal "components" should be retired!!
  - easier to record now temporal points (EEG started w/ few electrodes, many time points)
  - their original reason for being no longer relevant!!
  - easier to "paste" high level psychological functions onto a few waveform deflections
Derivation of Ill-posed Inverse

(from Dale & Sereno, 1993)

\[
x = As + n
\]

\(x = \) sensor data vector
\(A = \) forward soln matrix \((E + B)\)
\(s = \) source vector
\(n = \) sensor noise vector

\[
\text{Err}_w = \langle ||Wx - s||^2 \rangle
\]

\text{expected} = \sum_k \frac{p_k}{k}

\[
\text{assume } n, s \text{ normal, zero-mean with corresponding covariance matrices } C, R
\]

\[
\text{Err}_w = \langle ||W(As + n) - s||^2 \rangle
\]

\[
= \langle ||(WA - I)s + Wn||^2 \rangle
\]

\[
= \langle ||Ms + Wn||^2 \rangle \quad \text{where } M = WA - I
\]

\[
= \langle ||Ms||^2 + ||Wn||^2 \rangle
\]

\[
\text{diag } M = \text{noise variance (already squared)}
\]

\[
= \text{tr} (MRMT) + \text{tr} (WCW^T)
\]

\text{trace is sum of diag elements}

\[
\text{re-expand} = \text{tr} (WARA^TW^T - RATW^T - WAR + R) + \text{tr} (WCW^T)
\]

\text{Explicitly minimize by taking derivative w.r.t. } W, \text{ set to zero, solve for } W

\[
0 = 2WARA^T - 2RAT + 2WC
\]

\[
WARA^T + WC = RAT
\]

\[
W(AA^T + C) = RAT
\]

\[
W = RAT(AA^T + C)^{-1}
\]

\(W\) is inverse solution operator:

\[
\begin{bmatrix}
\text{Sources} \\
\text{Sensors}
\end{bmatrix}
\]

\(W\)

\text{equivalent to minimum norm and Tikhonov regularized inverse if } C, R \text{ are proportional to identity matrix (i.e., sensor noise & sources independent and equal variance).}
INVERSE SINC

\[ W = RA^T(ARA^T + C)^{-1} \]

\( \rightarrow \) "minimum norm" solution
(find \( \hat{x} \) w/ smallest norm = \( \| \hat{x} \| \))

- the minimum norm solution appropriately downplays deeper (= weaker scalp signal) sources since these are more likely to fall into the noise floor

- "problems" of minimum norm:
  - deeper sources get displaced to the surface
  - small superficial sources "win" because of approx. inverse square form of true solution
    \( \rightarrow \) smaller norm of distributed superficial sol.
  - can't fix by increasing priors of deep sources!!
    \( \rightarrow \) that will give deep sources given noise as input!!

\[ \text{inverse-2} \]
**Inverse Solutions to Ill-Posed Compared**

- **Sources** $s = Wx$
- **Sensors** $y$

**Sensor Data**

- How to use the inverse solution, $W$
- Same $W$ for all time points

**"Minimum Norm" Solution**

- i.e., norm $\|W\|_2$ of solution is smallest of infinitely many alternate solutions

**Linear Inverse Operator**

$W = \begin{pmatrix} R & A^T \end{pmatrix} \begin{pmatrix} A & R \end{pmatrix}^{-1}$

**Spatial Covariance**

- Sensor noise
- Forward solution

**Spatial Covariance**

- Sources

**From Error Minimization Derivation**

- Easier inverse $\Rightarrow$ Square in # of sensors (small)

**Alternate, Algebraically Equivalent Bayesian Derivation**

$W = (A^T C^{-1} A + R^{-1})^{-1} A^T C$

**Linear Inverse Operator**

$W = \begin{pmatrix} A^T \end{pmatrix} \begin{pmatrix} C & A \end{pmatrix}^{-1} + \begin{pmatrix} R & A^T \end{pmatrix} \begin{pmatrix} A & C \end{pmatrix}^{-1}$

- Both square in # of sources (large) hard inverses
PROBLEMS W/ SURFACE NORMAL

- Since nearby points on surface often have different orientation, surface normal constraint can help (since fwd soln A, B very different)

- But, since point spread function: point spread typically extends across sulci, artificial sign reversals occur

- Solutions

  1) Ignore sign \(\Rightarrow\) saves useful orientation info!

  2) Solve onto 3 orthogonal dipoles at each critical point instead of a single oriented dipole

- More appropriate when averaging across subjects, since detailed orientations vary a lot

- Also, fills in bottom of sulci (else unsigned stripes)
- insert FMRI values for Rii's
- but still allow other sites to have non-zero Rii's
- pathologies occur if solution restricted completely to FMRI points by setting non-FMRI Rii's to zero -> set to small number instead!

