Foundations of Neuroimaging
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29 Oct 2022
(124 pages)
Hardware

MAGNET HARDWARE

1 - $B_0$ field from superconducting magnet
2 - gradient coils
3 - body RF transmit/receive
4 - RF receive-only
5 - shim coils (in gradients)

$B_0 \rightarrow z$ (longitudinal)
$B_1 \rightarrow x, y$ (transverse)

max gradient: 
[80 mT/m, 200 T/h/sec]

(1) $B_0$ field
(2) Body gradient coils
(3) RF transmit body coil
(4) RF receive-only head coils

$I_T = 10,000$ Gauss
Earth: 0.25 - 0.65 G
25 - 65 mT

$\frac{Y}{2\pi} = 42$ MHz/T

RF transmitter (30 kW)
RF receiver

Circularly polarized $B_1$ field rotating \( \perp \) to $B_0$ at Larmor freq. ($B_1$ is several orders of magnitude smaller than $B_0$)

Three 1.5 million watt amplifiers to add ramps to $B_0$ field

non-superconducting water-cooled, external shield

opposite direction $B_0$ shield coils outside these not shown
superconducting coils in liquid helium (no power required after current injected to bring up field using induction)
**Spin & Precession**

- Nuclei act like a spinning sphere of matter with an embedded equatorial charge (nuclei w/ odd atomic weight or odd proton numbers)
- Moving charge creates magnetic field
  - Classical picture
    - Current loop from spinning charge (right-hand rule)
    - N.B.: Classically this would cause EM radiation, spin-down
- Stern-Gerlach experiment
  - Pass silver atoms thru strong mag. field → split into just 2 beams

**Microscopic picture**

- No strong magnetic field $B_0 = 0$
- Strong magnetic field, $B_0 = \uparrow$
- Strong $B_0$ plus oscillating $B_1$

**Macroscopic picture**

- Bulk magnetization
  - $M_z = \gamma h_B N$ (where $I = \pm \frac{1}{2}$)
  - $N$ = number of spins
  - $K_T$ = Boltzmann const.
  - $T_s$ = abs. temperature sample

**N.B.:**
- Compared to top & gravity magnetic relaxation is:
  - Frictionless spin doesn’t slow
  - Signed gravity can change precession dir.
  - Can stick under floor
  - Neighbors bumping causes decay ($T_2$)

**Precession**

- Distinguish precession (slow) from spin (fast)
- Treat classically, like spinning top
  - $\omega_L = \gamma h_B$ (Larmor freq.)

- Bulk equilibrium magnetization (parallel to $B_0$)
  - $M_z^0 = |\vec{M}| = \frac{\gamma^2 h^2 B_0 N}{4 K_T}$

- Gyromagnetic ratio
  - $\gamma = \frac{\omega_L}{B_0}$
  - $h$ = Planck’s const.
  - $k_B = \frac{\hbar}{2\pi}$ (gyromagnetic ratio)
  - $N_s = \frac{M_z^0}{2} \propto B_0$
  - $N_s = \frac{M_z^0}{2} \propto \text{number of spins}$

- Left-hand rule:
  - Thumb = B0
  - Fingers = precess.
**Bloch Equation**

- Time-dependent behavior of $\vec{M}$ in the presence of an applied magnetic field ($\text{excitation}$ $\Rightarrow$ $\text{relaxation}$)

$$(\text{laboratory frame}) \quad \frac{d\vec{M}}{dt} = \vec{M} \times \vec{Y} B - \frac{M_x \hat{i} + M_y \hat{j}}{T_2} - \frac{(M_z - M_z^0) \hat{k}}{T_1}$$

- In the Larmor-rotating coordinate system, a tilt who a phase shift in a standard $B_1$ excitation is rotation around $x$-axis

- Longitudinal and transverse relaxations

$\frac{dM_z(t)}{dt} = -\frac{M_z(t) - M_z^0}{T_1}$

$\frac{dM_x y(t)}{dt} = -\frac{M_x y(t)}{T_2}$

- Solution to equations above: time-dependent free precession e.g.'s

$M_z'(t) = M_z^0 \left(1 - e^{-t/T_1}\right) + M_z'(0) e^{-t/T_1}$

$M_x y'(t) = M_x y'(0) e^{-t/T_2}$

*Lab frame: same!*

$M_z'(0) = M_z^0 + \left[ M_z'(0) - M_z^0 \right] e^{-t/T_1}$

*Lab frame: times $e^{-t/T_2}$*

$M_z'(t) = 63\% M_z^0$

$M_x y'(t) = 37\% M_x y'(0)$
**VECTOR ADD, MULTIPLY**

- Adding vectors is easy
  \[ \vec{c} = \vec{a} + \vec{b} = [a_x + b_x, a_y + b_y] \]
  - Just add components (vector)
  - Applies to complex numbers
  - Generalizes to any D

- Multiple ways to multiply vectors: here are 3

  **Dot Product** (inner product) (= "scaled projection onto")
  \[ \vec{c} = \vec{a} \cdot \vec{b} = \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} a_x \\ a_y \end{bmatrix} = a_x b_x + a_y b_y + a_z b_z \]
  \[ p = \|\vec{a}\| \cos \theta \]
  \[ c = p \|\vec{b}\| \]
  \[ \|\vec{b}\| = 1 \]
  N.B. equals: \( \vec{a} \cdot \vec{a} \)
  \[ \frac{\sqrt{a_x^2 + a_y^2 + a_z^2}}{\sqrt{a_x^2 + a_y^2 + a_z^2}} \]
  \[ \text{length of } \vec{a} \]
  \[ c = \|\vec{a}\| \|\vec{b}\| \cos \theta \]
  \[ \leftrightarrow \text{zero if } \vec{a}, \vec{b} \text{ orthogonal} \]

  **Cross Product** (outer product) (can be generalized: see "geometric algebra")
  \[ \vec{c} = \vec{a} \times \vec{b} = \begin{bmatrix} 0 & b_z & -b_y \\ -b_z & 0 & b_x \\ b_y & -b_x & 0 \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = [a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_x b_y - a_y b_x] \]
  \[ \text{vector} \]
  \[ \text{right hand rule: curl fingers from } \vec{a} \text{ to } \vec{b}: \text{ thumb is } \vec{c} \]
  \[ \text{geometric algebra: bivector plane area} \]
  \[ \vec{c} = \vec{a} \times \vec{b} \]
  \[ \|\vec{c}\| = \|\vec{a}\| \|\vec{b}\| \sin \theta \]
  \[ \leftrightarrow \text{max if orthogonal} \]

  **Complex Multiply** (see also quaternions: geometric algebra generalization)
  \[ \vec{c} = \vec{a} \cdot \vec{b} = \begin{bmatrix} b_y & -b_x \\ b_x & b_y \end{bmatrix} \begin{bmatrix} a_x \\ a_y \end{bmatrix} = [a_x b_x - a_y b_y, a_x b_y + a_y b_x] \]
  \[ \text{vector} \]
  \[ \|\vec{c}\| = \|\vec{a}\| \|\vec{b}\| \]
  \[ \leftrightarrow \text{like real nums} \]

\[ \sum \text{of angles: } \theta_1 + \theta_2 \]
**Effects of \( \mathbf{M}, \mathbf{B}, \) and \( \Theta \) on Precession Freq.**

1. **Bloch 1st Term**
   
   \[
   \frac{d\mathbf{M}}{dt} = \mathbf{M} \times \mathbf{B}
   \]

2. **Cross Prod. Properties Review**
   
   \[
   \left| \frac{d\mathbf{M}}{dt} \right| = \left| \mathbf{M} \right| \left| \mathbf{B} \right| \sin \Theta
   \]

3. **Starting Condition**
   
   \( \Rightarrow \text{now see effects of changing } \left| \mathbf{M} \right|, \left| \mathbf{B} \right|, \Theta \)

4. **Change \( \mathbf{M} \) Length**
   
   \( \Rightarrow \frac{d\mathbf{M}}{dt} \) proportionally larger, so cancels effect of larger \( \mathbf{M} \)
   
   \( \Rightarrow \text{same precession freq. as starting cond.} \)

5. **Change \( \Theta \) Between \( \mathbf{M} \) and \( \mathbf{B} \)**

   \( \Rightarrow \frac{d\mathbf{M}}{dt} \) goes up (then down) as \( \sin \Theta \)

   \( \Rightarrow \text{but circumference also goes up as } \sin \Theta, \text{ cancelling again} \)

   \( \Rightarrow \text{same precession freq.} \)

6. **Change \( \mathbf{B} \) Length**

   \( \Rightarrow \frac{d\mathbf{M}}{dt} \) goes up, proportional to \( \mathbf{B} \)

   \( \Rightarrow \text{but circumference is same at starting cond.} \)

   \( \Rightarrow \text{increased precession freq. } (\omega = \gamma \mathbf{B}) \)
**Simple Matrix Operations**

**Basic idea**
- A matrix \([\text{rotates/scal}e\text{s}]\) a vector
- \(\vec{b} = M\vec{a}\)

**3D example**
\[
\begin{bmatrix}
    b_x \\
    b_y \\
    b_z
\end{bmatrix} =
\begin{bmatrix}
    M_{11} & M_{12} & M_{13} \\
    M_{21} & M_{22} & M_{23} \\
    M_{31} & M_{32} & M_{33}
\end{bmatrix}
\begin{bmatrix}
    a_x \\
    a_y \\
    a_z
\end{bmatrix}
\]

Add translate (after rotate/scale)
- Commonly used "hack" for aligning vols
- A 4D matrix \([\text{rotates/scal}e\text{s}][\text{then}][\text{translates}]\) (4th D = 1)
- N.B.: Have to keep track of order!!
  - Rotate/scal then trans ≠ trans then rot/scal
  - Change rot component: untranslate; rot, retranslate

**3 special cases (3D):** rotate around each major axis without changing length (scale = 1.0)

- Rotate around X-axis:
  \[ \hat{R}_x(\alpha) = \begin{bmatrix}
    1 & 0 & 0 \\
    0 & \cos\alpha & \sin\alpha \\
    0 & -\sin\alpha & \cos\alpha
  \end{bmatrix} \]
  - E.g., 90° flip

- Rotate around Y-axis:
  \[ \hat{R}_y(\alpha) = \begin{bmatrix}
    \cos\alpha & 0 & -\sin\alpha \\
    0 & 1 & 0 \\
    \sin\alpha & 0 & \cos\alpha
  \end{bmatrix} \]
  - E.g., 180° flip to avoid add 180° phase after 90° flip on x'

- Rotate around Z-axis:
  \[ \hat{R}_z(\alpha) = \begin{bmatrix}
    \cos\alpha & \sin\alpha & 0 \\
    -\sin\alpha & \cos\alpha & 0 \\
    0 & 0 & 1
  \end{bmatrix} \]
  - E.g., precession with 13θ along z'

**General case**
- Rotate around general Z'-axis:
  \[ \hat{R}_{z'}(\alpha) = \hat{R}_z(-\phi)\hat{R}_y(-\theta)\hat{R}_z(\alpha)\hat{R}_y(\phi)\hat{R}_z(\phi) \rightarrow \text{(quaternions are more efficient)} \]
SOLUTIONS TO SIMPLE DIFFERENTIAL EQ.

**Diff. Eq.:**
\[ dM_{x'y'}(t) = -\frac{M_{x'y'}(t)}{T_2} \]

**Solution:**
\[ M_{x'y'}(t) = M_{x'y'}(0_t) \cdot e^{-t/T_2} \]

How this works, and where \( M_{x'y'}(0_t) \) comes from...

**Goal:**
1) Find eq. whose derivative satisfies diff. eq.
2) Also find soln. (one of many) that passes thru init condition

Since our diff. eq. is:
- derivative of funct. = const. same funct.
- try exponential, since derivative \( e^x = e^x \)

**Diff. Eq.:**
\[ M(t) = -\frac{1}{T_2} \cdot M(t) \]

**One Soln:**
\[ M(t) = e^{-t/T_2} = e^{-t/2} \]

**Take Deriv. to Check:**
\[ M'(t) = \frac{-1}{T_2} \cdot e^{-t/T_2} \]

N.B. this function is the "unknown" like the \( x \) in \( x + 1 = 3 \)

**Diff. Eq.:**
\[ M'(t) = \frac{-1}{T_2} \cdot M(t) \]

**Another Soln:**
\[ M(t) = \text{const} \cdot e^{t/T_2} \]

So: any const is OK!

**Take Deriv. to Check:**
\[ M'(t) = \frac{-1}{T_2} \cdot \text{const} \cdot e^{-t/T_2} \]

For ex.: \( \text{const} = 2 \) \( \text{const} = -0.5 \)

Const is "initial condition" 

Information added to soln. (not from diff. eq.):
\[ M'(t) = M_{x'y'}(0_t) \cdot e^{-t/T_2} \]

Magnetization immedi. after RF (BI) ends

\[ M_{x'y'}(0_t) \]
VERIFY SOLUTION TO T1 REGROWTH

- Slightly more complex T1 sol' compared to T2 sol'n

T2 sol'n verify (comp prev)  T1 solution verify

\[ \frac{dM}{dt} = \frac{M_{xy}}{T2} \]

original diff eq.

\[ M'(t) = \frac{-1}{T2} \cdot M(t) \]

Make unknown funct M(t) more visible

\[ M'(t) = \frac{-1}{T1} \left( M(t) - M_o^z \right) \]

Init cond.

\[ M(t) = M_0^z \left( 1 - e^{-t/T1} \right) + M_2(0_t) e^{-t/T2} \]

const chain rule as before

\[ M'(t) = 0 + \frac{-1}{T1} M_0^ze^{t/T1} - \frac{-1}{T2} M_2(0_t) e^{-t/T2} \]

Chain rule

\[ M'(t) = \frac{-1}{T1} \left( -M_0^ze^{-t/T1} + M_2(0_t) e^{-t/T2} \right) \]

- Derivative in original T1 eq. says \( M(t) \) minus \( M_o^z \)

\[ M'(t) = \frac{-1}{T1} \left( M(t) - M_o^z \right) \]

solution \( \left[ M_o^z - M_0^z e^{-t/T1} + M_2(0_t) e^{-t/T2} \right] \)

- Which equals our re-calculated derivative:

\[ M'(t) = \frac{-1}{T1} \left( -M_0^z e^{-t/T1} + M_2(0_t) e^{-t/T2} \right) \]
Bloch Eq. - Matrix Version

\[ \frac{d\vec{M}}{dt} = \vec{M} \times \gamma \vec{B}_\phi \]

Differential Eq.:

\[ \frac{d\vec{M}}{dt} = \begin{bmatrix} \frac{dM_x}{dt} \\ \frac{dM_y}{dt} \\ \frac{dM_z}{dt} \end{bmatrix} = \begin{bmatrix} 0 & \gamma B_\phi & 0 \\ -\gamma B_\phi & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} \]

Solution:

\[ \vec{M}(t) = \begin{bmatrix} M_x(t) \\ M_y(t) \\ M_z(t) \end{bmatrix} = \begin{bmatrix} \cos \omega t & \sin \omega t & 0 \\ -\sin \omega t & \cos \omega t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} M_x(0) \\ M_y(0) \\ M_z(0) \end{bmatrix} = R_z(\omega t) \vec{M}(0) \]

Include Relaxation

\[ \frac{d\vec{M}}{dt} = \vec{M} \times \gamma \vec{B}_\phi - \frac{M_x t + M_y \gamma}{T_2} - \frac{(M_z - M_z^0)}{T_1} \]

Differential Eq.:

\[ \frac{d\vec{M}}{dt} = \begin{bmatrix} -\frac{1}{T_2} & \gamma B_\phi & 0 \\ -\gamma B_\phi & -\frac{1}{T_2} & 0 \\ 0 & 0 & -\frac{1}{T_2} \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ M_z^0/T_2 \end{bmatrix} \]

Solution:

\[ \vec{M}(t) = \begin{bmatrix} e^{-t/T_2} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \cos \omega t & \sin \omega t & 0 \\ -\sin \omega t & \cos \omega t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} M_x(0) \\ M_y(0) \\ M_z(0) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ M_z(0)(1-e^{-t/T_1}) \end{bmatrix} \]
**Excitation in the Rotating Frame**

- **Original Bloch eq. in laboratory frame**
  \[
  \frac{d\hat{M}}{dt} = \hat{M} \times \vec{B}^{B0} + \vec{B}^{B1} \text{ gradients}
  \]

- **Add on-resonance $B1$ to $\hat{M}$**
  \[
  \vec{B} = B1(x) (\cos \omega_0 t \hat{z} - \sin \omega_0 t \hat{j}) + B0 \hat{k}
  \]

  - **Lab frame < no gradient**
    - **Basic excite**
    - **Matrix version**
      \[
      \frac{d\hat{M}}{dt} = \begin{bmatrix}
      \frac{dM_x}{dt} \\
      \frac{dM_y}{dt} \\
      \frac{dM_z}{dt}
      \end{bmatrix} = \begin{bmatrix}
      0 & -w_0 & w(t) \cos \omega_0 t \\
      w_0 & 0 & 0 \\
      -w(t) \sin \omega_0 t & w(t) \cos \omega_0 t & 0
      \end{bmatrix} \hat{M}
      \]

  - **Substitution to convert to the rotating frame**
    \[
    \hat{M} = R_z(w(t)) \cdot \hat{M}_{rot}^{B0} \text{ (rotate from both } \hat{M} \text{ and } \vec{B})
    \]
    \[
    \vec{B} = R_z(w(t)) \cdot \vec{B}_{rot}^{B1} \text{ (off-res (appears as } B0)}
    \]

  - **After substitution any off-resonance appears as residual $B0$ ($B2$)**
    (see off-resonance notes page)

  - **Rotating frame < on-resonance**
    - **Basic excite, B1x-only no gradient**
      \[
      \frac{d\hat{M}_{rot}}{dt} = \begin{bmatrix}
      0 & 0 & 0 \\
      0 & 0 & w(t) \\
      0 & -w(t) & 0
      \end{bmatrix} \hat{M}_{rot}
      \]

  - **Rotating frame < off-resonance**
    - **General, B1x-only incl gradients**
      \[
      \frac{d\hat{M}_{rot}}{dt} = \begin{bmatrix}
      0 & 0 & 0 \\
      0 & 0 & w(t) \\
      0 & -w(t) & 0
      \end{bmatrix} \hat{M}_{rot}
      \]

  - **Gradient: $w(t) = RG_z \hat{z}$**
    - **B-res: appears as residual $B0$, lifting $\vec{B1}$ vect. out of x-y plane**
    - This means $\hat{M}$ vect. update will contain component that rotates $\hat{M}$ around z-axis (in rotating coords, z-phase)

  - **Rotating frame < on-resonance incl gradients**
    - **Small tip approx.**
      \[
      \frac{d\hat{M}_{rot}}{dt} = \begin{bmatrix}
      0 & 0 & 0 \\
      -w(t) & 0 & w(t) \\
      0 & 0 & 0
      \end{bmatrix} \hat{M}_{rot}
      \]

  - **Small tip ⇒ easier to solve!**
Bloch Eq. Summary
\[
\frac{d\hat{M}}{dt} = \hat{M} \times \hat{\mathbf{B}} - \frac{M_x \hat{i} + M_y \hat{j}}{T_2} - \frac{(M_z - M_0^c) \hat{k}}{T_1}
\]

(lab-frame)

(vector lengths not to scale!)

