Seminar III
A Boolean-valued Modal Set Theory

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Extension vs. Intension

$$\exists y\ [x = y \wedge \Phi(y)]\ \textit{vs.}\ \Box\Phi(x)$$

Different principles hold in different contexts:

$$\sigma = \tau \wedge \Phi(\sigma) \rightarrow \Phi(\tau)$$

vs.

$$\Box[\sigma = \tau] \wedge \Phi(\sigma) \rightarrow \Phi(\tau)$$

The prime example of an intensional mapping:

$$\Box[\![\Phi \leftrightarrow \Psi]\!] \leq [\![\Box\Phi \leftrightarrow \Box\Psi]\!]$$

Extensional Powersets

Definition: Given a complete **M**-set A the ***extensional powerset*** of A is the collection of P: A→ **M** where, for all x,y∈A, we have $P(x) \wedge [\![x = y]\!] \leq P(y)$.

And we can use the definition:

$$[\![P = Q]\!] = \bigwedge_{x \in A} (P(x) \leftrightarrow Q(x))$$

Theorem: The extensional powerset of A is a complete **M**-set.

Note: **A Principle of Comprehension follows for extensional predicates.**

Theorem: $\mathbb{R}_{\mathbf{M}}$ together with its extensional powerset satisfies the ***Dedekind Completeness Axiom***.

Intensional Powersets

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Note: A Principle of Comprehension follows.

Question: Should we be able to iterate this notion of powerset?

A Modal Boolean-Valued Universe

$$V^{(\mathbf{M})} = \{v : \mathbf{dom}\, v \to \mathbf{M} \mid \mathbf{dom}\, v \subseteq V^{(\mathbf{M})} \;\; \& \;\; \forall x,y \in \mathbf{dom}\, v\, [\, v(x) \wedge \Box [\![x = y]\!] \leq v(y)\,]\,\}$$

$$[\![u \in v]\!] = \bigvee \{ v(y) \wedge \Box [\![u = y]\!] \mid y \in \mathbf{dom}\, v \}$$

$$[\![u = v]\!] = \bigwedge \{ u(x) \to [\![x \in v]\!] \mid x \in \mathbf{dom}\, u \} \wedge \bigwedge \{ v(y) \to [\![y \in u]\!] \mid y \in \mathbf{dom}\, v \}$$

The new insight:

intensional ↘ $u \in v$ ↖ extensional

Note: All automorphisms in Γ extend to the model $V^{(\mathbf{M})}$.

For Technical Details See:

John L. Bell, **Set Theory: Boolean-Valued Models and Independence Proofs**, Third Edition, OUP 2005, xviii + 191 pp.

Nicolas D. Goodman, *A genuinely intensional set theory*, in: Stewart Shapiro (ed.), **Intensional Mathematics**, North-Holland 1985, pp. 63-80.

Nicolas D. Goodman, *Topological models of epistemic set theory,* **Annals of Pure and Applied Logic**, vol. 46 (1990), pp. 119-126.

A.G. Kusraev and S.S. Kutateladze, **Boolean Valued Analysis**, Kluwer 1999, xii + 322 pp.

What is MZF?

Substitution (A number of previous lemmata are needed.)

$$\Box[u = v] \wedge \Phi(u) \rightarrow \Phi(v)$$

Extensionality & Comprehension

$$\forall u,v\,[\, u = v \leftrightarrow \forall x\,[x \in u \leftrightarrow x \in v]]$$

$$\forall u\,\exists v \Box \forall x\,[x \in v \leftrightarrow x \in u \wedge \Phi(x)]$$

Singleton

$$\forall u\,\exists v \Box \forall x\,[x \in v \leftrightarrow \Box[x = u]]$$

Intensional Leibniz' Law

$$\forall x,y\,[\Box[x = y] \leftrightarrow \forall u\,[x \in u \rightarrow y \in u]]$$

Definable Modality

$$\{\varnothing\} = \{\varnothing \mid \Phi\} \leftrightarrow \Phi$$

$$\Box\,\Phi \leftrightarrow \forall u\,[\{\varnothing\} \in u \rightarrow \{\varnothing \mid \Phi\} \in u]$$

Two Membership Relations?

Extensional Membership

$$u \text{ € } v \leftrightarrow \exists y\, [u = y \wedge y \in v]$$

Extensional Comprehension

$$\forall u\, \exists v \Box \forall x\, [x \text{ € } v \leftrightarrow x \text{ € } u \wedge \exists y\, [x = y \wedge \Phi(y)]]$$

Extensional Singleton

$$\forall u\, \exists v \Box \forall x\, [x \text{ € } v \leftrightarrow x = u]$$

Extensional Leibniz' Law

$$\forall x,y\, [x = y \leftrightarrow \forall u\, [x \text{ € } u \rightarrow y \text{ € } u]]$$

Intensional Powerset

$$\forall v\, \exists w \Box \forall u\, [u \in w \leftrightarrow \Box [u \subseteq v]]$$

