Seminar II
The Measure Algebra & Higher-Order Logic

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A New Lewis Algebra?

Old. For every topological space **X**, the ***powerset*** P(**X**) is a cBa, and the ***lattice of open subsets*** Op(**X**) is a cHa and a ***subframe***.

Note: These examples include the Kripke models.

New (?). For the standard ***probability space*** Borel(**[0, 1]**) with ***Lebesgue measure***, the ***measure algebra*** Borel(**[0, 1]**)/Null is a ***cBa*** **(old)**, and the quotient Op(**[0, 1]**)/Null is a ***cHa*** that is a ***proper subframe*** **(new?)**.

Note: Call this cLa M. For $p \in$ **M write**

$|p|$ **for the measure of** p.

Proof of Completeness

Theorem. The measure algebra **M** is a cBa.

Proof: Let $X \subseteq M$ and let Y be the ideal generated by X. These two sets have the same upper bounds. (Why?) By Zorn's Lemma, let Z be a maximal family of non-zero, pairwise-disjoint elements of Y. Owing to the measure, Z is countable. (Why?) It then follows that the sup of Z exists and so

$$\bigvee Z = \bigvee Y = \bigvee X. \quad \text{(Why?)}$$

Theorem. The family **G** is a subframe of **M**.

Proof: Let $X \subseteq G$ and let Y be the set of elements $u = U/\mathrm{Null}$ where U is a rational interval and $u \leq v$ for some $v \in X$.

Then Y is countable and $\bigvee Y = \bigvee X \in G$. (Why?)

Of course, G is closed under finite meets. (Why?)

Structure of the Measure Algebra

$G_\delta = M = F_\sigma$ measurable

$\Box p = p$	G open	F closed	$\Diamond p = p$
$\Box\Diamond p = p$	$\Box F$ reg. open	$\Diamond G$ reg. closed	$\Diamond\Box p = p$
$\Box p = \Diamond p = p$	$G \cap F = C$ clopen	Boolean but uncountable	
$\lvert p \rvert \in \mathbb{Q}$	Q rational	not Boolean	
$G = B_\sigma$	B basic	countable Boolean from intervals with rational ends	

Note: Using the measure-algebra semantics, every modal logical formula has a probability. Owing to the continuous automorphisms of M, it turns out every pure statement without free variables has truth value either 0 or 1.

Proving $M \neq G \neq \Box F$

Theorem. There is a construction similar to that of the Cantor Discontinuum where $U \cup K = [0, 1]$ with U dense, open, K closed, nowhere dense, and of ***positive measure***, and $U \cap K = \varnothing$.

Corollary 1. **Let** $k = K/\mathrm{Null}$. **Then** $k \in \mathbf{M}$ **and** $k \notin \mathbf{G}$.

> **Proof:** **Suppose** $k \in \mathbf{G}$. **Let** V **be an open set,** $V/\mathrm{Null} = K/\mathrm{Null}$, **giving** $V - K \in \mathrm{Null}$. **But, this difference is open and so is** $\varnothing$. **So** $V \subseteq K$, **which implies** $V = \varnothing$. **But,** $K - V \in \mathrm{Null}$ **and** $K \notin \mathrm{Null}$.

Corollary 2. **Let** $u = U/\mathrm{Null}$. **Then** $u \in \mathbf{G}$ **and** $u \notin \Box\mathbf{F}$.

> **Proof:** **Suppose** $u \in \Box\mathbf{F}$. **Then** $u = \Box\Diamond u$. **But** U **is dense, so** $\Diamond u = 1$ **and hence** $u = 1$. **This implies** $k = 0$, **which is false.**

Proof of the 0-1 Law

Theorem. There is ***no*** $0 < p < 1$ invariant under the group Γ of all continuous, measure-preserving automorphisms of **M.**

Lemma. If $a,b \in \mathbf{B}$, $|a| = |b|$, then there is a $\tau \in \Gamma$ with $\tau(a) = b$.