- Full lab-frame picture is complex:
  - 3 component of \( \frac{d\hat{M}}{dt} \) update vector
  - Larmor freq. component 7-9 orders magnitude larger than \( T_2 \), \( T_1 \) decay
  - \( \hat{B}_1 \) is also rapidly wiggling

- Conceptual simplification in 4 stages:

1) **Lab frame**
   - Just precession
   - \( \hat{M} \) stopped
   - That is, \( \mathbf{B}_0 = 0 \)

2) **Rotating frame**
   - \( \hat{M} \) also stopped!
   - But \( \hat{M} \times \hat{\mathbf{B}} \) still works!!
   - "Precess" around \( \hat{B}_1 \) axis

3) **Add \( \hat{B}_1 \)**
   - Slow precess, now around tilted \( \mathbf{B}_{eff} \)

4) **Off-resonance**
   - Tilted plane
   - Apparent \( B_z \) comp. from residual precess. around \( z \) from off-resonance
RF Field Polarization

- Polarization (change in direction)

- Linearly polarized field
  \[ \vec{B}_1(t) = B_1 \cdot \cos \omega t \hat{x} \]
  Magn. strength: \( \{-1, 1\} \cdot 1 \)

- N.B.: \( \vec{B}_1 \) adds to much larger \( \vec{B}_0 \)

  \[ \frac{\vec{B}_0}{\vec{B}_0 + \vec{B}_1} \rightarrow \text{months} \]
  \[ \frac{\vec{B}_0 + \vec{B}_1}{B_0 + B_1} \rightarrow \text{wiggles L/R across time} \]

- Circularly polarized field (quadrature)
  \[ \vec{B}^{\text{circ}}_1(t) = B_1 \left( \cos \omega t \hat{x} - \sin \omega t \hat{y} \right) \]
  \[ = B_1 \cdot e^{-i\omega t} \]

- In the rotating coordinate syst, flipping around x-axis vs. y-axis is just difference in phase of RF

- Typical 90° flip (around x-axis)
  
- Typical 180° flip (around opposite y-axis)

- B1 generated/recorded by RF coil

- Magnetic field (vs. electric field)
**Phase-Sensitive Detection**

The method for moving very high-frequency Larmor oscillations down to a tractable frequency range.

\[ V(t) \approx 123 \text{ MHz} \]

1. Multiply by \( V(t) \)
2. Low-Pass Filter
3. \( S(t) \)

\[ S(t) \approx 50 \text{ kHz} \]

**Reference Signal** (123 MHz)

Demodulated signal \( \propto \) RF Coil Signal \( \cdot \) Reference (transmitter)

\[ \propto \sin[(\omega_0 + \delta \omega) t] \cdot \sin[\omega_0 t] \]

\[ \propto \frac{1}{2} [\cos \delta \omega t - \cos (2 \omega_0 + \delta \omega) t] \]

Filter this one out with low pass filter

- Two signals are made from a single receiving RF coil
- A quadrature coil can be treated the same way (OK to combine after adding \( \frac{\pi}{2} \) phase, then PSD)
- Quadrature coil has better S/N since noise in each part is uncorrelated (\( \frac{1}{2} \) better)

**One Freq - Freq Domain**

- Signal
- Reference
- Demodulated
- After filter

**Chirp - Time Domain**

- Chirp
- Center
- Demodulated
- Low freq signal

- Filter

\[ S(t) \text{ complex} = Mxy e^{-i \delta \omega t} \]
FID - FREE INDUCTION DECAY, T2*

- Signal (FID) resulting from RF pulse w/ angle $\alpha$
  \[
  \tilde{S}(t) = \sin \alpha \sqrt{\rho(w) \cdot e^{-t/T_2} \cdot e^{-i\omega t}} \, dw
  \]
  recorded complex signal

- An example spectral density ("Lorentzian inhomogeneity")
  \[
  \rho(w) = \frac{M_0^2}{(\Delta B)^2 + (w - \Delta B)^2}
  \]

- Combine $T_2$ + static terms
  \[
  \tilde{S}(t) = \pi \cdot M_0^2 \cdot \Delta B \cdot \sin \alpha \cdot e^{-\gamma \Delta B \phi} \cdot e^{-t/T_2} \cdot e^{-i\omega t}
  \]

- N.B. center freq., not original integration variable

- Overall decay rate including inhomogeneous $B\phi$
  \[
  \frac{1}{T_{2\ast}} = \frac{1}{T_2} + \frac{1}{T_{2\prime}}
  \]

- Recovery of $M_0^2 \sin \alpha$

- Illustration of exponential decay and oscillations.
ECHOES - spin echo

\[ t=0 \]

- Just after 90° x' pulse \( f_{lo} + f_{hi} \) have same phase

- Relaxation + phase dispersion of \( f_{lo} + f_{hi} \)
  (both from \( B > B_0 \))

- Remember brief RF just tips vectors while retaining length
  relaxation includes tips and shrinks \( (M_T) \) and grows \( (M_z, \text{echo}) \)

- 180° x' pulse works, too, but echo will have +\( \pi \) phase (left side in figs above)

- Echo generated even if second pulse not 180° (see next)

- FID decay (and echo growth/decay) described by \( T_2^* \), from inhomogeneity

- Reduction in height of echo compared to initial described by \( T_2 \), echo fixes 'star'

- Rotating coords

\[ t=T \]

- 180° y' pulse (y' pulse like x', pulse but RF has +90° phase)

\[ t=2T \]

- echo caused by re-phasing of \( f_{lo} + f_{hi} \)
  (w/decay due to \( T_2 \))

\[ z' \]

- N.B.: Bloch eqs in one voxel

\[ z' \]

- Rotate 180° around \( y' \)

\[ \vec{f}_{lo} + \vec{f}_{hi} \]

- arrows below plane

\[ \vec{f}_{lo} + \vec{f}_{hi} \]

- shaker from \( T_2 \)
**ECHOES — Spin echo**

\[
\alpha_1 - \tau - \alpha_2 - T
\] (both pulses along y' for simplicity)

**Effect of \( \alpha \) pulse**

\[
\begin{align*}
M_x' &\rightarrow M_x \cos \alpha - M_z \sin \alpha \\
M_y' &\rightarrow M_y \\
M_z' &\rightarrow M_x' \sin \alpha + M_z' \cos \alpha
\end{align*}
\]

(etc for \( \alpha_x, \alpha_z \))

**Effect of \( \tau \) delay**

\[
\begin{align*}
M_x' &\rightarrow (M_x \cos \omega \tau + M_y \sin \omega \tau) e^{-\tau/2} \\
M_y' &\rightarrow (-M_x \sin \omega \tau + M_y \cos \omega \tau) e^{-\tau/2} \\
M_z' &\rightarrow M_z (1 - e^{-\tau/2}) + M_z' e^{-\tau/2}
\end{align*}
\]

Immediately after \( \alpha \) pulse

\[
\begin{align*}
M_x'(\omega, 0) &= -M_z(\omega) \sin \alpha, \\
M_y'(\omega, 0) &= 0 \\
M_z'(\omega, 0) &= M_z(\omega) \cos \alpha
\end{align*}
\]

For one isochromat of freq. \( \omega \)

**After \( \tau \) delay**

\[
\begin{align*}
M_x'(\omega, \tau) &= -M_z(\omega) \sin \alpha \cos \omega \tau e^{-\tau/2} \\
M_y'(\omega, \tau) &= M_z(\omega) \sin \alpha \sin \omega \tau e^{-\tau/2} \\
M_z'(\omega, \tau) &= M_z(\omega) [1 - (1 - \cos \alpha_1) e^{-\tau/2}] \sin \alpha_2
\end{align*}
\]

**Time dependent free precession around \( z' \)**

(rewrite \( M_x'y'(\omega, \tau) \))

\[
\begin{align*}
M_x'y'(\omega, t) &= M_x'y'(\omega, \tau) e^{-(t-\tau)/T_2} e^{-i\omega(t-\tau)} \\
&= M_z(\omega) \sin \alpha_1 \sin^2 \frac{\omega_2}{2} e^{-t/T_2} e^{-i\omega(t+\tau)} \\
&- M_z(\omega) \sin \alpha_2 \cos^2 \frac{\omega_2}{2} e^{-t/T_2} e^{-i\omega t}
\end{align*}
\]

**For a large num. of freq's:**

\[
\begin{align*}
[M &\times \text{terms } 1, 2, 3 \text{ are dephasing} \Rightarrow \text{FID of echo} \\
&\text{term } 0 \text{ rephasing} \rightarrow \text{rephase at } t = 2\tau]
\end{align*}
\]

**Echo signal from 1 peak ampl.**

\[
S(t) = \sin \alpha_1 \sin^2 \frac{\omega_2}{2} \int_{-\infty}^{\infty} \rho(\omega) e^{-t/T_2(\omega)} e^{-i\omega(t-TE)} d\omega
\]

\[
A_E = \sin \alpha_1 \sin^2 \frac{\omega_2}{2} \int_{-\infty}^{\infty} \rho(\omega) e^{-TE/T_2(\omega)} d\omega
\]

\[
M_z \sin \alpha_1 \sin^2 \frac{\omega_2}{2} e^{-TE/T_2}
\]

(echo amplitude, ignoring freq. dependence of \( T_2 \))

**Echo parameters:**

\[
\begin{align*}
S_1(t) &= \frac{1}{2} \int_{-\infty}^{\infty} \rho(\omega) e^{-t/T_2(\omega)} e^{-i\omega(t-TE)} d\omega \\
S_2(t) &= \begin{cases} 
\frac{\rho(\omega)}{\pi} & \text{no } \frac{1}{2} \text{ factor} \\
\frac{\rho(\omega)}{\pi} & \text{add } \frac{1}{2} \text{ phase}
\end{cases}
\end{align*}
\]

etc for \( A_E \) ... like \( \Rightarrow \)
**ECHO TRAINS** — spin-echo trains

- It's (too) easy to make echoes...

\[
E_n = \frac{3^n - 1}{2}
\]

Echos after end of nth pulse
3 RFs \(\rightarrow\) 4 echos (here)
6 RFs \(\rightarrow\) 121 echos (!!)

Secondary echo: \(5E_{1,2}\) acts like RF pulse
\(x_3\) makes an echo from it

Two more conventional two-pulse spin echoes

Stimulated echo: combined effect of 3

\[
\begin{align*}
&\alpha_1: M_L \rightarrow M_T \\
&\alpha_2: \text{leftover } M_T \text{ flipped to } M_L \text{ (saved)} \\
&\alpha_3: \text{flip saved } M_L \rightarrow M_T \text{ which can then begin to cancel delays (after being held in limbo between 180° FID}_2 \text{ and FID}_3); \\
&\text{acts like 2-pulse echo}
\end{align*}
\]

- A useful multi-echo sequence (CPMG) is a 90° followed by 180° at 2τ spacing

- Typically, 90° and 180° applied in different axes \((x', \text{ then } y', y', ...\) which reduces phase errors due to imperfect 180° pulses (since slightly-off rotation around y' affects phase less)
EXTENDED PHASE GRAPHS

- Using full Bloch eq. solutions is tedious 😊 (need 1000 copies)
- Pictorial method for visualizing effects of a series of X pulses
- Starting point: initial RF creates new transverse pulse
  rotates a portion of existing transverse mag. into a position that results in rephasing and another portion into M_z.
  Third pulse can uncover and rephase transverse mag., helps QM view temporarily saved in longitudinal

Branching rule for effect of X 
RF pulse on transverse mag

Branching rule for effect of X 
RF pulse on longitudinal mag

Echo when phase path crosses zero

Each branch is weighted initially
Each branch decays
1) T_1 "stored" decays slower
2) T_2
3) M_z regrowth

Branch points:
- new M_z
- apply rule 1
- apply rule 2

Diagram showing phase dispersion and RF stim. points.
3 - Pulse Echo Amplitudes

- Assume $M_z^0 = 1$

---

**Echo** | **Time** | **Amplitude**
---|---|---
$SE_{1,2}$ ($t = 2T_1$) | $\sin \alpha_1 \sin^2 \frac{\alpha_2}{2} e^{-2\gamma_1/T_2}$ | $\alpha_1 = 90^\circ, \alpha_2 = 180^\circ e^{-2\gamma_1/T_2}$

**Special Cases**

- $2\alpha_1 = \alpha_2$
- $\sin^3 \alpha_1 e^{-2\gamma_1/T_2}$

$2^\circ$ ("Secondary") (t = 2T₂) ($t = 2T_1 + 2T_3$) | $-\sin \alpha_1 \sin^2 \frac{\alpha_2}{2} \sin^2 \frac{\alpha_3}{2} e^{-2\gamma_1/T_2}$

*N.B.: T₁*

$STE$ ("Stimulated") (t = 2T₁ + T₂) | $\frac{1}{2} \sin \alpha_1 \sin \alpha_2 \sin \alpha_3 e^{-\gamma_1/T_1} e^{-2\gamma_1/T_2}$

*N.B.: T₁*

$SE_{2,3}$ ($t = T_1 + 2T_2$) | $[1 - (1 - \cos \alpha_1) e^{-\gamma_1/T_1}] \sin \alpha_2 \sin^2 \frac{\alpha_3}{2} e^{-2(\gamma_1 + 2\gamma_2)/T_2}$

$SE_{1,3}$ ($t = 2(T_1 + T_2)$) | $\sin \alpha_1 \cos^2 \frac{\alpha_2}{2} \sin^2 \frac{\alpha_3}{2} e^{-2(T_1 + T_2)/T_2}$

---

- $T_2$-dependence in $STE$ (but also $SE_{2,3}$) from temporary "storage" of $M_T$ in $M_L$, then recovery by third pulse.
Hyper Echoes

1. $\phi_2 = 180^\circ$, $\phi_2 = 180^\circ$

2. $\alpha_y = 180^\circ$, $-\alpha_y = 180^\circ$

3. $\alpha_\phi = 180^\circ$, $-\alpha_\phi = 180^\circ$

-- Hennig & Scheffler (2001)
-- normalize $\vec{M}$ amplitude $\Rightarrow 1$
-- sphere surface defines 2D space for $\vec{M}$ moved by:

1) vector rotation of $\vec{M}$ around tilted axis in transverse $x$-$y$ plane by RF with flip, $\alpha$, and phase, $\phi = P(x, \phi)$

2) rotation around $z$ by phase evolution due to freq. offset, $w$ (B0 offset) and time, $t = \Phi(w, t)$

-- three symmetries:
- solid lines: phase eval $\Rightarrow 180^\circ$, phase or RF
- dashed lines: just $180^\circ$ equiv.

by combining long sequences observing these symmetries, can generate as strong echo even w/ many inserted $\alpha$-pulses in between

-- multi-echo example
-- can also use to prepare, then separate read-out

practical prob: $180^\circ$ pulses deposit a lot of RF (6x 90°) => prob at high fields

by arranging to get big echo in middle of $k$-space can get by with much less RF power

Practical use
**Gradient Echoes** - $T_2^*$, GE chains

- Initial negative gradient dephases spins
- After $t = \tau$ of positive gradient, spins rephase
- Does not correct for $T_2^*$ inhomogeneities
  - So echo amplitude is
    \[ A_E = e^{-t/T_2^*} \]
- The initial "FID" is not "free" since it is being actively dephased by gradient, so FID decay
  \[ \frac{1}{T_2} < \frac{1}{T_2^*} < \frac{1}{T_2^{**}} \]
  \[ A_E = e^{-t/T_2^{**}} \]

- Key difference between spin-echo (SE) and gradient echo (GE)
  - $B_0$ inhomogeneities not canceled
    \[ \Rightarrow \text{hence, echoes are } T_2^* \text{-weighted, not } T_2 \text{-weighted} \Rightarrow \text{more susceptible to inhomogeneities} \]

- Echo trains possible w/ gradient echo (CPMG-like)

- The faster the gradients are switched, the more echoes you get
- EPI hardware
  \[ \Rightarrow 64 \text{ echoes} \]
- Contrast (PD, T1, T2, T2*) depends on magnetization not getting back to equilibrium, and then differences in how far away each tissue type is at measurement time.

- Simple saturation/recovery w/ no echo.

- Initial conditions:
  \[ M_z \text{ before first pulse} = M_z^0 \]
  \[ M_z = 0 \text{ immed. after first pulse (i.e., } 90^\circ \text{ pulse)} \]

- From Bloch eq., \( M_z \) just before second pulse:
  \[ M_z^{(0)}(O_-) = M_z^0 (1 - e^{-TR/T_1}) + M_z^{(0)}(O_+) e^{-TR/T_1} \]
  \( M_z \) before current pulse
  \( M_z \) "regrowth-from-zero" term
  \( M_z \) "left-immed.-after-pulse" term (U.S. decaying)

- Given:
  \[ (1) \text{ } 90^\circ \text{ pulse} \]
  \[ (2) \text{ } no M_{xy} \text{ left} \] → pure tip: \( M_{xy} = M_z^0 \)

- Tip existing mag:
  \[ M_z^{(0)}(O_-) = M_{xy}^{(0)}(O_+) = M_z^0 (1 - e^{-TR/T_1}) \]
  Longitudinal mag just before pulse
  Transverse mag we can record after pulse
  Transverse mag depends on \( T_1 \)

- That is, the not-completely-regrown longitudinal magnetization, which depends on \( T_1 \), but which we cannot record, is completely converted to recordable transverse magnetization.