Extensional Powerset

$$\forall v\, \exists w \Box \forall u\, [u \text{ € } w \leftrightarrow u \subseteq v]$$

Foundation and Collection

Scedrov's Modal Foundation

$$\Box\forall x[\Box\forall y{\in}x.\,\Phi(y) \rightarrow \Phi(x)] \rightarrow \forall x.\,\Phi(x)$$

Foundation

$$\forall x[\forall y{\in}x.\,\Phi(y) \rightarrow \Phi(x)] \rightarrow \forall x.\,\Phi(x)$$

Goodman's Modal Collection

$$\Box\forall y\,\exists z.\,\Phi(y,z) \rightarrow \forall x\,\exists w\Box\forall y{\in}x\,\exists z[\Box z{\in}w \wedge \Phi(y,z)]$$

Collection

$$\forall y\,\exists z.\,\Phi(y,z) \rightarrow \forall x\exists w\forall y{\in}x\,\exists z \in w.\,\Phi(y,z)$$

Comment: **It seems plausible that stronger principles are valid and that the modalities can be generalized.**

A Refutation

Theorem. In $V^{(\mathbf{M})}$ the following has truth value 0:

$$\forall u,v\,[\, u = v \leftrightarrow \forall x\,[x \in u \leftrightarrow x \in v]].$$

Proof: Find $p \in \mathbf{M}$ with $0 < p < 1$ and $\Box p = 0$. **(How?)**

Let $a = \{\varnothing\}$ and $b = \{\varnothing \mid p\}$, and $u = \{a \mid p\}$ and $v = \{b \mid p\}$.

We have $[\![a = b]\!] = p$, and $[\![a \in u]\!] = p$ and $[\![a \in v]\!] = 0$.

It follows that $[\![u = v]\!] = \neg p$. We also calculate that

$[\![x \in u]\!] = [\![x = a]\!] \wedge p$ and $[\![x \in v]\!] = [\![x = b]\!] \wedge p$.

But then $[\![x \in v]\!] = [\![x = a]\!] \wedge p$ as well. From this we get:

$$[\![u = v \leftrightarrow \forall x\,[x \in u \leftrightarrow x \in v]]\!] = [\![u = v]\!] = \neg p.$$

The conclusion of the theorem then follows by the 0-1 Law for M.

Using Russell's Paradox

Theorem. For each stage $V_\alpha^{(\mathbf{M})}$ of the universe it is possible to find an element a of the model such that

$$[\![a = y]\!] = 0 \text{ for all } y \text{ in } V_\alpha^{(\mathbf{M})}.$$

Proof: Apply the ***Extensional Comprehension Principle*** to have an element a where for all x in the model:

$$[\![x \in a]\!] = [\![x \in \mathbf{V}_\alpha]\!] \wedge [\![\neg x \in x]\!],$$

where $\mathbf{V}_\alpha$ is the constant function 1 on $V_\alpha^{(\mathbf{M})}$.

Putting a for x, we have $[\![a \in \mathbf{V}_\alpha]\!] = 0$.

The desired conclusion then follows.

Another Refutation

Theorem. In $V^{(\mathbf{M})}$ the following has truth value 0:

$$\exists v \, \forall u \, [u \in v \leftrightarrow u = \varnothing].$$

Proof: Again, find $p \in \mathbf{M}$ with $0 < p < 1$ and $\Box p = 0$. Suppose we had v in the model where $[\![u \in v]\!] = [\![u = \varnothing]\!]$ for all u in the model. Now v is a function with **dom** $v \subseteq V_\alpha^{(\mathbf{M})}$ for some stage α. Find an a with $[\![a = y]\!] = 0$ for all y in $V_\alpha^{(\mathbf{M})}$. Take $u = \{a \mid \neg p\}$ which implies $[\![u = \varnothing]\!] = p$. We then have $p \leq [\![u \in \mathbf{V}_\alpha]\!] = \bigvee\{\Box[\![u = w]\!] \mid w \in V_\alpha^{(\mathbf{M})}\}$. But we find

$$\Box[\![u = w]\!] = \Box(\neg p \rightarrow [\![a \in w]\!]) \wedge$$
$$\Box\bigwedge\{w(y) \rightarrow [\![y \in u]\!] \mid y \in \mathbf{dom}\, w\} \leq \Box p,$$

But, this is impossible.

Note: We can also refute: $\forall v \, \exists w \, \forall u \, [u \in w \leftrightarrow u \subseteq v]$.

Pairs, Products, & Relations

Definitions: In $V^{(M)}$ the following are defined:

(i) $\{u\} = \{(u,1)\}$;
(ii) $\{u, v\} = \{(u,1),(v,1)\}$;
(iii) $(u, v) = \{\{u\}, \{u, v\}\}$; and
(iv) $a \times b = \{((x, y), a(x) \wedge b(y)) \mid x \in \mathbf{dom}\ a \wedge y \in \mathbf{dom}\ b\}$.