Proof of the Theorem (new?):

- **Find** $g \in \mathbf{G}$ **with** $p \leq g < 1$, **and so** $1\text{-}g > 0$.
- **Find** $b \in \mathbf{B}$ **with** $b \leq g$, $b \wedge p > 0$, **and** $|b| \leq |1\text{-}g|$.
- **Find** $h \in \mathbf{G}$ **with** $1\text{-}g \leq h$, **and** $|h \wedge g| < |b \wedge p|$.
- **Find** $a \in \mathbf{B}$ **with** $a \leq h$, **and** $|a| = |b|$.

Now $a \wedge p \leq h \wedge g$, **and so** $|a \wedge p| < |b \wedge p|$.

But let $\tau(a) = b$ **and so** $\tau(a \wedge p) = b \wedge p$. **Contradiction!**

Dorothy Maharam's Theorem

Theorem (1942). All separable, atomless, strictly positive, probability measure algebras are isomorphic and isometric.

Comments: It needs to be checked that this classical theorem also includes the **topological** isomorphism between Borel($[0, 1]$)/Null and Borel($[0, 1]^{\mathrm{I}}$)/Null for every countable power I using the usual **product measures and topologies**.

Other good spaces to investigate are Borel($\{0, 1\}^{\mathrm{N}}$)/Null and Borel($\mathbf{S}^n$)/Null for the n-dimensional spheres $\mathbf{S}^n$.
Such representations of the cLa **M** also indicate the **richness** of the automorphism group Γ.

John Oxtoby's Theorem

Theorem (1970). Any two topologically complete separable metric spaces with non-atomic Borel probability measures contain homeomorphic $\mathbf{G}_\delta$ sets of measure one where the homeomorphism is measure preserving.

Reference: John C. Oxtoby, **Homeomorphic Measures in Metric Spaces**, *Proceedings of the American Mathematical Society,* vol. 24 (1970), pp. 419-423.

Extension vs. Intension

$$\exists y\ [x = y \wedge \Phi(y)]\ \textit{vs.}\ \Box\Phi(x)$$

Different principles hold in different contexts:

$$\sigma = \tau \wedge \Phi(\sigma) \rightarrow \Phi(\tau)$$

vs.

$$\Box[\sigma = \tau] \wedge \Phi(\sigma) \rightarrow \Phi(\tau)$$

The prime example of an intensional mapping:

$$\Box[\![\Phi \leftrightarrow \Psi]\!] \leq [\![\Box\Phi \leftrightarrow \Box\Psi]\!]$$

What is an M-Set?

Definition. An **M**-set is a set A equipped with an **M**-valued equality $[\![x = y]\!]$, where, for all $x,y,z \in A$,

$$[\![x = x]\!] = 1 \;;$$
$$[\![x = y]\!] = [\![y = x]\!]; \text{ and}$$
$$[\![x = y]\!] \wedge [\![y = z]\!] \leq [\![x = z]\!].$$

A is **reduced** provided $[\![x = y]\!] = 1$ always implies $x = y$.

Note: There is a useful notion of complete M-set and a process of completion.

Note: Mappings between M-sets can be either extensional or intensional.

Singletons & Completeness

Definition: A ***singleton*** on an **M**-set A is a map $s\colon A \to \mathbf{M}$ where $\bigvee_{x \in A} s(x) = 1$, and for all $x, y \in A$, $s(x) \wedge [\![x = y]\!] \leq s(y)$, and $s(x) \wedge s(y) \leq [\![x = y]\!]$.

Definition: An **M**-set A is ***complete*** iff for every singleton $s\colon A \to \mathbf{M}$ there is a ***unique*** element $a \in A$ where $s(x) = [\![x = a]\!]$ for all $x \in A$.

Theorem: The singletons on a reduced **M**-set A form a ***complete M-set*** expanding A, where

$$[\![s = t]\!] = \bigvee_{x \in A} s(x) \wedge t(x).$$

Categories of M-Sets

Definition: Write $f\colon A \to B$ for an ***(extensional) mapping of complete M-sets*** as a function from A to B where for all $x,y \in A$ we have

$$[\![x = y]\!] \leq [\![f(x) = f(y)]\!].$$

Definition: Write $f\colon A \rightarrowtail B$ for an ***(intensional) mapping of complete M-sets*** as a function from A to B where for all $x,y \in A$ we have

$$\Box[\![x = y]\!] \leq [\![f(x) = f(y)]\!].$$

Question: What are good axioms for this kind of "double" category?