- Spectral density \( p(r) \) at p. density: underlying equilib. \( M_z^0 \)

\[ I(r) = C \cdot p(r) \cdot (1 - e^{-TR/T_1(r)}) \]

Recon. const. spectral dens.
IMAGE CONTRAST

Why imperfect 90° takes multiple flips till steady state

- initial fMRI images are usually discarded (why?)
  ➔ because they are brighter than all the rest
  ➔ because multiple flip required before steady state

N.B.: B1 imperfections guarantee this situation will occur
(e.g. at 3T, flip angle varies almost 25% across brain)

- at 3T, steady state
  for typical 1-2 sec
  TR images reached
  after 8 images
**IMAGE CONTRAST**

- Inversion recovery w/ no echo

**RF**

- $180^\circ$ pulse reverses longitudinal magnetization
  \[ M_z' = -M_z \]

- Recovery to end of first TI from long. part of Bloch eq.
  \[ M_z' = M_z^0 \left(1 - 2e^{-\frac{T1}{T1}}\right) \rightarrow \text{flipped into transverse by second pulse (1$^\circ$ 90)} \]

- Longitudinal then regrows from zero from first Bloch term
  \[ M_z' = M_z^0 \left(1 - e^{-\frac{(TR-T1)}{T1}}\right) \]

- After second $180^\circ$, just change sign again
  \[ M_z' = -M_z^0 \left(1 - e^{-\frac{(TR-T1)}{T1}}\right) \]

- Apply relaxation eq. again
  \[ M_z' = M_z^0 \left(1 - e^{-\frac{T1}{T1}}\right) - M_z^0 \left(1 - e^{-\frac{(TR-T1)}{T1}}\right) e^{-\frac{T1}{T1}} \]

\[ M_z' = M_z^0 \left(1 - 2e^{-\frac{T1}{T1}} + e^{-\frac{TR}{T1}}\right) \]

\[ \rightarrow \text{this is magnetization flipped to transverse, made recordable} \]
**IMAGE CONTRAST**

- **SE, IR-SE**

  - RF
  - 90° signal
  - 180° causes echo
  - slice select

  - G_x
  - phase encode
  - sweep
  - echo

  - G_y
  - center φ (10)
  - healthy
  - center φ (10)
  - center φ (10)

  - G_z
  - refocus

  - IFD
  - TE → TR

  - N.B. centering
  - G_x lobe also causes gradient echo

- **Steady state mag (2nd TR) just before 90°**

  \[
  M_z^o (O) = M_z^o \left(1 - 2e^{-\frac{(TR-TE/2)}{T1}} + e^{-\frac{TR}{T2}}\right)
  \]

- The echo signal (\(M_z^e\)) unlike in simple saturation recovery FID has an additional delay before it is recorded, so we have to take account of transverse mag relaxation

  \[
  A_e = M_z^o \left(1 - 2e^{-\frac{(TR-TE/2)}{T1}} + e^{-\frac{TR}{T2}}\right) e^{-\frac{TR}{T2}}
  \]

- If we assume TE much less than TR, then we can simplify:

  \[
  A_e = M_z^o \left(1 - e^{-\frac{TR}{T1}}\right) e^{-\frac{TE}{T2}}
  \]

- Similar equation for SE-IR

  \[
  A_e = M_z^o \left(1 - 2e^{-\frac{TI}{T1}} + e^{-\frac{TR}{T2}}\right) e^{-\frac{TE}{T2}}
  \]
*IMAGE CONTRAST*

GRE w/ small tip angle

- Use basic longitudinal relaxation from Bloch eq. again
  - & assume $M_x^{(n)}(0_-) = 0$ \(\Rightarrow\) transverse dephased before next pulse

  $$M_x^{(n)}(0_-) = M_x^0(1 - e^{-TR/T_2}) + M_x^{(n-1)}(0_+).e^{-TR/T_1}$$

- Assume we have a small tip angle:
  $$M_x^0 \cos \alpha \Rightarrow M_x^{(n)}(0_+) = M_x^{(n)}(0_-) \cos \alpha$$

- Assume we are in dynamic equilibrium:
  $$M_x^{(n)}(0_-) = M_x^{(n-1)}(0_-) = M_{x,y}^{ss}(0_-)$$

- Prepulse steady state longitudinal

  $$M_x^{ss}(0_-) = \frac{M_x^0(1 - e^{-TR/T_1})}{1 - \cos \alpha \cdot e^{-TR/T_1}}$$

- Post-pulse transverse magnetization

  $$M_{x,y}^{ss}(t) = \frac{M_x^0(1 - e^{-TR/T_1}) \cdot \sin \alpha \cdot e^{-t/T_2}}{1 - \cos \alpha \cdot e^{-TR/T_1}}$$

- Gradient echo amplitude

  $$A_E = \frac{M_x^0(1 - e^{-TR/T_1}) \sin \alpha \cdot e^{-TE/T_2}}{1 - \cos \alpha \cdot e^{-TR/T_1}}$$

- TI contrast mainly depends on flip angle $\alpha$, not TR $\Rightarrow \cos \alpha = 1$ \(\Rightarrow\) eliminates TI weight since denominator is numerator
- Saturate, wait for contrast\(_1\), invert, wait for contrast\(_2\), FLASH (center out)

A) \( M_2' \) (just after 90°) = 0 (perfect 90°)

B) \( M_2' \) (after TD) = \( M_2^0 \left( 1 - e^{-TD/T2} \right) \) (Blach term #1)

C) \( M_2' \) (just after invert) = \( \cos \phi \ M_2^0 \left( 1 - e^{-TD/T2} \right) \)

D) \( M_2' \) (after TI) = \( M_2^0 \left( 1 - e^{-T1/T2} \right) + \left[ \cos \phi \ M_2^0 \left( 1 - e^{-TD/T2} \right) \right] e^{-TI/T2} \)

E) \( M_2' \) (just after theta) = \( M_2^0 \left[ 1 - \left[ 1 - \cos \phi \left( 1 - e^{-TD/T2} \right) e^{-TI/T2} \right] \sin \alpha \)
**MAGNETIZATION TRANSFER CONTRAST**

- Protons in macromolecules & bind to membranes have wide range of resonant freqs ("bound") → $T_2 = 1$ msec

- Free protons in blood, CSF, water have narrow range of resonant freqs ("free") → $T_2 = 50$ msec

- Mag transfer pulse sequence
  1. Off center freq pulse to hit "bound" (but don't hit water too hard)
  2. Wait for magnetization transfer from saturated longitudinal $M_L$ of "bound" → $M_L$ of "free"
  3. Result of transfer → attenuation

  > N.B. this always happens a little (cf. $T_1$-weighted, $T_2$-weighted) something to keep in mind if hard pulse (wide freq)

- Used to increase contrast in TOF
  TOF (not explained) bright vessels from inflow fresh spins
  MT - contrast added: suppress tissue but not blood

- View w/ MIP: maximum intensity projection along lines

max → view as movie
**SIGNAL-TO-NOISE, CONTRAST-TO-NOISE**

- Signal-to-noise defined as: $\text{SNR} = \frac{\text{avg obj signal}}{\text{s.d. non-object region}}$
- Temporal SNR: $\text{tSNR} = \frac{S}{\text{s.d. non-object region}}$
- "Contrast" is a difference in signal intensity.
- Contrast-to-noise ratio:

$$C_{\text{AB}} = \frac{S_A - S_B}{\text{s.d. non-object region}}$$

$$\text{CNR}_{\text{AB}} = \frac{C_{\text{AB}}}{\text{s.d. non-object region}} = \frac{S_A - S_B}{\text{s.d. non-object region}}$$

**Spin-Echo:**

$$A_E = M_0 \cdot (1 - e^{-TR/T_1}) \cdot e^{-TE/T_2}$$

**Gradient Echo:**

$$A_E = \frac{M_0 \cdot (1 - e^{-TR/T_1})}{1 - \cos \alpha \cdot e^{-TR/T_1}} \cdot \sin \alpha \cdot e^{-TE/T_2\alpha}$$

**General Rules:** Spin-echo, Long TR GE

<table>
<thead>
<tr>
<th>Proton-density weighted</th>
<th>TR 440 (no T1 diffs)</th>
<th>TE 66 (no T2 diffs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T1-weighted</td>
<td>TR ≈ Ti (big T1 diffs)</td>
<td>TE 66 (no T2 diffs)</td>
</tr>
<tr>
<td>T2-weighted</td>
<td>TR 440 (no T1 diffs)</td>
<td>TE 66 (big T2 diffs)</td>
</tr>
</tbody>
</table>
SIGNAL-TO-NOISE S/N

- generalized dependence of SNR on 3D imaging parameters

\[
\text{SNR/voxel} \propto \frac{\Delta x \Delta y \Delta z}{\sqrt[3]{N_{ac} N_x N_y N_z \Delta t}}
\]

\[\text{voxel size} \quad \text{num. repeats} \quad \text{number of voxels of same size} \quad \text{read timestep}\]

- size (volume) of voxels (with the number of voxels held constant), linear effect on S/N

\[\Delta V \rightarrow \text{e.g., } 3 \times 3 \times 3 \text{ mm} \rightarrow 4 \times 4 \times 4 \text{ mm} \rightarrow \frac{64}{27} = 2.37 \text{ times better S/N}\]

- more voxels (with size of voxels, \(\Delta t\) per read step constant), \(\sqrt[3]{N}\) effect on S/N

\[
\Delta V \rightarrow \text{e.g., } 64 \times 64 \rightarrow 128 \times 128 \rightarrow \frac{\sqrt[3]{128 \times 128}}{\sqrt[3]{64 \times 64}} = 2 \text{ times better S/N}\]

- # acquisitions \(\sqrt{N}\) better S/N

\[\Delta V \rightarrow \text{e.g. } 1 \text{ acq} \rightarrow 2 \text{ acq} \rightarrow \frac{\sqrt{2}}{\sqrt{1}} = 1.41 \text{ times better S/N}\]

- larger timestep during readout, \(\sqrt{\Delta t}\) better S/N

\[\Delta t = \frac{1}{\text{BW}_{\text{read}}} \quad \text{digitization timestep during echo acquisition}\]

- \(\text{BW}_{\text{read}}\) determined by cutoff freq, analog low-pass filter

- \(\Delta t\) controls BW because low-pass cutoff has to be set higher for smaller (higher frequency detecting) \(\Delta t\)

- must filter out freq's > \(f_{\text{max}} = \frac{1}{2\Delta t}\) because they alias
**Complex Algebra**

- **Real/Imaginary**
  - add: \((r_1, i_1) + (r_2, i_2) = (r_1 + r_2, i_1 + i_2)\)
  - mult: \((r_1, i_1) \times (r_2, i_2) = (r_1r_2 - i_1i_2, r_1i_2 + i_1r_2)\)

- **Angle/Phase**
  - add: \((A_1, \phi_1) + (A_2, \phi_2) = (A_1 \cos \phi_1 + A_2 \cos \phi_2, A_1 \sin \phi_1 + A_2 \sin \phi_2)\)
  - multiply: \((A_1, \phi_1) \times (A_2, \phi_2) = (A_1A_2 \cos(\phi_1 + \phi_2), A_1A_2 \sin(\phi_1 + \phi_2))\)
  - divide: \((A_1, \phi_1) \div (A_2, \phi_2) = (A_1/A_2 \cos(\phi_1 - \phi_2), A_1/A_2 \sin(\phi_1 - \phi_2))\)

- **Complex to Real power**: \((A, \phi)^n = (A^n, n\phi)\)

**Fourier Transform**

- **Fourier Transform**
  - \(H(f) = \int h(t) e^{-i2\pi ft} \, dt\)
  - \(H(f) = \int h(t) e^{i2\pi ft} \, dt\)

**Convolution Theorem**

- \(F[g(x) \ast h(x)] = G(k) \ast H(k)\)

- The Fourier transform of two functions multiplied by each other equals the convolution of the Fourier transforms of each function:

- **Convolution**
  - \(f(x) = g(x) \ast h(x) = \int g(z) \cdot h(x-z) \, dz\)
  - \(F[f(x)] = G(k) \ast H(k)\)
  - \(G(k) = \int g(z) \cdot h(x+z) \, dz\)
Fourier transform (1)

\[ H(f) = \int_{-\infty}^{\infty} h(t) \cdot e^{-j2\pi ft} \, dt \]

- How to calculate \( H(f) \) for one \( f = 3 \):
  (real signal: only need 2 correlations)

\[ h(t) \quad \text{real signal} \]

\[ e^{-j2\pi ft} \quad \text{complex multiply} \]

\[ \cos(2\pi ft) \quad \sin(2\pi ft) \]

\[ \int_{-\infty}^{\infty} \quad \text{integrate/sum these multiplies across all } t \]

\[ \{ \text{real frequency domain} \}
\quad \{ \text{imaginary frequency domain} \} \]

\[ (A, \phi) \quad \text{amplitude frequency domain} \]

\[ \text{like correlating with } \sin \text{ and } \cos \text{ (at each freq) so we get phase (at each freq.)} \]
Fourier transform (1b)

\[ e^{i\phi} = \cos \phi + i \sin \phi \]
\[ e^{-i\phi} = e^{i(-\phi)} = \cos(-\phi) + i \sin(-\phi) \]
\[ = \cos \phi - i \sin \phi \]

- cos is an "even" function, sin is an "odd" function

An orthogonal decomposition
- think of discretely sampled \( \sin(bx) \), \( \cos(bx) \) as vectors
- \( \text{Corr}(\vec{v}_i, \vec{v}_j) \equiv \text{projection of } v_i \text{ onto } v_j \equiv \vec{v}_i \cdot \vec{v}_j

\[
\begin{align*}
\text{Corr}(\cos bx, \sin bx) &= 0 \\
&= \sin \text{ and } \cos \text{ of same frequency are orthogonal} \\
\text{Corr}(\sin bx, \sin bx) &= 0 \\
&= \text{different integer freqs of } \sin \text{ and } \cos \text{ are orthogonal} \\
\text{Corr}(\cos bx, \sin bx) &= 0 \\
&= \text{as above}
\end{align*}
\]

- in the continuous case, orthogonal functions defined as:
  \[ \int_{x=hi}^{x=lo} f(x) g(x) \, dx = 0 \]
UNDERSTANDING INVERSE FOURIER TRANSFORM AS ANOTHER CASE OF CORR w/ COS, SIN

- start with spike in image domain
- take example of spike at \(X = 0\)

\[
\begin{bmatrix}
\cos(x) & \cos(2x) & \cos(kx)
\end{bmatrix}
\text{ all freqs correlate w/ spike at } X = 0
\]

- if spike is moved away from zero, the frequency spectrum obtained by correlating cos with spikes oscillates

N.B. opposite direction sin spike are on imaginary axis

- to see why this is, since spike is all zero except at spike, the dot product for a given frequency only depends on the value of the \(e^{-i\omega x}\) cos and sin at location of spike

- pos pair (real) spikes same dist from origin
- pos/neg pair (imaginary) spikes, same dist orig.
- one spike at distance from origin

---

this is one way of thinking about what one point in k-space means, via correlating it w/ cos's, sin's to get periodic result in image space (inverse FT)
FOURIER TRANSFORM OF AN IMAGE (2)

1. Real image \( \xrightarrow{\text{Fourier Transform}} \) Imaginary image (Zero)
   - Spatial frequency

2. Amplitude image \( \xrightarrow{\text{Inversion}} \) Phase image (Zero)
   - View complex vectors directly

3. Complex vectors
   - Three equivalent representations of image & spatial frequency space
FOURIER TRANSFORM OF REAL IMAGE (2)

- what a single k-space point looks like for real image (polar coordinates A, ϕ instead of r, ϕ)

image space

(k-space)

(spatial freq. space)

offset of stripes is k-space phase

brightness of stripes proportional to k-space amplitude

distance from center is stripe spacing

angle of point perpendicular to angle of stripes

Inverse Fourier transform (image recon.)

amplitude

phase

(Should be all zero, not same as "stripe phase" above)

Cartesian dimension of k-space — x- and y- spatial freq.

N.B.: each dimension of spatial freq. space (k-space) from correlation w/ sin & cos — don't confuse k_x, k_y w/ sin, cos!

N.B.: increasing one 1D component increases the spatial freq of the 2D wave and rotates it
FOURIER TRANSFORM OF IMAGE (4)

- 3 equivalent representations of complex numbers in image space and spatial-freq. space (k-space)
- Example: cosinusoid in image space, then shifted in x-dir

REAL IMAGE

\[ I(x, y) = \cos(x) \]

FT of \( I(x, y) \)

\[ \text{Real component less than above because rot:} \]

\[ \text{Phase now 45° at } k_y = 1 \]
\[ \text{(-45° at } k_y = -1 \)
FOURIER TRANSFORM OF IMAGE (5)

- (cont.) center of k-space (real image)
- complex image

REAL IMAGE

\[ I(x, y) = 1 + \cos(x) \]

\[ \text{center of k-space:} \quad H(k) = \int h(x) e^{-2\pi ikx} dx \]

\[ \text{avg image brightness} = I(\text{real}) \]

FT OF REAL IMAGE

[the center of k-space is zero w/ pure sin or cos image b/c avg. brightness = 0]

COMPLEX IMAGE

\[ I(x, y) = \cos(x) - i \sin(x) = e^{-ix} \]

FT, FT\(^{-1}\)

"missing" spike results in single spike correlating with cos and sin

N.B.: this k-space is non-Hermitian:
- k-space will only have Hermitian symmetry if image is real:
- Hermitian symm. when complex conjugate (= complex num w/ sign flipped in image part) is equal to funct. value w/ arg.

1D: \[ H(k) = H^*(k) \]
2D: \[ H(-k_x, k_y) = H^*(k_x, k_y) \]

complex

spike only on one side of k-space

N.B.: this is like what an artifact "spike" does tho it would have real, phase

[ N.B. this is also exactly what a gradient does to image space! ]
FOURIER TRANSFORM OF IMAGE (G)

- (cont.) x- and y-spatial freqs.
- special case: real image from sum of reals

REAL IMAGE

\[ I(x,y) = \cos(x) + \cos(y) \]

N.B. adds but doesn’t rotate stripes

FT OF REAL IMAGE

\[ FT \rightarrow FT^{-1} \]

\[ I(x,y) = \cos(x+y) \]

rotates stripes!

\[ FT \rightarrow FT^{-1} \]

Remember, single k-space point transforms to complex img.
- but if Hermitian symmetry, imaginary components cancel
- Since all we want in image space reconstruction is real component, can just add real components of complex vectors at each image space point for every complex image corresponding to each k-space point

N.B.: the k-space phase will affect offset of real-valued image space cosinusoïd
- Therefore for real-valued image, we can visualize inverse FT as real-valued sum of offset real-valued cosinusoïds

N.B. cannot do this with MRI k-space data since phase errors (incl. multiple wraps) mess up real component — must use amplitude img
GRADIENT COILS

- gradient coils for x, y, z generate approximately a linear gradient in the strength of the z-component of the magnetic field B₀.

- for example, the x gradient coil induces a ramp in z-component of the magnetic field when moving in the x-direction:

  \[ B_{G,z} = G_x x \]

- Since a pure linear gradient of B₀, z in only the x, y, or z directions is not possible according to Maxwell equations, each gradient coil also induces a magnetic field that has components in the x- and y-direction (B₀,x and B₀,y).

- the other magnetic field components are usually ignored because they are so small relative to B₀,z, since B₀,z is added to B₀, and since B₀ is much stronger than B₀,x, B₀,y, and B₀,z.

- Since standard reconstruction methods assume the existence of "non-Maxwellian" gradient fields, spatial distortion is introduced.