Theorem: In $V^{(M)}$ we have:

(i) $\forall u,v\, [\{u\} = \{v\} \leftrightarrow \Box u = v]$;

(ii) $\forall u,v,s,t\, [\{u, v\} = \{s, t\} \leftrightarrow \Box [u = s \wedge v = t] \vee \Box [u = t \wedge v = s]]$;

(iii) $\forall u,v,s,t\, [(u, v) = (s, t) \leftrightarrow \Box [u = s \wedge v = t]]$; and

(iv) $\forall a,b,t\, [t \in (a \times b) \leftrightarrow \exists x,y\, [x \in a \wedge y \in b \wedge \Box\ t = (x, y)]]$.

Relational Comprehension

$$\forall a,b\, \exists w \subseteq (a \times b)\, \Box\, \forall x \in a\, \forall y \in b\, [(x, y) \in w \leftrightarrow \Phi(x, y)]$$

Embedding M-Sets

Theorem. Ordinary sets u in the two-valued universe V can be embedded into the modal universe $\mathrm{V}^{(\mathbf{M})}$ by the following well-founded definition: $\underline{u} = \{(\underline{x}, 1) \mid x \in u\}$.

Definition. Given a reduced **M**-set A with equality $[\![x = y]\!]$, define maps s_a: $\underline{A} \rightarrow \mathbf{M}$ for all $a \in A$ by $s_a(\underline{x}) = [\![x = a]\!]$ for all $x \in A$. Note that in $\mathrm{V}^{(\mathbf{M})}$ we have $[\![s_a = s_b]\!] = [\![a = b]\!]$ for all $a, b \in A$.

Then define $\mathcal{E}(A) = \{(s_a, 1) \mid a \in A\}$.

Theorem. In the modal universe $\mathrm{V}^{(\mathbf{M})}$, the element $\mathcal{E}(\mathbb{R}_{\mathbf{M}})$ plays the rôle of the ***real numbers*** in the set theory.

Applying Ergodic Theory?

Recall: **In the measure-algebra model of MZF, every continuous, measure-preserving automorphism of M induces an automorphism of the whole universe $V^{(M)}$. Γ is the group of all such automorphisms.**

Furstenberg's Multiple Recurrence Theorem.

Let $\tau \in \Gamma$, and let $[\![\Phi(a)]\!] \neq 0$, where $\Phi(a)$ has no other parameters. Then for all ***k*** there exists an ***n*** such that

$$[\![\Phi(a) \wedge \Phi(\tau^{n}(a)) \wedge \Phi(\tau^{2n}(a)) \wedge \Phi(\tau^{3n}(a)) \wedge \ldots \wedge \Phi(\tau^{kn}(a))]\!] \neq 0.$$

Two Sub-Universes

$$U^{(\mathbf{M})} = \{v : \mathbf{dom}\, v \to \mathbf{M} \mid \mathbf{dom}\, v \subseteq U^{(\mathbf{M})} \ \& \ \forall x,y \in \mathbf{dom}\, v\, [\, v(x) \wedge \Box [\![x = y]\!] \leq \Box v(y)\,]\,\}$$

$$W^{(\mathbf{M})} = \{v : \mathbf{dom}\, v \to \mathbf{M} \mid \mathbf{dom}\, v \subseteq W^{(\mathbf{M})} \ \& \ \forall x,y \in \mathbf{dom}\, v\, [\, v(x) \wedge [\![x = y]\!] \leq v(y)\,]\,\}$$

Note: (i) The universe $U^{(\mathbf{M})}$ models an **inuitionistic G-valued** set theory.

(ii) The universe $W^{(\mathbf{M})}$ models the usual **M**-valued, **extensional Boolean-valued** set theory.

(iii) Both universes are **definable** in the modal universe $V^{(\mathbf{M})}$.

Truth by Degrees?

Comment: There are *many* subframes of M. For example $\mathbf{D} \subseteq \mathbf{G} \subseteq \mathbf{M}$, defined as $\mathbf{D} = \{\,[0, r]/\mathrm{Null} \mid r \in \mathbb{R}\,\}$, is closed (in M) under *arbitrary* sups and infs.

The modal operator Δ defined by

$$\Delta p = \bigvee \{ d \in \mathbf{D} \mid d \leq p \}$$

is, of course, stronger than $\Box$ but *not intensional*.

Questions: But is Δ at all interesting?

Would propositions with values in D be interesting? **Suggestions welcome!**

Are You Ready for Multiverses?

Observation: Large cBa's usually have many subframes (= abstract topologies). Each one gives a model for MZF. And indeed one cBa may give rise to many of these. For example:

M measurable
G open
S cylindric (using higher dimensions)
D real-valued degrees
E broad degrees (small, medium, large)
T binary degrees (all or nothing, 0 or 1)

And we have both modal and intuitionistic versions.