M Itself as an M-Set

Definition. Make **M** into an **M**-set by defining the **M**-valued equality as $[\![p = q]\!] = p \leftrightarrow q$ for all $p, q \in$ **M**.

Theorem. **M** as an **M**-set is complete.

Theorem. **M** with $\Box[\![p = q]\!]$ is **not** complete.

Questions: (1) Are there other interesting intensional mappings on M other than $\Box$ and $\Diamond$?

(2) Can they be used for modeling other known modal logics?

Completeness and non-Completeness

M with $[\![p = q]\!]$: Let $s\colon M \to M$ be a singleton. We find $s(p) \le p \leftrightarrow s(1)$, and because $s(p) \wedge (p \leftrightarrow s(1)) \le s(s(1))$, we have $s(p) \le s(s(1))$. It follows that $s(s(1)) = 1$. But then $p \leftrightarrow s(1) \le s(p)$ holds. Whence, $s(p) = p \leftrightarrow s(1)$.

M with $\Box[\![p = q]\!]$: Let $s\colon M \to M$ be a singleton with this new equality. If $s(p) = \Box[\![p = q]\!]$ for all $p \in M$, then also $s(p) = \Box\, s(p)$ for all $p \in M$. Take an $r \in M$ with $0 < r < 1$ where $\Box r = r$ and $\Box\neg r = 0$. Define an s by setting $s(p) = (r \wedge \Box p) \vee (\neg r \wedge \Box\neg p)$. It is easy to prove that s is a singleton. But, we find $s(0) = \neg r$.

M as a Complete G-Set

M with $\Box[\![p = q]\!]$: Let $s: M \to G$ be a G-singleton. Define $a = \bigwedge_{q \in M}(s(q) \to q)$. Therefore, $a \leq s(p) \to p$, and so $s(p) \wedge a \leq p$. Thus, $s(p) \leq a \to p$, and $s(p) \leq \Box(a \to p)$. Now $s(p) \wedge s(q) \leq p \to q$, and so $s(p) \wedge p \leq s(q) \to q$. Thus, $s(p) \wedge p \leq a$, and $s(p) \leq p \to a$, so $s(p) \leq \Box(p \to a)$. Whence, $s(p) \leq \Box(a \leftrightarrow p)$. Since $s(p) \wedge \Box(a \leftrightarrow p) \leq s(a)$, we have $s(p) \leq s(a)$. Therefore, $s(a) = 1$. But also $s(a) \wedge \Box(a \leftrightarrow p) \leq s(p)$, and thus $s(p) = \Box(a \leftrightarrow p)$.

Note: In G-valued logic we will be able to show that M is a cBa. **Question:** What does this buy us?

Boolean-valued Integers

Theorem. The set $\mathbb{N}_{\mathbf{M}} = \{\theta \mid \theta{:}P(\mathbb{N}) \to_{frm} \mathbf{M}\}$ can be made into a ***complete*** **M**-set by defining equality as

$$[\![\theta = \eta]\!] = \bigvee_n (\theta(\{n\}) \wedge \eta(\{n\})).$$

Corollary. $\mathbb{N}_{\mathbf{M}}$ is the ***completion*** of $\mathbb{N}$.

Theorem. Equality on $\mathbb{N}_{\mathbf{G}} = \{\theta \mid \theta{:}P(\mathbb{N}) \to_{frm} \mathbf{G}\} \subseteq \mathbb{N}_{\mathbf{M}}$ satisfies: $[\![\theta = \eta]\!] = \Box[\![\theta = \eta]\!] = \Diamond[\![\theta = \eta]\!]$.

Theorem. All the arithmetic operations $\bullet{:}\mathbb{N}_{\mathbf{M}} \times \mathbb{N}_{\mathbf{M}} \to \mathbb{N}_{\mathbf{M}}$ can be defined by:

$$[\![\theta \bullet \eta = \zeta]\!] = \bigvee_{n,m} (\theta(\{n\}) \wedge \eta(\{m\}) \wedge \zeta(\{n \bullet m\})),$$

and the set $\mathbb{N}_{\mathbf{G}}$ will also be closed under them.