- the Maxwellian terms B₀,x, B₀,y are known; can be included in the reconstruction process.

\[ \Delta \phi G_x(x) \approx -\frac{x^2 G^2 x}{2B_0} \]
SLICE SELECTION ($G_2$)

- slice select gradient on during RF stim

$$B_z f = \frac{x}{\pi t} (B_0 + B_{G_2})$$

- protons here can only be excited by a narrow band of radio frequencies

- to apply a pulse containing a narrow band of frequencies, we use a sinc pulse envelope (Fourier transform of a narrow freq. band)

- this excites protons in a narrow slab

- since the slice-selection gradient introduces (space-dependent) phase shifts (see freq encode) these have to be removed by a post-excitation rephasing $z$-gradient

- approximation from assuming tip occurs instantaneously in middle

- valid for small tip: $90^\circ \rightarrow 52^\circ$

- in practice: adjust to max, use crusher to kill spurious transverse on $180^\circ$
PULSES FOR SLICE SELECTION

- Fourier transform approach to slice-selective pulse (linear approx. even tho tipping is non-linear)

\[ \hat{B}_1(t) \propto \int_{f=-\infty}^{f=\infty} p(f) \cdot e^{-i2\pi ft} df \]

Frequency selection function

\[ \hat{B}_1(t) = A \cdot f_w \cdot \text{sinc}(\pi f_w t) \cdot e^{-i2\pi ft_c} \]

Amplitude controlling flip angle (control slab width)

Sinc envelope width determined by freq. width, \( f_w \) (N.B. wider \( f_w \) is narrower sinc)

Modulation (complex) at center freq., \( f_c \)

Larmor oscillations, at center freq.

Sinc envelope width inversely proportional to \( f_w \)

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SLICE SELECT RF PULSES

Interleaved Acquisition → better S/N b/c imperfect slice profile

Common RF pulses

- non-selective pulse ("hard" pulse)
- standard slice select sinc
- Gaussian

→ pulses need to be "apodized" (have "foot" removed)
→ multiply by function so begin/end of pulse is differentiable

Fat Saturation

- fat protons have chemical shift causing resonant freq offset
- add phase offset not due to gradients, RF
- fix by off-water-resonance 90° (saturation) pre-pulse centered on fat freq
→ need high quality (narrow-freq) pulse to avoid saturate water!

HOWTO

1) fat sat pulse
2) wait T2 so fat signal decays, but no T1 regrowth of fat
3) RF stim for water "protons - p" interest

Adding Another Gradient Tilts Slice

with 3 gradients on, can excite arbitrary angle plane
translate plane by changing either gradient amplitude or RF freq band: B0
WHY "FREQUENCY-ENCODING" IS A MISNOMER

- comes from original analogy in Lauterbur (1973):

<table>
<thead>
<tr>
<th>Spectroscopy</th>
<th>Imaging</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) chemical shift change freq ⟷ gradient changes freq.</td>
<td></td>
</tr>
<tr>
<td>2) stimulate w/broadband RF ⟷ same</td>
<td></td>
</tr>
<tr>
<td>3) time-sample FID containing multiple freqs ⟷ same</td>
<td></td>
</tr>
<tr>
<td>4) FT of FID to get spectrum ⟷ FT of FID to get Δx offsets</td>
<td></td>
</tr>
</tbody>
</table>

- this is technically correct (FT of FID) but highly misleading
  - e.g., phase-encoding (turning a different gradient ON and OFF before recording FID) seems to be something completely different since OFF gradient can't affect freqs in FID

- the "k-space" perspective is a "Copernican Turn"
  - idea is that data is not a set of samples of a time domain signal generated by multiple chemical-shift like frequencies
  - rather, it is a set of samples of a frequency-domain signal, each sample generated by multiple spatial locations (which are analogous to multiple time points)
  - i.e., the 'direction' of the FT (Fourier transform) is reversed:

<table>
<thead>
<tr>
<th>signal</th>
<th>result</th>
</tr>
</thead>
<tbody>
<tr>
<td>spectroscopy</td>
<td>samples of oscillations in time-domain ⟷ frequency-domain spectrum of shifts</td>
</tr>
<tr>
<td>MRI</td>
<td>samples of spatial freq. in freq.-domain ⟷ spatial object (like a time-domain signal)</td>
</tr>
</tbody>
</table>

- the original analogy only 'works' because FT ≈ FT⁻¹ (except sign change)
FREQUENCY ENCODING (1)

- Frequency encode gradient \( G_x \) causes precession rates to vary linearly in \( x \)-direction

\[ \begin{align*}
\text{precession} & \quad \text{frequency} \quad \uparrow B_z \\
\text{in} & \quad \text{\( x \)-direction} \quad \rightarrow \text{correct} \quad (\text{remember that strength of} \ G_x \text{causes variation of slope of} \ B_z \text{in} \ x \text{-direction})
\end{align*} \]

- Different frequency signals are mixed together, and recorded as a 1-D signal over time

\[ \rightarrow \text{correct, but remember, we are recording summed local magnetization vectors after de-modulation} \]

- A Fourier transform, which can convert back and forth between \( x \)-position (cf. time) and spatial frequency (cf. temporal freq.) is done on signal

\[ \rightarrow \text{correct} \]

- Spatial frequencies get confused/conflicted with precession frequencies

\[ \rightarrow \text{wrong} !! \]

- Therefore, the Fourier transform is used to convert position-dependent precession frequencies into spatial position

\[ \begin{align*}
\text{conceptually wrong} !! & \quad \rightarrow \text{FT actually converts spatial frequencies to spatial position} \\
& \quad \rightarrow \text{the spatial frequency increases for each time point in the readout} \\
& \quad \rightarrow \text{the precession freq ramp is constant each timestep}
\end{align*} \]

N.B.: Gradient ramp does not need to be on during recording!!
FREQUENCY ENCODING (2)  connect intuition - why phase critical

- "Frequency"-encode gradient ($G_x$) turned on during
during echo causes precession rates
to immediately vary with x-position

- at beginning of gradient on, the phase of
signal coming from each x-position is the same
Summed phase angle is what we measure

- early after gradient on, phase advances (because
of faster precession frequency) arise with greatest
phase advance at largest x-position

- later during gradient on, phase advances cause
multiple wraparounds of phase angle across space

- in practice, the lowest spatial frequency (= 0)
occurs in the middle of the gradient on time
because the phase is "rounded" negatively by
a preparatory gradient (to the highest negative
spatial frequency) before data collection occurs

$\delta$ is spatial frequency

Individual RF data samples (after
demodulation)
FREQUENCY ENCODING (3) why each datapoint is 1 spatial freq

Standard Fourier transform: (Temporal freq $\leftrightarrow$ time)

$$H(\omega) = \int_{-\infty}^{\infty} h(t) \cdot e^{-i \frac{2\pi \omega}{T} t} dt$$

"$k$" is often used instead of "$\omega$" for the frequency variable

Imaging equation: (Spatial freq. $\leftrightarrow$ space)

$$S(\omega) = \int_{-\infty}^{\infty} I(x) \cdot e^{-i \frac{2\pi \omega}{T} x} dx$$

Sum across x of object

Signal strength at one x-position (brightness of image point)

Spin density (spectral density)

Oscillations come from readout phase wrapping, where $f$ is single spatial freq (e.g. 5/3) and $x$ goes across object

$$f = \frac{G_x}{SE} = \frac{G_x (t-TE)}{}$$

To make image, do inverse Fourier transform of recorded signal $S(\omega)$

Don't confuse with instantaneously changed precession freqs which are constant across entire readout time (for each x position)
ALTERNATE DERIVATION (incl. effects of $G_x$) SIGNAL EQ

- oscillators at $w = \frac{\partial}{\partial x}$ at each position (just $x$ for now)

$$S(t) = n(x) e^{-i\phi(x)} dx$$

- by definition, freq. $w$ is rate of change of phase, $\phi$

$$\frac{d\phi(x,t)}{dt} = w(x,t) = \frac{\partial B(x,t)}{\partial x}$$

integrating $\phi(x,t) = \int_0^t w(x,t) dt = \int_0^t B(x,t) dt$

- assuming phase initially 0, B affected by gradients

$$B(x,t) = B_0 + G_x(t) x$$

$$\phi(x,t) = \int_0^t B_0 dt + \left[ \int_0^t G_x(t) dt \right] x$$

$$= \omega_0 t + 2\pi K_x(t) x$$

$k$ is time integral of gradient waveform

- demodulation removes the $B_0$-caused carrier frequency $e^{-i\omega_0 t}$ from the first equation

$$S(t) = \int_x m(x) e^{-i 2\pi K_x(t) x} dx$$

amplitude of each oscillator

gradient-controlled phase
- Turn on gradient after excitation but before readout

- Different levels of $G_y$
  \[ B_{z,y} \]
  \[ y \]
  \[ y \]
  \[ y \]

- Higher levels of $G_y$ (slope of $B_z$ in $y$-direction!)
  \[ \rightarrow \text{higher spatial freq. (more phase wraps) in } y \text{-direction} \]

- Phase wraps persist after phase-encode gradient off

- Read-out gradient ($G_x$) phase wraps then add to phase-encode phase

2D Imaging Equation

\[
S(k_x, k_y) = \iint \text{I(x,y)} \cdot e^{-2\pi (k_x x + k_y y)} \, dx \, dy
\]

- Signal recorded at single time point (one readout point)
  \[ \downarrow \]
  complex signal (from phase-sensitive detection)

- Sum across x-y plane
  \[ \downarrow \]
  image (= strength of magnetization at each x-y point)

- Phase (vector of unit length and particular angle which is function of $G_x$ and $G_y$)
  \[ \downarrow \]
  phase angle (of scalar magnetization) in rotating frame, set by gradients

- Ignoring relaxation, spatial frequency $k_x$ and $k_y$ have no "inertia"—they stay wherever the gradients last left them
3-D IMAGING

- use $z$-gradient for 2nd phase-encoding instead of slice selection

- excitation of whole slab (slice-select is whole brain)

- simple spin echo example (in real life, usu. done with echo trains [FSE] or small flip angle to allow short TR [SPGR])

Simple 3D spin echo example

$S(k_x, k_y, k_z) = \ldots$

$I(x, y, z) = e^{-2\pi (k_x x + k_y y + k_z z)} dx dy dz$

- i.e., freq-encode phase, first phase-encode phase, and second phase-encode phase all just add (= 3D rotation of phase stripes)

- S/N much better than 2-D because each entire excited volume contributes to signal from each pulse instead of just slice

$\rightarrow$ phase stripes created throughout volume vs. slice:

N.B., this ignores relaxation effects for now

one readout point in image space

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PHASE & FREQ, 2D & 3D

Since the phase-encode gradient and the freq encode gradient both affect phase, the result is a rotation of phase "stripes" when the two add.

Phase encode f=4
Freq encode f=0

Phase encode f=4
(unchanged in y-axis)
Freq encode f=10

N.B.: Stripes have sharp edges from phase wrap (not sinusoid since Q from 2-comp quadrature!)

Stripes here represent complex value

Phase of whole image summed to one (complex) number by RF coils

Successive read out steps:

More rotation
Higher spatial freq

Small phase encode Gy

Large phase encode Gy

Small phase encode Gz

Large phase encode Gz

E.g., after y-gradient, spins at a point might be 2 cye ahead while after x-gradient spins at same plt 8 cye ahead: but counting wraps in y-direction, still only 2 ahead

3D phase encode w/ G_y and G_z starts rotated in y-z plane
**Gradients Move K-Space Location of Data Point**

- K-space (spatial-frequency space) location is set by integral of gradient over time up to recording point:

\[
K = \frac{1}{x} \int_0^t G(t) \, dt
\]

Spatial freq recorded at function of \( t \) - gradient strength as \( t \) recorded time up to recording point.

Simple form of integral w/ boxcar gradient:

\[
K = G t
\]

(k is area under curve).

---

**Frequency-encode**

- RF 90°
- \( G_x \)
- Samples

\[ \rightarrow \]

\[ K_y \]

\[ K_x \]

---

**Frequency-encode gradient echo**

- RF 90°
- \( G_x \)

\[ \rightarrow \]

\[ K_y \]

\[ K_x \]

Neg \( G_x \)

---

**Frequency-encode spin-echo (plus gradient echo!!)**

- RF 90° + 180°
- \( G_x \)
- TE

\[ \rightarrow \]

\[ K_y \]

Pos \( G_x \)

---

**Phase-encode then frequency encode gradient echo**

- RF 90°
- \( G_y \)
- \( G_x \)

\[ \rightarrow \]

\[ K_y \]

\[ G_x + G_y \]

---

**N.B. 180° wave to conjugate point.**
**Image Reconstruction**

\[ S(k_x, k_y) = \sum_{y} \int_{x} I(x, y) e^{-i 2\pi (k_x x + k_y y)} dx \, dy \]

- \( I(x, y) \): signal (complex) in screen image
- \( S(k_x, k_y) \): brain spin density (real) in transverse magnetization
- \( e^{-i 2\pi (k_x x + k_y y)} \): gradient caused phase maps (complex) in transverse magnetization
  - N.B.: assumes perfect sinusoids! (they're not)

\[ I(x, y) = \sum_{k_y} \int_{k_x} S(k_x, k_y) e^{i 2\pi (x k_x + y k_y)} dk_x \, dk_y \]

- ideally \( I(x, y) \) is real
- in practice \( I(x, y) \) is complex
- use amplitude image: \( |A| \, e^{i \phi} \)

Adding exponents is the same as multiplying two \( e^{i 2\pi k \cdot} \)'s:

\[ \int_{k_y} \int_{k_x} S(k_x, k_y) e^{i 2\pi k_x x} e^{i 2\pi k_y y} dk_x \, dk_y \]

Same as two sequential 1D FFTs (actual code):

\[ \int_{k_y} \left[ \int_{k_x} S(k_x, k_y) e^{i 2\pi k_x x} dk_x \right] e^{i 2\pi k_y y} dk_y \]

- in practice, finite number of samples \( N \) and \( M \) are collected
- \( k_x \) and \( k_y \) directions of \( k \)-space (integral \( \rightarrow \) discrete sum)
  - \( b/c M/2 \) belongs to next replica

\[ I(x, y) = \sum_{m=-M/2}^{M/2-1} \left[ \sum_{n=-N/2}^{N/2-1} S(n, m) e^{i 2\pi \frac{n}{N} \Delta k_x x} e^{i 2\pi \frac{m}{M} \Delta k_y y} \right] e^{i 2\pi m \Delta k_y y} \Delta k \]

- sampling interval in \( k \)-space
- Sampling
- Aliasing, FOV

- Limited in range of frequencies sampled (Δk)
- Limited in rate of sampling
- Causes overlap of replicas
- Finite frequency range
- Insufficient samples in time domain

Thus, finer sampling of time gives fewer
replicas in k-space

- Limited frequency range
- Finite samples in space
- Insufficient samples in k-space

- Lower Nyquist frequency than
- Smaller sampling rate
UNDER/OVER SAMPLE

\[ \text{FOV}_x = \frac{1}{\Delta k_x} \]

\[ s_x = \frac{\text{FOV}_x}{N} = \frac{1}{N \Delta k_x} \]

- FOV (distance to repeat) is reciprocal of spatial frequency sampling interval
- Pixel size is FOV divided by K-space sample count

3 more examples (not included: less samples to same spatial freq (bottom last page))

- Basic Image
- Same num samp. to 2x spatial freq.
  (i.e. gradients stronger or time ON longer)
- 2x num. samples to same spatial freq.
  (i.e. gradients weaker or time ON shorter)
- 2x number samples to 2x spatial freq.
  (i.e. gradients stronger or time ON longer)

- N = 10
  - k_x = 5
  - \( \Delta k_x = 1 \)
  - FOV = 1
  - \( s_x = 0.1 \)

- N = 10
  - k_y = 10
  - \( \Delta k_y = 1 \)
  - FOV = 2
  - \( s_x = 0.05 \)

- N = 20
  - k_x = 5
  - \( \Delta k_x = 0.5 \)
  - FOV = 2
  - \( s_x = 0.05 \)

- N = 25
  - k_y = 10
  - \( \Delta k_y = 1 \)
  - FOV = 1
  - \( s_x = 0.05 \)

- Basic image
- Square pix
- X-pix half width
- Replicas intrude
  [Scanner makes square image "wrap" occurs]
- Square pix
  - Twice x-pix count
  - So FOV = 2x
- This is "phase oversamp"
  [Scanner crops to square
  replicas move out]
- X-pix half width
  - Twice x-pix count
  - Same FOV
- This is decrease pixel size w/o change FOV
Fourier Transform Solution to Replicas

1. Image/brain space
2. Sampled data spatial frequency

Convolving:

\[ \text{convolve} \]

\[ \Rightarrow \]

\[ \Rightarrow \]

\[ = \text{equals} \]

\[ = \text{equals} \]

Useful FTs:

- Rect
  \[ \text{rect} \left( \frac{x}{w} \right) \xrightarrow{\mathcal{F}} W \cdot \text{sinc} \left( \pi w K \right) \]

- Gaussian (special case)
  \[ e^{-\pi x^2} \xrightarrow{\mathcal{F}} e^{-\pi K^2} \]
  \[ \text{larger} \Rightarrow \text{narrower} \]

- Gaussian (adj. width)
  \[ e^{-a x^2} \xrightarrow{\mathcal{F}} \sqrt{\frac{\pi}{a}} e^{-\frac{\pi k^2}{a}} \]

- Comb
  \[ \sum_{n=-\infty}^{\infty} \delta(x - n) \xrightarrow{\mathcal{F}} \Delta K \sum_{p=-\infty}^{\infty} \delta(k - p \Delta K) \]

- Limit approach to Fourier transform of comb
  \[ F_{\text{fov}} = \frac{1}{\Delta K} \]
  \[ \Delta K = \frac{1}{\text{fov}} \]
**Point Spread Function**

\[ \hat{I}(x) = \Delta k \sum_{n=\{-N/2, N/2\}} S(n \Delta k) e^{i 2\pi n \Delta k x} \]

- Set true image to \( S \)-function, then measured signal is:
  \[ S(m \Delta k) = 1 \]
- Substitute into recon to get PSF:
  \[ h(x) = \Delta k \sum_{n=\{-N/2, N/2\}} e^{i 2\pi n \Delta k x} \]
- Simplify
  \[ h(x) = \Delta k \frac{\sin (\pi N \Delta k x)}{\sin (\pi \Delta k x)} \Rightarrow \text{periodic} \]
- That is, image is reconstructed from a sum of sinc's, because the FT of a boxcar pixel in \( k \)-space is an image sinc

---

**Image**

- How PSF modifies ideal (infinite \( k \)) image
  - Convolve
  - \( \Rightarrow \) ringing

**FT**

- Under rect.
  - Narrower sinc
  - \( \Rightarrow \) FT
  - (Taken outside zero crossings)
  - Acquisition window
    - truncates high spatial freq. data
GENERAL LINEAR INVERSE RECON FOR MRI

\[ S(k_x) = \int I(x) e^{-i \frac{2\pi}{k_x} x} \, dx \]