- this allows extracting time course from sources visible in EEG/MEG and FMRI

- N.B.: sources that are only visible in EEG/MEG will be dispersed to small distributed values at a large number of vertices

visible in both EEG/MEG and FMRI
visible only in EEG/MEG and not FMRI -> distributed at small amplitude across many vertices
**Noise Sensitivity Normalization**

Forward: \[ x = As \] well-posed (Liu, Dale, and Belliveau, 2002)

Inverse: \[ s = Wx \] ill-posed

Solve: \[ x = As + n \] for \( s \)

\[
W = RA^T(ARA^T + C)^{-1}
\]

- Multiply inverse operator by noise sensitivity matrix, \( D \) (diagonal)

\[
D_{ii} = \frac{1}{\text{diag}(WCW^T)}
\]

\[
W_{\text{norm}} = DW
\]

\[
S_{i,\text{norm}} = \left(W_{\text{norm}}x\right)_i = (DWx)_i = \frac{W_i x}{\sqrt{(WCW^T)_i}} = \sqrt{\frac{(W_{xx}^TW_i^T)}{(WCW^T)_i}}
\]

If assume Gaussian white noise, noise covariance \( C \), is multiple of \( I \), so

\[
W_{i,\text{norm}} = \frac{W_{i,\text{orig}}}{\|W_{i,\text{orig}}\|}
\]

i.e., scale each row of \( W \) by single value — the norm of that row

row of \( W \) is:

That is, if inverse sol’n for deep source is reduced by interaction of inverse square nature of far and min norm, dividing by norm of row of inverse (same) will increase/rescue deep source
**Noise Sensitivity Normalization**

1. **Shallow source (unit strength)**
   - Forward big
   - Inverse small

2. **Deep source (unit strength)**
   - Forward small
   - Inverse reduced because of minimum norm
   - \(W\) should be bigger than for superficial source
   - Min norm reduces lot because of inverse square

\[
S_i = \frac{\text{orig. } X}{\|W_i\|_{\text{orig}}}
\]

- Effect on inverse solution: more like significance is actual power.
- Effect on point-spread function is to equalize shallow & deep.
- Shallow spread out more than min norm.
- Deep shrunk to same as shallow.

**Point-spread functions**

- Noise normalized

---

N.B.: The diagram illustrates the relationship between shallow and deep sources, showing how their effects are normalized and how their point-spread functions are adjusted.
**Conclusions**

- More EEG or more MEG better.
- EEG better than MEG (cf. radial) (EEG faster, less currently less accurate).
- Biggest gain from adding small # EEG (no MEG) (e.g., 20) to many MEG (no EEG) (e.g., 150).
- Easier to add many MEG, so: optimal < 30 EEG < 300 MEG.

EEG/MEG forward-solution-scaling factor error causes more cross-talk.
Music (1)
(from Dale & Sereno, 1993) (cf. Mosher & Leahy)

Using sensor covariance

\[
D = \langle \mathbf{x} \mathbf{x}^T \rangle = \sigma_n^2 \mathbf{I} + \sum_i \sum_j \sigma_i \sigma_j \mathbf{C}(i,j) \mathbf{A}_i \mathbf{A}_i^T
\]

where \( \mathbf{A}_i \) is a spatial pattern across sensors, and the covariance structure is estimated from recording at one time point

\[
\sim \left[ \begin{array}{c} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{array} \right] \left[ \begin{array}{c} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{array} \right]^T
\]

\[
\sum_i \mathbf{C}(i,j) \mathbf{A}_i \mathbf{A}_i^T
\]

\[
\mathbf{D} = \mathbf{U} \Lambda \mathbf{U}^T
\]

** Columns of \( \mathbf{U} \) matrix are orthogonal basis vectors of "spatial pattern" space

*one out is spatial pattern across sensors*

...find, order most significant spatial patterns in sensors over time...

Project forward solutions onto these spatial patterns

("project" = dot prod = similarity) for each point in brain (fud defines spatial sensor pattern for unit source)

\[
\mathbf{e}_i = \mathbf{A}_i \mathbf{U} \Lambda \mathbf{U}^T \mathbf{A}_i \\
\Rightarrow \text{big single number if forward solution looks like } \mathbf{U}'s
\]
MUSIC

(2) how to weight the minimum norm inverse

\[ R_{ii} \approx \frac{A_i^T A_i}{A_i^T U \Lambda U^T A_i} \]

cf. - like parallel resistance

\[ R_{\text{parallel}} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \ldots} \]

- any low resistance \((\xi)\) decreases overall resistance (small \(R_{ii}\))

\[ \text{i.e., if forward soln has appearance like any low eigenvalue spatial pattern, it gets devalued} \]

\[ W = R A^T (A R A^T + C)^{-1} \]
**Music**

- How it works: Take advantage of spatial information that changes over time.

One time point

![Diagram of one time point with arrows showing changes over time.]

Multiple time points

![Diagram of multiple time points with arrows showing changes over time.]

N.B.: Problem if assumption about lack of perfect correlation is violated.

E.g., if two widely separated sources (i.e., different field solns) are highly correlated, MUSIC will eliminate both since no Single field soln will look like that "Z-separated dipole" pattern (e.g., L/R A-T).

"Dual music" hack to fix...

If sources really were superficial, these spatial patterns would appear in D.

If not, superficial sources would be eliminated, leaving deep...