Extensional vs. Intensional Induction

(Ext) In $\mathbb{N}_{\mathbf{M}}$ we have as valid:

$$[\Phi(0) \wedge \forall x[\Phi(x) \rightarrow \Phi(x+1)]] \rightarrow \forall x\, \exists y\, [x = y \wedge \Phi(y)]$$

(Int) In $\mathbb{N}_{\mathbf{G}}$ we have as valid:

$$[\Phi(0) \wedge \forall x[\Phi(x) \rightarrow \Phi(x+1)]] \rightarrow \forall x.\, \Phi(x)$$

Question: Why is this interesting or useful?

Automorphisms of Structures

Theorem. The group Γ of all continuous, measure-preserving automorphisms of $\mathbf{M}$ induces ***automorphisms*** of $\mathbb{N}_{\mathbf{M}}$ leaving $\mathbb{N}_{\mathbf{G}}$ and all the arithmetic operations invariant.

Theorem. For modal arithmetic formulae $\Phi(a,b,c,\ldots)$, we have $\tau(⟦\Phi(a,b,c,\ldots)⟧) = ⟦\Phi(\tau(a),\tau(b),\tau(c),\ldots)⟧$, for $a,b,c,\ldots \in \mathbb{N}_{\mathbf{M}}$, $\tau \in \Gamma$.

Theorem. The only elements of $\mathbb{N}_{\mathbf{M}}$ ***invariant*** under all transformations in Γ are the ***standard integers*** of $\mathbb{N}$. The only ***invariant extensional operations*** on $\mathbb{N}_{\mathbf{M}}$ are those coming from ***standard arithmetic operations*** on $\mathbb{N}$.

Boolean-valued Baire Space

Theorem. The set $\mathbb{B}_{\mathbf{M}} = \{\theta \mid \theta : \mathrm{Op}(\mathbb{N}^{\mathbb{N}}) \to_{\mathrm{frm}} \mathbf{M}\}$ can be made into a ***complete*** **M**-set by defining equality as

$$[\![\theta = \eta]\!] = \bigwedge_{U \in \mathrm{Op}(\mathbb{N}^{\mathbb{N}})} (\theta(U) \leftrightarrow \eta(U)).$$

Theorem. We can identify $\mathbb{B}_{\mathbf{M}}$ up to an **M**-isomorphism with the space $\{\phi/\mathrm{Null} \mid \phi : \mathbf{[0, 1]} \to_{\mathrm{meas}} \mathbb{N}^{\mathbb{N}}\}$, where we have $[\![\phi/\mathrm{Null} = \psi/\mathrm{Null}]\!] = \{t \in \mathbf{[0, 1]} \mid \phi(t) = \psi(t)\}/\mathrm{Null}$.

Hint: The opens of $\mathbb{N}^{\mathbb{N}}$ are **generated** by the sets $\{f \mid f(n) = m\}$.

Theorem. The space $\mathbb{B}_{\mathbf{M}}$ is **M**-isomorphic to the **M**-valued function space $\mathbb{N}_{\mathbf{M}}{}^{\mathbb{N}_{\mathbf{M}}}$.

Boolean-valued Reals

Theorem. The set $\mathbb{R}_{\mathbf{M}} = \{\alpha \mid \alpha : Op(\mathbb{R}) \to_{frm} \mathbf{M}\}$ can be made into a complete **M**-set by defining equality as

$$[\![\alpha = \beta]\!] = \bigwedge_{U \in Op(\mathbb{R})} (\alpha(U) \leftrightarrow \beta(U)).$$

Theorem. $Op(\mathbb{R} x \mathbb{R})$ is the ***frame-coproduct*** of $Op(\mathbb{R})$ with itself.