Signal eq. \( \rightarrow \) fwd problem

\[ I(x) = \int S(k_x) e^{+i \frac{2\pi}{k_x} x} \, dk_x \]

Recon eq. \( \rightarrow \) inv. problem

\[ s = F \cdot i \]

\[ s = \begin{bmatrix} x \ y \end{bmatrix} \begin{bmatrix} F \\ i \end{bmatrix} \]

Linear "forward solution"
Matrix vectors have complex entries
Can build in any measurable priors

\[
F_{x,y,t} = g(x,y) e^{-i\phi(x,y)} e^{-\frac{(T\pm m \Delta T + T_E)/T_2}{i T_2}} e^{-i \Delta B(x,y) T \pm m \Delta t} e^{-i 2\pi (m \Delta k_x \pm n \Delta k_y) / \Delta x y} \]

cal gain at this location
coil phase
T2 decay
B0 error (x,y dep.)
Freq + phase (complex)

Multi-coil

\[
\begin{bmatrix} s \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} k_x \\ k_y \end{bmatrix} \begin{bmatrix} F_{coil 1} \\ F_{coil 2} \end{bmatrix} \]

Naturally incorporates undistorted field map
different sensitivities function for each coil
contains additional info about some loc.
But, need reference scan, lo-res ok
(need phase corrections for each coil)

\[ i = F^+ s \]

over-determined

More Penrose inverse

\[
F^+ = (F^T F)^{-1} F^T \quad (x,y)^2 \rightarrow "small" \]

\[
F^+ = F^T (F F^T)^{-1} \quad (x,y \cdot \text{cobs})^2 \rightarrow 16 \times \text{bigger for 4 coils} \]

\[
i = \begin{bmatrix} (F^T F)^{-1} F^T \end{bmatrix} s \]

Slice-by-slice
Assume slice select swamps others
FAST SPIN ECHO (FSE)  RARE, FSE, 3DFSE

- one 90° pulse followed by multiple 180° pulses (e.g., 8) each with a different phase-encode gradient
- each phase “winder” is “unwound” because leftover phase would be re-focused away by 180° (vs. EPI where it persists between blips)

- the “effective TE” is the TE when center of k-space is collected (largest effect on contrast, largest echo)
- each subsequent echo has more T2 decay: \( E_n = e^{-nTE/T2} \) \( n = 1, 2, \ldots, M \)

- by arranging to collect \( k_y = 0 \) early, PD-weighted instead of T2-weighted

- possible to correct different T2-weighting of echoes by estimating T2 curve from \( k_y = 0 \) echo train

- 3DFSE — like 2D except
  - wind/unwind added to thick slice select (w/curshers on 180°)
  - \( G_z \) (phase 2)
  - \( G_y \) (phase 1)

N.B. all those 180° pulses deposit a lot of RF power: \( 90° + 180° = 45x \) power 30°
MULTI-SLAB 3DFSE, PROBLEMS

- RF in
- $G_z$
- $G_y$
- $G_x$
- RF out

$90^\circ$ $180^\circ$ 8.5 ms min $180^\circ$ $180^\circ$

→ echo train e.g. 20
→ etc to fill 3D k-space

$G_z$ is "partition"
$G_y$ is "phase encode"
$G_x$ readout needs no pre-wind since $180^\circ$ does it

→ $T_{eff}$ is time from $90^\circ$ to echo thru center of k-space

- echoes die out quickly by $e^{-t/\tau}$
- since echoes after $90^\circ$ limited to <30, can't fill 3-D k-space in a reasonable time
- SAR constraint $SAR \propto B_0^2 \theta^2 A_b$
  $\Rightarrow 180^\circ$ pulses deposit 4-6x power of $90^\circ$
- "multi-slab" is halfway between slices and single-slab

- problem at slice boundaries - esp. movement
- multislab requires slice selective RF pulses $\Rightarrow$ longer than non-selective 'hard' pulses

$4$ ms RO

hard to get under 8 msec inter-echo spacing

limits speed of covering k-space
**SINGLE-SLAB 3D FSE (SPACE)**


- regular FSE (180° pulse train)
- sub 180° pulses cause each successive pulse to also generate a stimulated echo (STE)
  - this "storage" in Z-axis preserves magnetization for longer time
- smaller flip angles allow much larger echo trains
  - enough to collect whole plane of 3-D k-space +
- different than hyper echoes (not symmetric)
- contrast must consider STE

**Single-slab 3DFSE pulse seq.:**

\[
SE = \sin\alpha, \sin^2\frac{\alpha}{2} e^{-\frac{2\pi}{T_2}}
\]

\[
STE = \frac{1}{2} \sin\alpha \sin\frac{\alpha}{2} \sin\frac{\alpha}{3} e^{-\frac{2\pi}{T_2}}
\]

N.B.: T1!!

**RF**

- Hard (non-selective) pulse not 180°

**G**

- Distance between pulses

**FID**

- Spin echo plus stimulated echo from 3 RF pulses

**Echo train num:**

- Like partition num, but involves both k_x, k_y

- Long TE: 26, 16, 6

- Short TE: 15, 15, 15

N.B.: time to exit k-space is \(\approx 5X\)

- Apparent contrast time b/c of "storage"
  - \(e.g. TE_{eff} = 585\) ms looks like FSE \(TE = 140\) ms
**FAST GRADIENT ECHO (FLASH)**

- Small tip so TR can be greatly reduced (e.g., 10 msec, less than T$_2$)
- 'leaves' undecayed transverse magnetization "unwound" and re-used "spoiled" before next shot

**STEADY-STATE COHERENT (GRASS, FISP)**

- Unwind phase from phase-encode M$_T$ before next pulse (there because TR<TE)
- Unwind read gradient, too

\[ S = k \sin \frac{T1}{T2} e^{-T2/T1} \]

- T$_2$/T$_1$-weighted contrast (bright CSF)
- Brain 0.91
- Fat 0.3
- Soft 0.7

**STEADY-STATE SPOILED (SPGR, FLASH)**

- Spoil with random gradient (but this still allows some x refocusing)
- Spoil with gradient plus incremented phase of RF pulses (RF spoiling)
- Good gray-white contrast (T1-weighted)

**NON-STEADY STATE, MAGNETIZATION-PREP (MP-RAGE)**

- Preparation pulse \( 180^\circ \times \) strongly Ti-weighting
- Contrast varies in spatial-frequency-dependent way

- Longitudinal mag. not affect much by low angle pulses
**Quantitative TI — intro, methods**

**Motivation**
- Image values are arbitrary/relative (cliffseg's, manufacturers)
- Uncorrected coil fall-off (receive inhomogeneity) can result in 2-3x differences in voxel brightness
- Uncorrected variation in local B1 field can cause contrast variation
  - At 3T, B1 can vary by 25% across the brain
  - This can invert contrast in a fast gradient echo

**Pre-scan normalise**
- Collect low-res GE image, receive w/ body coil (no coil fall-off)
- Set parameters to get low GM/WM contrast
- Collect data scan (e.g., MPRAGE) w/ surface coils, strong GM/WM
- Use ratio between scans to generate smooth correction field

**T1 divided by T2**
- MPRAGE → Strong T1-contrast
- SPACE → T2-weighted (no T1 weighting)
- T1/T2 removes coil fall-off
- Problems: noise in regions of low signal

**MP2RAGE**

**N.B. SSFP-like in partition, phase-encode directions**
- Convert to 0.5 to 0.5 image: \( S = \text{real} \left( \frac{\tilde{S}_{TI_1} \cdot \tilde{S}_{TI_2}}{||\tilde{S}_{TI_1}||^2 + ||\tilde{S}_{TI_2}||^2} \right) \)
- Calc. PD & T1 from above
QUANTITATIVE T1 - HELMS 2-FIP ANGLE METHOD

- start w/ gradient echo signal e.g., dropping T2 decay \( e^{-\frac{TE}{T2}} \)
  \[
  S_{\text{Ernst}} = A \cdot \sin \alpha \cdot \frac{1 - e^{-\frac{TR}{T1}}}{1 - \cos \alpha \cdot e^{-\frac{TR}{T1}}}
  \]
  Ernst eq.
  \[\cos \alpha_E = e^{-\frac{TR}{T1}}\]
  "Ernst angle"
  \[\alpha_E = \cos^{-1}(e^{-\frac{TR}{T1}})\]

- simplify/linear re/estimate
  \[TR < T1\]
  linear approx. of exponentials
  Taylor expansion simplification of \( \sin, \cos \), drop small terms

- \( S \approx A \cdot \alpha \cdot \frac{TR/T1}{\alpha^2/2 + TR/T1} \) (max: \( \alpha^2/2 = TR/T1 \))

- solve for TD and \( \alpha \) (proton density) given signals from 2 diff flip angles

- \( T1_{\text{est.}} = 2TR \frac{S_1/\alpha_1 - S_2/\alpha_2}{S_2 \alpha_2 - S_1 \alpha_1} \)

- \( A_{\text{est.}} = \frac{S_1 S_2 (\alpha_2/\alpha_1 - \alpha_1/\alpha_2)}{S_2 \alpha_2 - S_1 \alpha_1} \)

- tiny error for flip \( \leq 15 \text{ deg.} \)

- problem: flip angle varies a lot at 3T (e.g., 25%) from nominal/requested (e.g., 1.5T)

2 flip angles

\( S_1, S_2 \) - spin-echo and stimulated echo (EV)

- estimate TE/T2
  \[S = k \cdot \sin^3 \alpha \cdot e^{-\frac{TE}{T2}}\]
  \[S_{\text{SE}} = k_2' \cdot \sin^2 \alpha \cdot \sin 2\alpha \cdot e^{-\frac{TE}{T2}} \]
  \[\alpha = \cos^{-1} \left( \frac{S_{\text{SE}} \cdot e^{-\frac{TM}{T1}}}{S_{\text{SE}}} \right)\]

- B1 map
  \[RF_{\text{in}} \rightarrow z = 90^\circ \cdot TE/2 \rightarrow 2z \cdot TE/2 \rightarrow \alpha \cdot TE/2 \rightarrow \text{partition (3D)} \]

- slice
  \[\text{phase} \rightarrow \text{blips} \rightarrow \text{blips} \rightarrow \text{read} \]

- T2* - add EPI-like echo train to each FLASH excit.
Echo Planar Imaging (EPI)

- Single shot EPI collects all k-space lines (e.g. 64) after a 90° RF pulse using a train of gradient echoes.

- Since there is only one RF pulse per slice, spins never get reset to all-the-same (= zero freq., center of k-space)

- Therefore, the recording point (Δt) in k-space (= spin phase stripe pattern) stays wherever the x and y gradients last left it

- That explains why successive y phase-encode steps are achieved without changing the size of the G_y "blips"

- Echoes are T2*-weighted (gradient echo)

- Contrast mainly determined by echoes near center of k-space, which are only recorded after about 32 echoes.
**SPIN ECHO EPI**

why SE-BOLD may be selective for capillary bed

- Standard EPI is a gradient echo method, which results in $T_2^*$-weighting

- Deoxyhemoglobin is paramagnetic, which reduces signal in a $T_2^*$-weighted image due to greater dephasing

- The excess of oxygen (probably the result of the need to drive $O_2$ into tissue, which requires more $O_2$ in the blood than is actually used) leads to the positive BOLD effect

- Spin echo corrects (cancels) static $T_2^*$ ($T_2$) dephasing, incl. deoxy

- If all spins stayed in the same position, spin echo would eliminate the BOLD effect by eliminating dephasing

- Diffusion exposes spins to different fields (reducing gradient echo dephasing)

- Magnetic field gradients produced by large vessels are smoother across space than those produced by small vessels

- For $TE \approx 100$ ms, spins diffuse 10's of $\mu m$, which is larger than diameter of small capillary, meaning that spins will likely experience different fields over time

- Therefore, spin echo will be less successful at canceling BOLD effect near small vessels (BOLD effect will be reduced near large vessels where diffusion less likely to expose spin to different fields here)

- This argument only works for extravascular spins --- intravascular signal in BOLD is large (despite being only 4% by volume) because large gradient produced around red blood cells

- Measure intr/extra v/ bipolar pulse which kills signal in faster moving blood in moderate and larger vessels

- Over half of SE-BOLD at 1st is venous...
SPIN ECHO EPI

- EPI is a multi-gradient echo pulse sequence.

- "Spin-echo EPI" uses a 180° pulse to add a single spin-echo to the contrast-controlling gradient echo through the center of k-space.

- "Asymmetric spin-echo EPI" arranges for the spin-echo to occur T msec before the gradient echo, which gives more T2*-weighting (for ky=0 echo).

- The 180° pulse rephasing reduces the T2* signal, which is why the partially rephased asymmetric spin echo has been more commonly used.

- At higher fields, spin echo EPI is more promising because signal to noise is higher so we can take spin echo hit.

- Contribution from venous blood is reduced, since blood T2 is so short, we can let it decay away before recording.
- Coil fall-off intuitively contains info about resolution if same brain location imaged by different coils w/ diff. fall-offs
  - but what does this look like in k-space?
  - Slow variation in RF-field fall-off (e.g., 1-4 cyc/FOV) causes a blur in acquired data in k-space (N.B. not addition!)
  - To see this, consider multiplication by coil fall-off function in image space, which equals convolution (w/ FT of that function) in k-space - at all spatial frequencies!!

- Simple example w/ "brain" consisting of one spatial freq:

\[ \text{Image domain} \quad \text{Spatial freq. domain} \]

- N.B. inverse FT of k-space data "smeared" in spat freq. Space is sharp image w/ fall-off (not blurred img.)
  - "Smear" means normally orthogonal spat. freq.'s "leak" to adj. freqs.

- GRAPPA - construct k-space "kernel" to fill in missing k-space lines by training on fully-sampled data from near k-space center

- SENSE - general linear inverse approach

- N.B.: neither would work unless normally orthog. spat. freqs. blurred!
- excite multiple slices at once
- function of $G_z$ blips is to shift slices in $G_y$ direction
- this occurs because for given slice, a phase pedestal is added to the entire slice
  $\Rightarrow$ this "Fourier Shift Theorem"
  $\Rightarrow$ [N.B.: different than $B_0$ defect-induced incremented phase errors]
- problem w/all up $G_z$ blips $\Rightarrow$ phase error builds up

**trick #1**
- start w/2 slices, one at $z=0$, other above
  $\Leftrightarrow$ if $\pi\ (180^\circ)$ phase shift used, blip up/down same! (no effect at $z=0$)
  $\Rightarrow$ i.e., move top or bottom replica

**trick #2**
- for multiple slices not all at $z=0$, phase no longer same for even/odd
  $\Leftrightarrow$ but can add phase to equilibrate to k-space before recon.

**trick #3**
- for more than 2 slices:
  $\downarrow$ $\uparrow$ $\uparrow$ $\downarrow$ $\uparrow$ $\downarrow$ $\uparrow$ $\downarrow$ etc.
MULTI BAND/BLIPPED CAIPI (cont.)

- relation between leave-one-out aliasing and nominally fully-sampled SMS

---

**Orig Image**

- one slice

---

**K-space**

- fully sampled
- skip alternate
- skip 2
- k-space undersampled
- k-space is undersampled
- leave alternate lines out, wraps image
- SENSE/GIRAPPA can fix (fill in)
  b/c local coil view smears
  k-space data to adjacent spatial frequencies
- nominally, w/ SMS we record every line of k-space
- but equivalent to leave alternate out b/c our multi-slice
  FOV was not big enough

---

**Slice-GRAPPA**

- reg GRAPPA -> fill in missing lines
- slice GRAPPA -> fill in multiple k-spaces
  for each overlapped slices
  by training on fully-sampled data
  at beginning of scan

---

**Interslice Leakage Block**

- when training GRAPPA kernel on fully-sampled data,
  also minimize interslice leakage (split-slice-GRAPPA)
- can also do regular GRAPPA on top of this
- reason: for diffusion, loss in S/N from undersample
  cancelled by shorter TE readout (bigger signal)
- also gain from reduced image distortion
  from shorter ky readout
ECHO-VOLUME IMAGING EVI

- multi-shot (like FLASH) but acquiring one plane of 3-D k-space per shot (can do spin echo, too)

---

- entire k-space must be filled before 3D image is reconstructed
- since entire volume is excited each shot, potentially higher S/N
- must use smaller flip angle to avoid killing $M_L$ since entire volume excited every partition (e.g. every 80 msec)

- main issue is movement artifact since data assembled from many shots over several secs
- breathing-induced B0 problems in different partitions may cause blur
SPIRAL IMAGING

- by using smoothly changing gradients (sinusoids) less gradient power required than w/ trapezoids (less eddy currents)

earlier EPI hardware like this: sinusoidal gradient waveform from resonant circuit w/ non-uniform sampling to get constant Δk

- sinusoids in both Gx and Gy give spiral k-space trajectory

RF

Gx

Gy

90°

Sample all quantizations of each spatial frequency while slowly increasing spatial freq.

- constant angular velocity goes too fast at large k_x, k_y
- constant linear velocity better but impractical near k_x=0, k_y=0
- compromise: start constant angular, end constant linear

Constant angular velocity

\[ w(t) = \omega_0 \tau \]

\[ k(t) = A t e^{i \omega_0 t} \]

\[ G(t) = \frac{1}{A} \frac{d}{dt} k(t) = A e^{i \omega_0 t} + iA \omega_0 e^{i \omega_0 t} \]

\[ G_x(t) = A \cos \omega_0 t - A t \omega_0 \sin \omega_0 t \]

\[ G_y(t) = A \sin \omega_0 t + A t \omega_0 \cos \omega_0 t \]

Constant linear velocity

\[ w(t) = \omega_0 \tau \]

\[ k(t) = A \tau e^{i \omega_0 t} \]

\[ G(t) = \frac{1}{A \tau} \frac{d}{dt} k(t) = \frac{A}{A \tau} e^{i \omega_0 t} \cos \omega_0 t + \frac{A}{2} \omega_0 e^{i \omega_0 t} \]

\[ G_x(t) = \frac{A}{A \tau} \cos \omega_0 t + \frac{A}{2} \omega_0 \cos \omega_0 t \]

\[ G_y(t) = \frac{A}{A \tau} \sin \omega_0 t + \frac{A}{2} \omega_0 \sin \omega_0 t \]
SPIRAL 3D IR FSE (from Eric Wong)

- 3D: block select vs. slice select
- FSE: multiple echoes from one 90°
- Spiral: multiple spirals vs. multiple lines
- Interleaved spirals (like FSE interleaves)
- True IR (vs. MPRAGE) all echoes after 90° derive from mag w/ same TI contrast (vs. non-steady-state)
- Possible to preserve sign
- High, uniform contrast, but lots of waiting (TI), high BW

180° (prep1) TI ~ 700 msec
180°
180°
180°
RF

G_z

G_y

G_x

Sig.

FID

echo_1

echo_2

Loop order

3D k-space

(“stack of spirals”)

Spiral interleaves

k_z echoes

k_z echoes

k_z echoes (after one 90°)
Phase Errors & Echo-Centering Errors

- Anything that causes a deviation of the $B_z$ field strength from the expected value $(B_0, z + G_{x,z}x + G_{y,z}y + G_{z,z}z)$ changes precision frequency and therefore, expected phase angle.

- Incorrect phase of spatial frequency stripes results in a shift in space in the magnitude image after reconstruction.

**Phase Shifts theorem**

- Phase shift in spatial freq. domain causes spatial shift in image domain.
- First defense: freq. prescan
- Refine w/shimming and $B_0$-mapping/phase unmasking before reconstruction

Echo Centering Error

- If realignment of all spins $(k_x = k_y = 0)$ doesn't occur at the middle of read gradient, echo is shifted.
- Since echo is in spatial frequency domain, this is frequency shift.

- Spatial frequency shift results in wrapping in phase image after reconstruction.

**Fourier freq. shift theorem**

- Frequency shift in freq. domain causes phase shift in spatial.

Mathematical Formulas:

1. $I(x) = \sum_{k_x} e^{-i 2\pi k_x x} S(k_x)e^{i 2\pi k_x x}$
2. $I(x - x_0) = \int_{k_x} e^{i 2\pi k_x x} \cdot S(k_x) e^{i 2\pi k_x x} dk_x$
FAST SCAN ARTIFACTS

EPI vs. Spiral

Brain-induced field defects lead to phase errors

**EPI**
- $G_x$ readout gradient strong → [field defects smaller percentage, less deformation of $k_x$ (vertical stripe components)]
- $G_y$ "blips" are weak and total $G_y$ record time much longer (5 times) than standard readout (50 ms vs. 10 ms)
- An extra gradient in the $x$-direction, for example, unmaps phase as a function of $x$-position
- But $G_x$ big, so effect on freq.-encode direction is much less than on phase-encode direction, whose error accumulates

The lack of blurring has lead to a preference for EPI, despite the substantial image shifts

**Spiral**
- With centripetal spirals, phase errors accumulate in a radial direction
- Thus, higher spatial frequencies have more error (= more shearing)
- For spurious $x$-direction gradient as above, there is a radial blurring, rather than a vertical shift because higher frequency phase stripes misaligned relative to low spatial freq.

For defects with more complex contours in the $y$-direction (than linear, as above) the vertical shifts (in EPI) will vary with $y$-position, and may result in signals from different $y$-positions being reconstructed on top of each other
- Local estimates of R6 required to correct images
- Felding method: multiply TEIs to estimate R6 from phase slope
- Postspread-function: extra phase encode to estimate PSF (should be S-function)

Collect one data point

- Some defect makes leftward dot in vertical phase
- Some defect makes rightward dot in vertical phase
- Localized AG6 defect often arises from air pockets embedd in tissue
- Localized AG6 defect often arises from air pockets embedd in tissue
- Indentation in inferior temporal lobe
- Can under estimate epithelium

Localization of AG6 defect, effect on recon
LOCALIZED Bφ DEFECT, EFFECT ON RECON

- when local Bφ defect disturbs image space phase stripes during signal acquisition, estimates of local spatial freq. are affected (compressed stripes = higher spatial freq.)

- if each successive ky line recorded w/ same echo time (e.g., w/ single line phase encoding), this will correspond to constant spatial freq. offset in k-space

- a k-space freq. offset only results in image space phase shift (Fourier freq. shift theorem), which is invisible in amplitude image (cf. echo cont. error)

- however, with w/EPI, static Bφ defect causes more and more local displacement of image phase stripes for each additional ky line

  - that is, later lines have greater spatial freq. offset
  - effectively stretches k-space in ky direction
  - same num samples to higher spatial freq., shrinks FOV (squishes voxels — see FOV page)

- when image is reconstructed, region with local Bφ defect shifted oppositely

- Thus, local shift effect due to combination of 3 things:
  1) static local ΔBφ defect
  2) successive increases in phase error for successive spatial freq. measurements during long EPI readout
  3) small size of ky phase encode blips relative to Bφ defect (much less of this effect in freq. encode direction)

- Respiration (which affect Bφ) in 3D FLASH might cause similar effect within kz partition (if successive spatial freqs.)
GRADIENT NON-LINEARITIES

- Ideally the $G_x$, $G_y$, and $G_z$ gradient coils attempt to impress a linear variation onto to $z$-component of the $B$ field — $B_z$ — in the $x$, $y$, and $z$-directions.

- In practice, gradient coils are non-linear (esp. printed-circuit-like).

- Non-linearities are worse in smaller coils, but also in higher performance coils, designed for post-processing correction of distortions.

- Non-linearities result in phase errors, which result in 3-D image distortion.
  
  - A non-linear slice-select gradient will excite a curved slice.
  
  - Non-linear phase and frequency encode gradients will distort in-plane features.

- Some scanners correct these differently.
  
  - For 3-D scans (all directions), 2-D scans (just in-plane), and EPI scans (no corrections!)

- This can result in errors approaching 1 cm in functional overlaps.

- Different coil designs have different directions of distortion (!).

- The assumption of non-Maxwellian gradients results in additional phase errors.

- These can also be corrected since the $B_x$ and $B_y$ components are known.

- These effects do not build up over time in phase-encode direction since they only occur when gradients are turned on.

- Fourier shift theorem.

- These distortions are predictable and can be corrected.

- That is, the assumption that gradients cause no field in the $B_x$ and $B_y$ direction.
**SHIMMING AND B₀-MAPPING**

- Passive iron shims inserted to flatten B₀ field
- Additional coils (usu. in the gradient coils) can be statically energized in an attempt to flatten the B₀ field (a few ppm good)
- Primary use is to compensate for defects in flatness present without a sample in the magnet (geometric imperfections, impurities in metal, etc.) (= several hundred ppm)

Linear shim coils impose gradients in x, y, and z
Higher order shims impose higher order spherical harmonic field components (e.g., z²)

- Secondary use is to compensate for inhomogeneities caused by introducing the sample into the B₀ field
- Local resonance offsets caused by B₀ defects estimated from images
  - e.g., sample phase at multiple echo times

- Fit defective field using combination of fields generated by shim coils. Then add these corrections to base shim currents
  - This only corrects spatially gradual field defects
  - Local defects due to air in sinuses much higher order than shims
- After shimming, field map measured again

- Image voxel displacements calculated from resonance offset map are used to unwrap the reconstructed magnitude image
- For EPI images, assume displacements all in phase-encode direction (since freq encode gradient is strong relative to defects)
**NAVIGATOR ECHOS**

- **1D navigator**
  - B₀ drift problem
    - slow up/down drifts in B₀ continuously occur
    - a pedestal in B₀ is pedestal in phase (not gradient)
      - which causes spatial shift (Fourier shift theorem)
    - in EPI, mainly affects phase-encode dir b/c small B₀
      - result is successive volumes drift in phase encode dir
  - Gradient balance problem
    - unequal L/R readout gradients cause L/R shift in position of even/odd lines in k-space
      - causing N/2 (Nyquist) ghosting → another phase error

- **3D navigator**: collect 3D sphere in k-space
  - rotation of object → rotation of k-space amplitude pattern
  - translation of object → phase shift of k-space phase (Fourier shift)
  - sample at sufficient radius to pick up high spatial freq features
  - N.B.: excite whole volume
  - do N/S hemispheres separately (less T₂*, cancel EPI-like error accumulation)

*Welch et al. (2002) MRM*

\[
\begin{align*}
\text{RF} & \Rightarrow 90° \\
G_z & \Rightarrow \text{crush} \\
G_y & \Rightarrow \text{100% phase} \\
G_x & \Rightarrow \text{12 msec} \\
\end{align*}
\]

\[
\begin{align*}
x(n) &= \sin \left( \frac{\pi}{T} \sin^{-1} z(n) \right) \left[ 1 - z^2(n) \right] \\
y(n) &= \cos \left( \frac{\pi}{T} \sin^{-1} z(n) \right) \left[ 1 - z^2(n) \right] \\
z(n) &= \frac{2n - N - 1}{N} \\
\end{align*}
\]

- can be used for prospective motion correction (rotate, translate w/ gradients)
- better estimate, because of speed, than Full TR of EPI images (27 ms vs. 2-4 sec)
- may need to smooth rot, trans estimates across time (e.g. Kalman filter)
RF FIELD INHOMOGENEITIES $B_1$ inhomogeneities

- receive coil inhomogeneities alter the amplitude of the received signal, altering the reconstructed proton density in a spatially varying way
  - variations can be used (cf. GRAPPA, SENSE) and/or corrected

- transmit coil inhomogeneities affect the flip angle in a spatially varying way (can affect contrast: FLASH)
  - potentially worse (why local transmit is still in progress)
  - usu. fixed by using a large transmit coil (e.g. body coil)

- RF penetration at higher fields ($\leq$ higher RF frequencies)
  - is less uniform:
    1) decreased RF wavelength (closer to size of head) at higher freq.
    2) increased permittivity ($\varepsilon$) and conductivity ($\sigma$) at higher field

- 2nd advantage of the falloff in signal recorded with a small, receive-only RF coil is better signal-to-noise (less noise received from other parts of brain)

- different sensitivity functions from different coils can be used to scan less lines in $k$-space (GRAPPA/SENSE/SPACE-RIP)

- record lo-res volume (b/c coil fall-off is smooth) through both body coil and small coil(s)

- divide small coil body coil at each voxel to determine receive field

- use receive field to normalize main image(s)

[ see also: qT1, MP2RAGE, T1/T2]
**Diffusion (DTI)**

**Diffusion - Weighted Imaging**

- Simple diffusion weighting
  - RF
  - Select
  - Encode
  - Prepare
  - Readout

- Diffusion weighting
  - [Image]

- Apparent diffusion coefficient: \( A(D) = \frac{S_0}{S} = e^{bD} \)
  - where \( b = y^2 G^2 \delta^2 (T - \delta t/3) \)

1) **Anisotropic Diffusion (Gaussian)**

- Measure \( D \) along multiple axes
- Have to measure tensor, not scalar
  - Even for determining one primary direction

\[
D = \begin{bmatrix} u_x & u_y & u_z \end{bmatrix} \begin{bmatrix} D_{xx} & D_{xy} & D_{xz} \\ D_{yx} & D_{yy} & D_{yz} \\ D_{zx} & D_{zy} & D_{zz} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}
\]

**Diffusion Surface (non-Gaussian)**

- Need to measure diffusion in many directions (>6) to properly characterize even 2 main directions

2) **Length Scale by multiple b-values**

- Fit line to semi-log signal as function of \( b \)
- If not straight line: multi-exponential, e.g.,
  - hi ADC/fast extra vs. lo ADC/slow/intercellular

**Voxel Tract Tracing**

1. Markov process
2. Crossing fibers
3. "Freeway ramp" probe
4. Sharp turns into gyrice
PRACTICAL DIFFUSION-WEIGHTED PULSE SEQ

- spin-echo 'Stejskal-Tanner' EPI seq. (standard on scanners)
  - allows longer TE
  - 90° flips M_z so rephase gradient, same sign as dephase

- eddy-currents are long time constant currents in metal of scanner that distort B field -> spatial image distortion

- "doubly-refocused" spin echo sequence (DSE) can cancel the effects of eddy current (w/ partic. time constants)
  - also, keep crushers orthogonal to diffusion-encoding gradients

Nagy et al. (2014) MRM

\[ T_{RSE} = 0 = T_1 - T_2 - T_3 + T_4 \]
PERFUSION – ARTERIAL SPIN LABEL

- Basic idea: tag blood below area of interest and collect control & tagged images. Assume directional input flow!

  - Continuous ASL (CASL) — continuously tag a plane.
  - Pseudo-continuous ASL (pCASL) — greatest on, blood gets adiabatically inverted as it passes through location with coned resonant tag.
  - Pulsed ASL (PASL) — e.g., EPSTAR, FAIR, PICORE, QUIPPS II

- Small diffs between control and tag (~1%)
  - Requires accurate balancing of control & tag images, control mag. transfer

- Contrast problems:
  - Transit delays — biggest confounding factor
  - Relaxation rate differs
  - Venous clearance (vs. microvessels, which get stuck!

  Solutions for quantitative
  - Invert delay so all spins arrive into low-velocity capillaries
  - Kill end of tag to reduce signal variation of tag

- QUIPPS II — Quantitative Perfusion

  1) Pre-saturate spins in target slices
  2) Tag — 180° pulse below slices (to control off-resonance)
  3) Saturate tagged block to end tag (TI1)
  4) EPI or spiral images of target slices (TI2)

  $\Delta M \approx \text{flow} \times \left[ 2 M_0 \text{TI}, e^{-\frac{TI_2}{T1A}} \right]$

  - Can extract flow and BOLD adjacent substrata minimize movement artifact

  1) Alternate tag and control, GRE TE = 30 ms
  2) Dual echo spiral, $k = 0$ early => hi S/N flow TE = 30 ms => BOLD
- Original CASL (continuous arterial spin labeling) requires RF on continuously to adiabatically invert blood flowing through one plane.
  - Can only image one slice (b/c dephasing from gradient).
  - Hard to keep RF on continuously on modern scanner (esp. BC).
  - Can use special purpose RF transmit (separate xmit channel).

A) Original CASL

\[
\begin{align*}
\text{RF} & \quad \text{\[image formation module ("readout") \rightarrow [multiple possibilities]\]}
\end{align*}
\]

B) pCASL - pseudo-continuous arterial spin labeling

\[
\begin{align*}
\text{RF} & \quad \text{begin readout}
\end{align*}
\]

- Problem: multiple pulsers create aliased slice planes.
  \[
  \text{RF}(t) = \frac{1}{\Delta t} \text{comb}(t/\Delta t) \ast \text{rect}(t/8)
  \]
  [use: convolution of 2 funct equals multiplying their FTs]
  \[
  \mathcal{F}[\text{RF}(t)] = \text{comb}(6\Delta t) \ast \delta \text{ sinc}(\pi \delta)
  \]
  \[
  \text{for const } G_z: \quad z = \frac{n}{V G_z \Delta t}
  \]
  \[\Rightarrow\] aliased labeling planes at: \( b = n/\Delta t \) in frequency space, modulated by broad sinc()
  - Use Hamming or hyperbolic sech to reduce replicas
    \[
    \text{\textit{Hamming}} \xrightarrow{\text{FT}} \text{\textit{6}} \xrightarrow{\text{\textit{Hamming}}} \text{\textit{hyperbolic sech}} \text{\textit{(cf. Gaussian)}}
    \]

C) pCASL w/ shaped gradients

- Tag pulses have phase offset respecting gradient:
  - Control identical except every other has \( \pi \) phase.
  - Control identical except every other has \( \pi \) phase.
**OFF RESONANCE EXCITATION**

**Main Idea:** Examine evolution of \( \vec{M} \) vector in rotating coord syst set to "off-resonance" \( \vec{B}_1 \) field freq \( (\omega_f) \), not Larmor freq of \( \vec{M} \) \( (\omega_0) \)

- Normally, if rotating coord syst freq set to Larmor freq \( (\omega_f=\omega_0) \), an actually precessing \( \vec{M} \) will be stationary (ignoring decay) \( \Rightarrow \) implies effective \( B_z=0 \) in rotating

- Now, move \( \vec{M} \) to rotating coord syst at \( \vec{B}_1 \) freq lower than \( \omega_0 \) (assume \( \vec{B}_1=0 \) at \( 0 \)): existing \( \vec{M} \) will now appear to precess around \( z \)-axis:

  - N.B.: this is precession in already rotating coord syst.
  - freq of precession in rotating coord syst \( \omega_0 - \omega_f \)
  - Larmor rot freq \( \omega_0 \)
  - "incorrectly set rotating coord syst freq"

- Thus, viewing \( \vec{M} \) vector in off-resonance rotating coord syst makes it look like additional \( \vec{B}_z \) field is causing "extra" precession

- "Extra" \( \vec{B}_z \) component is proportional to \( \Delta \omega_0 \) offset
  - \( \Rightarrow \) can be pos or neg: rot coord freq too low \( \Rightarrow \) pos \( \vec{B}_z \)
  - rot coord freq too high \( \Rightarrow \) neg \( \vec{B}_z \)

- Extra \( \vec{B}_z \) adds to \( \vec{B}_1 \) resulting in slow precession around tipped axis: \( \vec{B}_{\text{eff}} \) (effective)

- Extra \( \vec{B}_z \) from any gradient \( \Rightarrow \) same effect on \( \Delta \omega_0 \)
  - Changes \( \omega_0 \) instead of \( \omega_f \)

\[
\vec{B}_{\text{eff}} = \frac{\Delta \omega_0}{Y} \hat{k} + B_z \hat{k} + B_1 \hat{i}
\]

- Effective \( \vec{B} \) in rotating frame set to \( \vec{B}_1 \) freq
- Apparent "extra" \( \vec{B}_z \) from Larmor-
  - \( \vec{B}_1 \) freq mismatch (pos or neg)
  - If on-res., \( \vec{B}_{\text{eff}} \) tilt (if on-res., \( B_{\text{eff}} \) tilt)

- Transverse \( \vec{B}_z \) from optional gradient
- RF stim

- RF: sweep freq
  - \( \omega_0 \): constant
- RF: const freq
  - \( \omega_0 \): sweeps because spins flow along gradient direction

**Adiabatic RF pulse:**

**Flaw-driven CASL tag**
**SPECTROSCOPY + IMAGE**

- **chemical shift**: small displacement resonant freqs due to variable shielding of target nucleus (e.g., $^1$H) by surrounding electron orbitals
  
  - e.g., acetic acid:
    
    ![Diagram of acetic acid with molecular structure and electron orbitals](image)
  
  - how we get chemical shift spectrum:
    
    ![Diagram of chemical shift spectrum](image)
  
  - data before FT is a series of time-domain samples of the mix of shifted-freq offsets
    
    - FT turns data into "shift spectrum"
      
      ![Diagram of pulse sequence](image)

**Pulse Sequence**

- since we are already using phase (f=0) encoding for space, we need an "extra dimension" w/ all gradients OFF!
  
  - use spin-echo to undo built-up chemical shifts, then record chemical-NMR-like signal $\rightarrow$ and FT-it like chemists do!

<table>
<thead>
<tr>
<th>Signal</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>NMR</td>
<td>time domain FT oscillation samples (shift)</td>
</tr>
<tr>
<td>MRI</td>
<td>spatial FT spatial object freq. samples (like time domain signal)</td>
</tr>
</tbody>
</table>

- N.B.: opposite "direction" of FTs.
PRESS, MEGA-PRESS

usu. single voxel by using 3 orthog. slice selects
(tho can add PEs gradients & more excitations to get multiple vox)

PRESS — 3 orthog. slice select

RF_in
G_2
G_y
G_x

RF_out

SE_{1,2} (like FID)

It's OK, now to think
of multiple refs here!