Theorem. Using $+ : \mathbb{R} x \mathbb{R} \to \mathbb{R}$ and $(+) : Op(\mathbb{R}) \to_{frm} Op(\mathbb{R} x\ \mathbb{R})$, then for $\alpha, \beta : Op(\mathbb{R}) \to_{frm} \mathbf{M}$ we have $(\alpha, \beta) : Op(\mathbb{R} x \mathbb{R}) \to_{frm} \mathbf{M}$, and so we can define $(\alpha + \beta) = (\alpha, \beta) \circ (+) : Op(\mathbb{R}) \to_{frm} \mathbf{M}$.

Note: Other continuous functions can be handled in the same way. Many laws of algebra then follow automatically.

Note: We can also define: $[\![\alpha \leq \beta]\!] = \bigwedge_{r \in \mathbb{Q}} (\alpha((r, \infty)) \to \beta((r, \infty)))$.

Random Variables as Reals

Theorem. For the measure algebra **M** we can identify

$$\mathbb{R}_{\mathbf{M}} = \{\, f/\mathrm{Null} \mid f : \mathbf{[0,1]} \to_{\mathrm{meas}} \mathbb{R} \,\}$$

as the **M**-***valued reals***, where we have

$$[\![\, f/\mathrm{Null} = g/\mathrm{Null} \,]\!] = \{\, t \in \mathbf{[0,1]} \mid f(t) = g(t) \,\}/\mathrm{Null};$$

$$[\![\, f/\mathrm{Null} \leq g/\mathrm{Null} \,]\!] = \{\, t \in \mathbf{[0,1]} \mid f(t) \leq g(t) \,\}/\mathrm{Null};$$

and

$$f/\mathrm{Null} + g/\mathrm{Null} = (f + g)/\mathrm{Null}.$$

Note: We can similarly treat all other measurable operations on the **M**-valued reals.

Automorphisms of Reals

Theorem. The group Γ of all continuous, measure-preserving automorphisms of $\mathbf{M}$ induces ***automorphisms*** of $\mathbb{R}_{\mathbf{M}}$ leaving $\mathbb{R}_{\mathbf{G}}$ and all the standard, continuous operations invariant.

Theorem. The only elements of $\mathbb{R}_{\mathbf{M}}$ ***invariant*** under all transformations in Γ are the ***standard reals*** of $\mathbb{R}$. The only ***invariant internal open subsets*** $\mathbb{R}_{\mathbf{M}}$ are those coming from ***standard opens*** of $\mathbb{R}$.

Do we have Random Numbers?

— building on ideas of Robert Solovay and Alex Simpson —

Definition. $\mathrm{Rand}_{\mathbf{M}}$ = the set of elements of $\mathbb{R}_{\mathbf{M}}$ ***avoiding*** all the ***standard closed*** subsets of $\mathbb{R}_{\mathbf{M}}$ of ***measure zero.***

⌘ **Theorem. Random reals exist!** ⌘

Program of Research. Investigate how this notion of randomness extends to structures built from the real and complex numbers (e.g. **vector spaces and Clifford algebras**).

Extensional Powersets

Definition: Given a complete **M**-set A the ***extensional powerset*** of A is the collection of $P: A \to \mathbf{M}$ where, for all $x, y \in A$, we have $P(x) \wedge [\![x = y]\!] \leq P(y)$.

And we can use the definition:

$$[\![P = Q]\!] = \bigwedge_{x \in A} (P(x) \leftrightarrow Q(x))$$

Theorem: The extensional powerset of A is a complete **M**-set.

Note: **A Principle of Comprehension follows for extensional predicates.**

Theorem: $\mathbb{R}_{\mathbf{M}}$ together with its extensional powerset satisfies the ***Dedekind Completeness Axiom***.

Intensional Powersets

Definition: Given a complete **M**-set A the ***intensional powerset*** of A is the collection of P: A$\rightarrow$ **M** where, for all $x,y \in A$, we have $P(x) \wedge \square[\![x = y]\!] \leq P(y)$.

And we use the definition

$$[\![P = Q]\!] = \bigwedge_{x \in A} (P(x) \leftrightarrow Q(x))$$

Theorem: The intensional powerset of A is a complete **M**-set.

Note: A Principle of Comprehension follows.

Question: Should we be able to iterate this notion of powerset? The topic for Seminar III.