FT to get shift spectrum

MEGA-PRESS — add "editing" RFs to suppress Solvent (water)

RF_in

MEGA 180° pulses set to freq.

of solvent

these only spoil the

water and allow the

broad band 180° to

refocus the spins

of interest

G_1, G_3 — asymmetric

spoilers to dephase

spins in bandwidth

of selective MEGA pulses

G_4 — G_1 symmetric

around 2\(\pi\) 180°

only record 2\(\pi\) half

FT to get shift

spectrum

(Not There)
**Phase-encoded Stimulus & Analysis**

**Map Polar Angle**

**Map Frequency**

**Map Eccentricity**

**Map Prox/Distal Axis, Road Maps**

**Periodic Stimuli (Phase-encoded)** - e.g., 8 cycles at 64 sec/cycle

**Calculate Significance**
- Ratio between amplitude at stimulus frequency (=signal) and average of amplitudes at other frequencies (=noise)
- Ignore harmonics, low freq (=movement)

**Smooth**
- Vector average of complex significance (A, φ) with that of nearest neighbor surface points

**Display**
- Plot phase using hue and saturation to indicate significance

**Delay Correction**
- Record responses to opposite directions of stimulus (ccw/cw, in/out, up/down)
- Vector average after reversing angle of one, penalizes inconsistent more than just avg of angles

**Typically 0.5 - 5% amplitude**

**Strongly periodically activated single voxel time course**

**Remove constant (avg) and linear trend**

**Real**

**Imaginary**

**FFT, convert to A, φ**

**A**

**B**

**φ**

**τ**

**Freq = total TR's/2**

**Reversed CCW**

**CCW significance (complex)**
**Convolution**

\[ h(x) = f(x) \otimes g(x) = \int_{-\infty}^{\infty} f(z) \cdot g(x-z) \, dz \]

- Definition of convolution \((f*g)(x)\)
- Commutative

**Graphically**

- Place kernel at \(x\)
- Reverse kernel \(\Rightarrow\) multiply, sum
- Current \(x\)

**Why reverse makes sense**

\[ h(x) = \int_{-\infty}^{\infty} g(z) \cdot f(x-z) \, dz \]

**Impulse response function (HDR)**

- Impulses (exp. design)
- Impulse occurred a while ago \(\Rightarrow\) small effect
- Impulse occurred recently \(\Rightarrow\) larger effect

**Intuitive non-reversed view of convolution output**

- Start here!

**N.B. Cross-correlation**

- Same as convolution except no reversal
  \[ g(x+z) \quad \text{instead of} \quad g(x-z) \]

- N.B. Auto-correlation
  - Same, except no-reversal and use same function for both \(f, g\)

**How to calculate convolution output for this time point** (only 3 terms in sum, all others zero)
**General Linear Model (GLM)**

\[ \hat{y} = \hat{X} \hat{h} + \hat{S} \hat{b} + \hat{n} \]

- **Data** = design * HDR + drifts * weights + noise

- **Goal**: solve for the hemodynamic response functions, \( \hat{h} \)

- **Design temporalization**:\( \begin{bmatrix} t_{\text{temp}} \\ t_{\text{temp}} \end{bmatrix} \)

- **Experimental design**

- **X**: known

- **S**: unknowns

- **Fixed**: assume white noise

- **Simpler: preconvolve**
  1) convX off fixed \( \hat{h} \)
  2) solve for \( \beta \)
  \[ \hat{y} = X \beta \]

- **Orthogonal col's most efficient**
  - **Projection matrix** that removes part of vector that lies in \( S \) space

- **Matrix notation** for discrete convolution of stimulus pattern with hemodynamic resp. functs.

- **Factor**
  - \( \begin{bmatrix} t_{\text{temp}} \\ s_{\text{lin}} \end{bmatrix} \)

- **Basic vectors of signal space**

- **Basic vectors of interference space**

- **Maximum likelihood estimate** (Liu et al. 2001 Neuroimage)

1) assume white noise, solve for \( \hat{h} \)

\[ \hat{h} = (X^T P_s^+ X)^{-1} X^T P_s^+ y \]

where \( P_s^+ = I - S (S^T S)^{-1} S^T \)

or

\[ \hat{h} = (X^T X)_-^{-1} X^T y \]

where \( X_- = P_s^+ X \)

2) significance (how to construct F-ratio)

\[ F = \frac{N - K - \ell}{k} \begin{bmatrix} y^T (P_{ks} - P_s) y \\ y^T (I - P_{ks}) y \end{bmatrix} \]

- **P_{ks}**: projects data on est + nuisance subspace
- **P_s**: projects data onto nuisance subspace

3) **see diagram next page for geometric interp**
GEOMETRIC INTERPRETATION OF GENERAL LINEAR MODEL

- with no nuisance functions ($S$), we could look at orthogonal projection of data onto experimental design and compare that to error to determine significance

\[ \hat{y} = X\hat{h} + \hat{\varepsilon} \]

\[ \hat{y} = P_x \hat{y} \]

Projection matrix $P_x$ operates on $\hat{y}$ to give projection of data into experiment space $X$

- when nuisance functions, $S$, are considered, problem: $S$ may not be orthogonal to $X$

For example: linear trend not orthogonal to std. block design

[remember: "orthogonal" means dot prod. $\equiv 0$ corr $= 0$]

Orthogonal projection

Oblique projection

Calc. dot prod

Equals

Multiply

Sum: not 0

Geometric Picture

(orig from Schaff & Friedlander, 1994 IEEE)

$X_S$: space of data modeled by all exp. design and nuisance

Oblique projection onto nuisance ($E_y$)

Orthogonal projection onto nuisance ($P_y$, data explained by $S$)

Data orthogonal to nuisance

Error ($\varepsilon$) not explained by exp. design and nuisance ($F$ denote)

\[ [(I - P_y) y] \]

Orthogonal projection onto exp. design ($P_y$, data explained by reference)

Same as projection onto reference only in special case where $S \perp X$
WHY USE SURFACES?

- raw MRI data is a 2D flat slice or a 3D volume
  \( \rightarrow I(x,y) \) or \( I(x,y,z) \)

but...

1) the neocortex (and cerebellar cortex) are thin, folded 2D sheets
   - cortex starts as smooth "balloon" \( \rightarrow \)
   - major sulci, temporal lobe form \( \rightarrow \)
   - great size increase, "crinkles" form

2) neocortex contains many topological maps along its surface
   - retinotopy
   - tonotopy
   - somatotopy
   - musculotopy
   - plus higher level maps \( \rightarrow \approx \frac{2}{3} \) of its area

3) surface displays allow seeing (almost) all of data at once
   - only 30% exposed
   - everything visible
   \( \rightarrow \) [flattened]

4) differences in major sulci make 3D-based alignment difficult
   - e.g. STS, monkey-like IPS vs. postcentral
   - plus extra
   \( \rightarrow \) extremely anisotropic def.

5) idiosyncratic sulcal crinkles

- these introduce additional noise into alignment in 3D
- exact position of crinkles unlikely to have functional implications (the 3D align might respect them)
SEGMENTATION & SURFACE RECON

1) MNI auto-Talairach \( \rightarrow \) generates 4x4 matrix

- make average brain target (blurry)
- blur target (further), blur single brain (a lot), gradient descent on \( x \) cross
- repeat w/ less blurring of avg target and current brain
- problem: variable neck cut-off \( \rightarrow \) much better than standard! < fit to bounding box

2) Intensity Normalization (output: "T1")

- histogram of pixel values in 10 mm thick H+R slices
- smooth histogram
- peak fit to get initial estimate of white matter
- discard outlier peaks across slices
- fit splines to peaks across slices
- interpolated scaling factor \( \rightarrow \) to H+R
- scale each pixel so WM peak is 110
- refine estimate to interpolate in 3D

\[ \text{5-10 times} \]

- find points in \( 5 \times 5 \times 5 \) within 10% of WM, get near scale for them
- build Voronoi to interpolate scales "set above"
- smooth Voronoi boundaries (3 iterations)
- re-scale each voxel

3) Skull Stripping (output: "brain")

- "shrink-wrap" algorithm
- start with ellipsoidal template \( \rightarrow \) sub-tessellated icosahedron
- minimize brain penetration and curvature
- curvature: spring force \( \text{from center-to-neighbor vec sum} \)
- brain penetration

\[ \text{apply force along surface normal that prevents surface from entering gray matter} \]
SEGMENTATION & SURFACE RECON

Spring force in detail

- Implementing a "force" is like directly constructing the operator that minimizes something (without first defining the 'something'.)
- More formally, we would define cost function, then take its derivative (gradient) to minimize it.

Shrinkwrap update e.g., (skull strip, original Dale & Sereno surface refinement)

\[
\mathbf{r}_{\text{center}}(t+1) = \mathbf{r}_{\text{center}}(t) + F_{\text{smooth}}(t) + F_{\text{MRI}}(t)
\]

For each vertex, \( \mathbf{r}_{\text{center}} \)

\[
F_{\text{smooth}} = \lambda_{\text{tang}} \sum_{\text{neigh}} (\mathbf{I} - \mathbf{n}_{\text{center}} \mathbf{n}_{\text{center}}^T) \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}})
\]

Identity 3x3 stronger than normal (0.5)

Expand by distribution to neighbor vertex minus projection of neighbor onto normal = tangential!

\[
F_{\text{MRI}} = \lambda_{\text{MRI}} \mathbf{n}_{\text{center}} \sum_{d} \max \left[ 0, \tanh \left[ I(\mathbf{r}_{\text{center}} - d \mathbf{n}_{\text{center}}) - I_{\text{brain}} \right] \right]
\]

Max force, saturated at 1.0 max pixel product = 1.0.

\( d \) sample points into brain along the direction of normal.

Snapshot of surface and "core sample" from one vertex.

Outside (dark), skin (light), skull (dark-light-dark), etc.
4) Non-isotropic filtering (output: "win") — "floss" and "speckle"
   - preliminary hard threshold
   - find ambiguous/boundary voxels
     → 20% or more of 26 immediate neighbors different
   - find plane of least variance
     → to avoid expensive calc below...

for each ambiguous voxel
[for each direction (from icosahedral supertessellation)]
consider $5 \times 5 \times 5$ volume around 1 voxel
find plane of least variance in this hemisphere
   - median filter w/ hysteresis (1 vox thick)
   → if 60% of within-slab differ, reverse classification
   → "flosses" sulci without blurring

5) Find cutting planes
   - midbrain, to separate hemispheres (SAG)
   - callosum, to separate hemispheres (SAG)
   - midbrain, to avoid fill into cerebellum (T1)

6) Region-growing to define connected parts (output: "filled")
   - inside-out, outside-in, inside-out — for each hemisphere

   - "wormhole filter" ($3 \times 3 \times 3 = $ center + 26)
     → fills (unfilled) voxel if 66% neighbors differ
     → eliminates structures within 1-D structure
7) Surface Tessellation (output: rh.orig, lh.orig)

- variable num neighbors possible!
- quads to triangles

- find filled voxels bordering unfilled
- make ordered list of neighboring vertices
  \( \rightarrow \) 3D cross-products oriented properly

- long list of values associated with each numbered vertex
  e.g. 
  - position (orig, morphed)
  - area (orig, morphed)
  - curvature (intrinsic, Gaussian)
  - sulcussness (summed \( L \) movement during unfolding)
  - cortical thickness
  - fMRI data \( \leq \)
  - EEG/MEG dipole strength

- separate fMRI data set must be aligned, sampled

  \( + \) fMRI voxels larger
  Sample at each surface vertex
  nearest-neighbor "soap bubble" smoothing
  to interpolate data onto hi-res mesh

- some quantities only well-defined on surface
  \( \rightarrow \) gradient of magnitude of cortical map measure (e.g., eccentricity)
SEGMENTATION & SURFACE RECON

Smooth, inflate, final surfaces

- smoothing/inflation/WM, pial done as derivative of energy functional

\[ J = J_{\text{tangential}} + \lambda_{\text{normal}} J_{\text{normal}} + \lambda_{\text{image}} J_{\text{image}} \]

- total scalar error to minimize
- scalar tangential error (fixed by redistributing vertices)
- scalar curvature error (fixed by reducing curvature)
- smaller scalar image error

\[ J_{\text{normal}} = \frac{1}{2} \sum_{\text{centers}} \sum_{\text{neighbors}} \left[ \mathbf{n}_{\text{center}} \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}}) \right]^2 \]

- across all vertices, curvature error
- \( \lambda \) so no coefficient on derivative

\[ J_{\text{tangential}} = \frac{1}{2} \sum_{\text{centers}} \sum_{\text{neighbors}} \left[ \mathbf{t}^x_{\text{center}} \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}}) \right]^2 + \left[ \mathbf{t}^y_{\text{center}} \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}}) \right]^2 \]

- "squishing" of mesh
- x-direction in tangent plane
- y-direction in tangent plane

\[ J_{\text{image}} = \frac{1}{2} \sum_{\text{centers}} \left[ I_{\text{tag}}(\mathbf{r}_{\text{center}}) - I_{\text{tag}}(\mathbf{r}_{\text{center}}) \right]^2 \]

- image error
- target brightness at current location

- take directional derivative of energy functional (to find steepest uphill)
- move each vertex in the opposite (negative) direction w/ self-intersect test

\[ \frac{\partial J}{\partial \mathbf{r}_{\text{center}}} = \lambda_{\text{image}} \left[ I_{\text{tag}}(\mathbf{r}_{\text{center}}) - I_{\text{tag}}(\mathbf{r}_{\text{center}}) \right] \nabla I(\mathbf{r}_{\text{center}}) \]

- x-component of tangential
- \( \mathbf{t}^x_{\text{center}} \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}}) \)
- \( \mathbf{t}^y_{\text{center}} \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}}) \)

N.B.: eq. 9 in Dale, Fischl, Sereno different and incorrect!
SULCUS-BASED CROSS-SUB. ALIGN

- use summed perpendicular vertex move during inflation as vtx measure of "sulcus-ness"
- add term to error function, J: "sulcus-ness" error
  \[ J_{\text{sulc}} = \frac{1}{2\#\text{vtx}} \sum_{\text{vertices}} \left( S_{\text{subj}}^{\text{act}} - S_{\text{targ}}^{\text{targ}} (\text{vtx}) \right)^2 \]
  find neg of steepest uphill direction of change in sulcus-ness of target

- take deriv. of J w.r.t. vertices
  \[ \frac{\partial J_{\text{sulc}}}{\partial \text{vtx}} = \lambda_{\text{sulc}} \left( S_{\text{subj}}^{\text{act}} - S_{\text{targ}}^{\text{targ}} (\text{vtx}) \right) \]
  sulcus-ness of movable subj vtx

- bootstrap morph to one brain make avg of n targs re morph to avg of targs

- smooth wm, inflated, sphere, registered sphere

Sub
\[ \rightarrow \text{infl} \rightarrow \text{sph} \rightarrow \text{mor} \rightarrow \text{inter} \]

Sub
\[ \rightarrow \text{infl} \rightarrow \text{sph} \rightarrow \text{mor} \rightarrow \text{inter} \]

Sub
\[ \rightarrow \text{infl} \rightarrow \text{sph} \rightarrow \text{mor} \rightarrow \text{inter} \]

- each subj's native surf has diff # vertices
- interpolate values (coords, surface measures, stats) to each icosahedral vertex from neighboring vertices of native mesh (dashed lines)

- average surface made from folded/inflated avg coords
  - folded: loses area from sulcal wrinkles (at average "inflated")
  - inflated: retains orig area, correct sulc/gyrus ratio ("inflated-avg")

- can use sampled-to-icos individual subj coords to draw icosahedral surface in shape of an indiv. brain

> N.B.: morph will have changed local vertex density compared to more uniform native mesh (use native for sing. subj.)
SOURCE OF EEG/MEG

PSPs

- no distant signal from axon spike

anisotropic cables
- aligned spatially
- coherent/biased stim one end

GM
WM

isotropic
"closed field" (invisible at distance)
N.B.: spikes only detected by 15µm microelectrode in gray matter!

Head

1) - Local dipole
2) - EEG through skull, skin
3) - Swearing because skull 1/80 conductive of brain

the effect on current flow is like a higher order (larger) "cable"

MEG

- Radial dipoles lost
- Tangential dipole generates Gabor-like scalp distribution of B field
INTRACORTICAL CIRCUITS & ORIGIN OF EEG

Cell Types
- Excitatory (spiny)
  - Pyramidal
  - Spiny stellate (e.g., V1 layer 4)
- Inhibitory (smooth)
  - Basket
  - Double bouquet
  - Chandelier
  - Clutch

Circuits
- Huge complexity
- First principal components: input → layer 4 → layer 2/3 → feedforward
- Microelectrode recording (e.g., 10 μm tip):
  - High pass → spikes
  - Low pass → local field potentials
- Spikes only recordable in gray matter
- White matter spikes only recordable with pipette w/ very fine tip b/c inward & outward currents are spatially close in axon/spike (> 1 μm)

Intra/Inter Cortical Connections Cartoon

"Lower" (e.g., V1)

1. Feedforward
2. Input
3. Motor output
4. Feedback

"Higher" (e.g., V2)

1. Feedforward
2. Layer 2/3
3. Layer 5/6
4. Output feedback

Spike is inverted here
Axon initial segment
Ascending input (e.g., dLGn)
GRADIENT, DIVERGENCE, CURL

**Gradient (\(\nabla\))** (generalized derivative)

\[
\nabla \mathbf{F}(\mathbf{r}) = \frac{\partial F_x(\mathbf{r})}{\partial x} \mathbf{i} + \frac{\partial F_y(\mathbf{r})}{\partial y} \mathbf{j} + \frac{\partial F_z(\mathbf{r})}{\partial z} \mathbf{k}
\]

This turns a scalar function defined at each \(x, y, z\) point \(\mathbf{r}\) into a vector field.

**Divergence (\(\nabla \cdot \mathbf{F}\))** (derivative "dot product")

\[
\nabla \cdot \mathbf{F}(\mathbf{r}) = \frac{\partial F_x(\mathbf{r})}{\partial x} + \frac{\partial F_y(\mathbf{r})}{\partial y} + \frac{\partial F_z(\mathbf{r})}{\partial z}
\]

This turns a vector field defined at each \(x, y, z\) point \(\mathbf{r}\) into a scalar field.

**Curl (\(\nabla \times \mathbf{F}\))** (derivative "cross product")

\[
\nabla \times \mathbf{F}(\mathbf{r}) = \left( \frac{\partial F_z(\mathbf{r})}{\partial y} - \frac{\partial F_y(\mathbf{r})}{\partial z} \right) \mathbf{i} + \left( \frac{\partial F_x(\mathbf{r})}{\partial z} - \frac{\partial F_z(\mathbf{r})}{\partial x} \right) \mathbf{j} + \left( \frac{\partial F_y(\mathbf{r})}{\partial x} - \frac{\partial F_x(\mathbf{r})}{\partial y} \right) \mathbf{k}
\]

This turns a vector field defined at each \(x, y, z\) point \(\mathbf{r}\) into another vector field.

**Vector identities**

\[
\nabla \times \nabla F = 0 \quad \text{curl of the gradient of any scalar field is zero}
\]

\[
\nabla \cdot (\nabla \times \mathbf{A}) = 0 \quad \text{divergence of the curl of any vector field is zero}
\]

\[
\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}
\]
Potential ($\Phi$), Electric Field ($\nabla \Phi$) = CSD ($\nabla \cdot (-\nabla \Phi) = \nabla^2 \Phi$)

low-frequency field approximation
- Electric fields uncoupled from magnetic ($\mu$, electromagnetic radiation)
- Pre-Maxwellian approx. (EEG freq's $\ll 1$ MHz)
- Calculate electric fields as if magnetic fields don't exist
- Calculate magnetic fields strictly from distribution of currents
- Ignore capacitive effects, too

Scalar potential, $\Phi$ (what we measure with electrode)

\[ \mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t} \]

Electric field vector

Scalar field ($\Phi$) into vector field ($\mathbf{E}$)

1. $\mathbf{E}$ defined as force (vector) acting on unit charge at a given point in space (as result of arbitrary distribution of other charges)

2. Current density, $\mathbf{J}$ (not curr. source dens.!) is proportional to $\mathbf{E}$→still a vector!

3. Two defs of $\mathbf{A}$: $\mathbf{A} = \nabla \times \mathbf{A}$ (Joule's law) $\sigma$ is conductivity

CSD is Laplacian of $\Phi$ (= div $\mathbf{E}$)

\[ \nabla \cdot (-\nabla \Phi) = \text{scalar field} = - \left[ \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} \right] \equiv -\nabla^2 \Phi \]

3D CSD gold standard (rat BAER paper)

$\Phi$ data $\nabla$ $\nabla \Phi$ $\nabla \cdot (-\nabla \Phi)$

Scalar field source/sink movie as function of $z$
1D AND 2D CURRENT SOURCE DENSITY EXPTS.

1D CSD
- Raw, event-related signal relative to ground, $\frac{d}{dx}$ (e.g., skull)
- Rationale: CSD changes much more slowly parallel to cortex than perpendicular to cortical sheet
- Assume approx. constant ($\approx 0$) parallel to cortex
- Recording sites (e.g., by slowly withdrawing electrode tip)
- LFP (local field potential)
- Both types of data can be recorded from same electrode

2D CSD
- 2D array of electrodes on pial surface or on scalp
- Rationale: All electrodes record along same surface, so assume depth profiles are constant
- $\nabla^2$ means find spatial (i.e., 1D depth) curvature of potential
- Discrete approx.: center $- \frac{\text{above} + \text{below}}{2}$
- N.B. in example above, even though all 3 potentials are positive, smaller value of center point implies sink!
INTRACORTICAL C.S.D.

- e.g. click evoked rat A-I
  (Sukov & Barth, 1998)

- phase-locked CSD
  Gamma shifts with each cycle
**MAXWELL EQUATIONS**

**Electrostatics, Magnetostatics**

- **low freq limit**
- **N.B., these are all defined at a (every) point in space**

**108**

- **Divergence**
- **Conductivity**
- **Impressed currents**

- **Incidentally, this Maxwell equation violated by a linear gradient in \( B_z \), in \( y \), \( \partial \phi / \partial x \), div [**B**] = 0

- **Currents due to ionic flow**
- **that appear out of nowhere**
- **(Nearst battery)**

- **Curl**
- **Magnetic field**
- **Impressed currents**
- **Conductivity**

- **Propagation of potentials, magnetic fields instantaneous (no capacitance)**
- **Simultaneous eqs to solve \( \nabla \cdot \vec{J} \) are sources, \( \phi, B \) are data**

- **Linear**

<table>
<thead>
<tr>
<th>( \vec{J}<em>p ) + ( \vec{J}</em>{\text{inc}} )</th>
<th>( \phi )</th>
<th>( B )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \phi )</td>
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<td>( \vec{B} )</td>
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</tbody>
</table>
WHY WE CAN IGNORE MAGNETIC INDUCTION

\[ \vec{E} = -\nabla \Phi - \frac{\partial \vec{A}}{\partial t} \]  
[from Nunez, 1981]

\[ \vec{B} = \nabla \times \vec{A} \]

"vector potential"

\[ \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \]

magnetic field
field component due to charge distribution

\[ \vec{H} = \frac{1}{\mu} \vec{B} \]  
permeability \( \mu \)

\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]

take \( \nabla \times \) of both sides
use \( \vec{B} = \mu \vec{H} \)

\[ \nabla \times \nabla \times \vec{E} = -\mu \frac{\partial}{\partial t} \left( \vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \]

if linear in conductivity and dielectric, too, and fields periodic w/ f

\[ \nabla \times \nabla \times \vec{E} = -2\pi f \mu (\sigma + 2\pi f \varepsilon) \vec{E} \]

to neglect: 
\[ \frac{2\pi f \mu (\sigma + 2\pi f \varepsilon) |\vec{E}|}{|\nabla \times \nabla \times \vec{E}|} \ll 1 \]

1) \(|\nabla \times \nabla \times \vec{E}| \approx |\vec{E}|/L^2 \) where \( L \) is dist over which \( \vec{E} \) varies significantly
2) \( \mu \approx \) tissue similar to empty space
3) assume conductive (large) \( \sigma \), dielectric unit, and EEG freq

\( \mu \) number is about \( 10^{-6} \) \( \to \) small
MOMOPOLE, DIPPOLE FORWARD SOL’N

\[ \frac{\Phi_1}{4\pi\sigma} = \frac{S}{4\pi\sigma} \left( \frac{1}{r_1^3} - \frac{1}{r_2^3} \right) \]

potential recorded for source monopole
distance, source to measuring point \((= ||\vec{r}||)\)

approximations for "far enough away" measurements
(subtracting two \(1/r^3\) gives inverse square)

\[ \frac{\bar{B}_2}{(\mu_0/4\pi)} \approx \frac{\vec{J} \cdot \vec{r}}{r^3}, r \gg d \]

N.B.: both assume inside infinite isotropic conductor

\[ \Phi_i(t) = e_i s_i(t) \] (source)

\[ \bar{b}_i(t) = \bar{m}_i s_i(t) \] (source)

\[ \Phi_i(t) = \sum_j e_j s_j(t) \] (all sources)

\[ \bar{b}_i(t) = \sum_j m_j s_j(t) \] (all sources)

**Since \(\vec{r}\) also in numerator, this now inverse square**

N.B.: reverse \#6 Current flow +

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Linear superposition with fixed electrodes and sensors
Forward Solution (1)

- well-posed (one answer)
- linear: \( b(A) + b(B) = b(A+B) \)
- approximations due to unknown electrical properties of head

- 3-shell spherical analytic
  - skull conductivity
  - "smearing" (cf. cable theory)
  - remember, we only need to be able to calc. weight for each dipole/electrode pair independently

- 3-shell boundary element
  - arbitrary shape
  - homogeneous conductivity
  - solution = infinite homogeneous + matrix of correction factors
  - for magnetic, only need one shell b/c currents thru skin/skull too small to make sig. \( B \)

- finite element
  - most general computational intensive w/ small grid
  - many unknown parameters to estimate
FORWARD SOLN (2)

\[ V_i = \sum_j E_{ij} S_j + n_i \]

**matrix form**

\[
\begin{bmatrix}
    v_1 \\
    v_2 \\
    \vdots \\
    v_n
\end{bmatrix} =
\begin{bmatrix}
    E_{11} & \cdots & E_{1s} \\
    \vdots & \ddots & \vdots \\
    E_{n1} & \cdots & E_{ns}
\end{bmatrix}
\begin{bmatrix}
    S_1 \\
    S_2 \\
    \vdots \\
    S_s
\end{bmatrix} +
\begin{bmatrix}
    n_1 \\
    n_2 \\
    \vdots \\
    n_n
\end{bmatrix}
\]

**lower case bold \rightarrow vector**

**upper case bold \rightarrow matrix**

\[ v = ES + n \]

**E fixed across time**

**v, s, n vary**

**note:** only one current source for each column in the E+B matrix!
WHY LOCALIZE?

- most of ERP literature based (indeed) on temporal "components"

but:

1) underlying local cortical generators (from microelectrode LFP, CSD)
   - extended in time (400 msec), visible from every scalp electrode
   - multiphasic in every cortical area
   - temporally non-static depending on stimulus
     - e.g. simple contrast, brightness diffs
     - can modulate retinal delay by 50 msec!

2) thus, any "component" consists of sum of activity from multiple cortical areas at different hierarchical levels

3) stimulus manipulations will change temporal overlap
   - may cause "component" peak to disappear without changing cortical areas being activated

4) verified by intracortical LFP/CSD (Schroeder et al., 1998)

macaque monkey

Intra cortical data

- these areas span the visual system from bottom to top, accounting for roughly 50% of the entire macaque monkey dxt

- by contrast, the spatial signature of the signal from one cortical area is static — a better area-based "component"

- temporal "components" should be retired!!

- easier to record now (EEG started w/ few electrodes, many time points)
- easier to "paste" high level psychological functions onto a few waveform deflections
Derivation of Ill-posed Inverse

(from Dale & Sejnow, 1993)

\[ x = As + n \]

\( A = \) forward sulci matrix \((E + B)\)

\( s = \) source vector

\( n = \) sensor noise vector

\[ Err_w = \langle \| Wx - s \|^2 \rangle \quad \text{expectation:} \quad \langle \sum_k P_k K \rangle \]

Assume \( n, s \) normal, zero-mean \( W \) corresponding covar. matrices \( C, R \)

\[ Err_w = \langle \| W(As + n) - s \|^2 \rangle \]

\[ = \langle \| (WA - I)s + Wn \|^2 \rangle \]

\[ = \langle \| Ms + Wn \|^2 \rangle \quad \text{where} \quad M = WA - I \]

\[ = \langle \| Ms \|^2 \rangle + \langle \| Wn \|^2 \rangle \quad \text{diag is noise variance (already squared)} \]

\[ = \text{tr}(MRR^T) + \text{tr}(WCW^T) \quad \text{trace is sum of diag elements} \]

[re-expand]

\[ = \text{tr}(WARA^TW^T - RATA^TW^T - WAB + R) + \text{tr}(WCW^T) \]

Explicitly minimize by taking derivative w.r.t. \( W \), set to zero, solve for \( W \)

\[ 0 = 2WARA^T - 2RATA^T + 2WC \]

\[ WARA^T + WC = RATA^T \]

\[ W(ARA^T + C) = RATA^T \]

\[ W = RATA^T(ARA^T + C)^{-1} \]

\( W \) is inverse solution operator:

\[ \begin{bmatrix} s \end{bmatrix} \]

\[ \begin{bmatrix} W \end{bmatrix} \]

Equivalent to minimum norm and Tikhonov regularized inverse if \( C, R \) are proportional to identity matrix (i.e., sensor noise & sources independent and equal variance)
Small superficial sources win because smaller norm of distributed superficial source

"Minimum norm solution appropriately downplays deeper sources, since those are more likely to fall into the noise floor"

\[ W = R A T ( A R A T + C )^{-1} \]

Find \( \Sigma \) of smallest norm = \( \text{null} \)
**Inverse Solutions to Ill-Posed Compared**

\[ \mathbf{s} = \mathbf{Wx} \]

- How to use the inverse solution, \( \mathbf{W} \)
- Same \( \mathbf{W} \) for all time points

"Minimum norm" solution

i.e., norm \( \| \mathbf{x} \| \) of solution is smallest of infinitely many alternate solutions

**Linear inverse operator**

\[ \mathbf{W} = \mathbf{RA}^T (\mathbf{A} \mathbf{R} \mathbf{A}^T + \mathbf{C})^{-1} \]

from error minimization derivation

\[ \begin{bmatrix} \mathbf{W} \\ \mathbf{R} \end{bmatrix} = \begin{bmatrix} \mathbf{R} \\ \mathbf{A}^T \end{bmatrix} \begin{bmatrix} \mathbf{A} \\ \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{A}^T \\ \mathbf{C} \end{bmatrix} \]

\[ \mathbf{ARA}^T \Rightarrow \text{Square in \# of sensors (small)} \]

**Alternate, algebraically equivalent Bayesian derivation**

\( \mathbf{W} = (\mathbf{A}^T \mathbf{C}^{-1} \mathbf{A} + \mathbf{R}^{-1})^{-1} \mathbf{A}^T \mathbf{C} \)

\[ \begin{bmatrix} \mathbf{W} \\ \mathbf{C} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^T \\ \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{C} \\ \mathbf{A} \end{bmatrix} + \begin{bmatrix} \mathbf{A}^T \mathbf{C}^{-1} \mathbf{A} \end{bmatrix} \]

\[ \mathbf{ATC}^{-1} \mathbf{A} \Rightarrow \text{Both square in \# of sources (large)} \]
PROBLEMS W/SURFACE NORMAL

- Since nearby points on a surface often have different orientation, surface normal constraint can help (since forward soln A, B very different)

- But, since point spread function typically extends across sulci, artefactual sign reversals occur

- Solutions

  1) Ignore sign $\rightarrow$ saves useful orientation info!

  2) Solve onto 3 orthogonal dipoles at each critical point instead of a single oriented dipole

    $\rightarrow$ more appropriate when averaging across subjects, since detailed variations vary a lot

    also, fills in bottom of sulci (else unsigned stripes)
- Use inverse-2

**FMRI Constrained Inverse**

- insert **FMRI** values for **Rii**'s
  - but still allow other sites to have non-zero **Rii**'s

- Pathologies occur if solution restricted completely to **FMRI** points by setting non-FMRI **Rii**'s to zero

- This allows extracting time course from sources visible in EEG/MEG and FMRI

- N.B.: sources that are only visible in EEG/MEG will be dispersed to small distributed values at a large number of vertices

visible in both EEG/MEG and FMRI

visible only in EEG/MEG and not FMRI
NOISE SENSITIVITY NORMALIZATION

Forward: \( x = As \) well posed (Liu, Dale, and Belliveau, 2002)
Inverse: \( s = Wx \) ill posed
Solve: \( x = As + n \) for \( s \)

\[
W = R A^T (A R A^T + C)^{-1}
\]

- multiply inverse operator by noise sensitivity matrix, \( D \) (diagonal)

\[
D_{ii} = \frac{1}{\text{diag} \sqrt{W C W^T}}
\]

\[
W_{\text{norm}} = D W
\]

\[
s_{i, \text{norm}} = (W_{\text{norm}} x)_i = (D W x)_i = \frac{W_i x}{\sqrt{(W C W^T)_i}} = \sqrt{\frac{(W x x^T W^T)_i}{(W C W^T)_i}}
\]

If assume Gaussian white noise, noise covariance, \( C \), is multiple of \( \mathbf{I} \), so

\[
W_{i, \text{norm}} = \frac{W_{i, \text{orig}}}{\|W_{i, \text{orig}}\|}
\]

i.e., scale each row of \( W \) by single value — the norm of that row
row of \( W \) is:

\[
\text{Inverse solution coefficient for one source} \Rightarrow \text{scale} (\text{divide}) \text{ by norm of this row}
\]

That is, if inverse sol'n for deep source is reduced by interaction of inverse square nature of fluid and min norm, dividing by norm of row of inverse (1 same) will increase/reduce deep source.
**Noise Sensitivity Normalization (2)**

- **Shallow source** (unit strength)
  - Feed big
  - Inv small

- **Deep source** (unit strength)
  - Feed small
  - Inv reduced because of minimum norm
    - (N.B. W should be bigger than for superficial source but min norm reduces it of inverse square)

\[
S_i = \frac{\hat{W}_i \cdot \hat{x}_{\text{orig}}}{\| \hat{W}_i \|_{\text{orig}}} 
\]

- Effect on inverse solution: more like significance is actual power
- Effect on point-spread function is to equalize shallow & deep
  - Shallow spread out more than min norm
  - Deep shrunk to same as shallow

**Point-spread functions**

- Noise normed
  - Feed data
  - Speed in
Conclusions

- More EEG or more MEG better
- EEG better than MEG (cf. radial) (EEG far better currently less accurate)
- Biggest gain from adding small # EEG (n MEG) (e.g. 30) to many MEG (n EEG) (e.g. 150)
- Easier to add many MEG, so: optimal < 30 EEG < 300 MEG
- EEG/MEG forward-solution-scaling-factor error causes more cross talk
** MUSIC (1)**
(from Dale & Sereno, 1993) (cf. Mosher & Leahy)

Using **sensor covariance**

\[
D = \langle xx^T \rangle = \sigma^2 I + \sum_{i} \sum_{j} \sigma_i \sigma_j \text{Cov}(i,j) A_i A_j^T
\]

recording at one time point

\[
\sim [x_1, \ldots, x_n] [x_1, \ldots, x_n]^T
\]

\[
D = U \Lambda U^T = \begin{bmatrix} U_1 & U_2 & \cdots & U_n \end{bmatrix} \begin{bmatrix} \Lambda_1 & & & \\ & \Lambda_2 & & \\ & & \ddots & \\ & & & \Lambda_n \end{bmatrix} \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}
\]

**columns of U matrix are orthogonal basis vectors of "spatial pattern" space (one part is spatial pattern across sensors)**

**find, order most significant spatial patterns in sensors over time**

**Project forward solution onto these spatial patterns**

("project" = dot prod = similarity) for each point in brain

\[
\tilde{z} = A_z^T U \Lambda U^T A_z^T \Rightarrow \text{big single number if forward solution looks like } U\text{'s}
\]

**One num for each source all eigenvectors of sensor spatial patterns | fund for 1 column (gain vector) of A matrix**
How to weight the minimum norm inverse.

\[ R_{ii} \approx \frac{A_i^T A_i}{A_i^T U \Lambda U^T A_i} \]

\[ W = R A^T (A R A^T + C)^{-1} \]

- Like parallel resistance:
  \[ R_{\text{parallel}} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \ldots} \]

- Any low resistance \( R_i \) decreases overall resistance (small \( R_{ii} \))

- I.e., if forward soln has appearance like any low eigenvalue spatial pattern, it gets devalued
- how it works: take advantage of spatial information that changes over time

- how it fixes min norm problem

N.B.: Problem if assumption about lack of perfect correlation is violated
e.g., if two widely separated sources (i.e., diff fund solns) are highly correlated, MUSIC will eliminate both since no single fund soln will look like that "2-seprated dipole" pattern (e.g., L/R A-I)

"dual MUSIC" hack to fix...