Seminar I
Boolean and Modal Algebras

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A Very Brief Potted History

Gödel gave us two translations: **(1)** classical into intuitionistic using not-not, and **(2)** intuitionistic into S4-modal logic.

Tarski and **McKinsey** reviewed all this algebraically in propositional logic, proving completeness of **(2)**.

Mostowski suggested the algebraic interpretation of quantifiers.

Rasiowa and **Sikorski** went further with first-order logic, giving many completeness proofs (*pace* **Kanger**, **Hintikka** and **Kripke**).

Montague applied higher-order modal logic to linguistics.

Solovay and **Scott** showed how **Cohen's** forcing for ZFC can be considered under **(1)**. **Bell** wrote a book (now 3rd ed.).

Gallin studied a Boolean-valued version of Montague semantics.

Myhill, Goodman, Flagg and **Scedrov** made proposals about modal ZF.

Fitting studied modal ZF models and he and **Smullyan** worked out forcing results using both **(1)** and **(2)**.

What is a Lattice?

$0 \leq x \leq 1$	Bounded
$x \leq x$	Partially
$x \leq y \,\&\, y \leq z \Rightarrow x \leq z$	Ordered
$x \leq y \,\&\, y \leq x \Rightarrow x = y$	Set
$x \vee y \leq z \Leftrightarrow x \leq z \,\&\, y \leq z$	With sups
	&
$z \leq x \wedge y \Leftrightarrow z \leq x \,\&\, z \leq y$	With infs

What is a Complete Lattice?

$$\bigvee_{i\in I} x_i \le y \Leftrightarrow (\forall i\in I)\ x_i \le y$$

$$y \le \bigwedge_{i\in I} x_i \Leftrightarrow (\forall i\in I)\ y \le x_i$$

Note:

$$\bigwedge_{i\in I} x_i = \bigvee\{y \mid (\forall i\in I)\ y \le x_i\}$$

What is a Heyting Algebra?

$$x \leq y \to z \Leftrightarrow x \wedge y \leq z$$

What is a Boolean Algebra?

$$x \leq (y \to z) \vee w \Leftrightarrow x \wedge y \leq z \vee w$$

Alternatively using Negation

$$x \leq \neg y \vee z \Leftrightarrow x \wedge y \leq z$$

Distributivity

Theorem: Every Heyting algebra is ***distributive***:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

Theorem: Every complete Heyting algebra is

$(\wedge\bigvee)$*-distributive*:

$$x \wedge \bigvee_{i \in I} y_i = \bigvee_{i \in I} (x \wedge y_i)$$

Note: The dual law does **not** follow for complete Heyting algebras.

Proof Outline

To show that

$$x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z),$$

it is sufficient to show that

$$y \vee z \leq x \rightarrow ((x \wedge y) \vee (x \wedge z)).$$

So, it is sufficient to show that

$$y \leq x \rightarrow ((x \wedge y) \vee (x \wedge z)),$$

(and also for z). But this last comes down to

$$x \wedge y \leq (x \wedge y) \vee (x \wedge z).$$

Now reverse the argument. Q.E.D.

Note: The $(\wedge\vee)$-law has a similar proof.

First-Order Algebraic Semantics

$$[\![aRb]\!] = \text{given}$$

$$[\![\Phi \wedge \Psi]\!] = [\![\Phi]\!] \wedge [\![\Psi]\!]$$

$$[\![\Phi \vee \Psi]\!] = [\![\Phi]\!] \vee [\![\Psi]\!]$$

$$[\![\Phi \rightarrow \Psi]\!] = [\![\Phi]\!] \rightarrow [\![\Psi]\!]$$

$$[\![\exists x.\Phi(x)]\!] = \bigvee_{a \in A} [\![\Phi(a)]\!]$$

$$[\![\forall x.\Phi(x)]\!] = \bigwedge_{a \in A} [\![\Phi(a)]\!]$$

Note: A number of details are being ignored here.

Semantical Completeness

A sentence Φ is provable in intuitionistic *first-order* logic if, and only if,

$$[\![\Phi]\!] = 1$$

whatever the interpretation in a complete Heyting algebra.

Note: The proof from left to right is obvious! And the result holds for classical and modal logic.

Generic Completeness

There is – relative to the choice of language – a ***single*** algebra such that

if $[\![\Phi]\!] = 1$ for this algebra,

then Φ is provable in intuitionistic (classical) (modal) first-order logic.

The proof goes through the Lindenbaum algebra and the MacNeille completion of lattices.

MacNeille Completion I.

The completion embeds a lattice into the lattice of those ideals that are equal to the lower bounds of all their upper bounds.

Hint: **Think of Dedekind cuts.**

The Good: The completion preserves all the ***existing*** sups and infs.

The Bad: The MacNeille completion of a distributive lattice is ***not always*** distributive!

MacNeille Completion II.

- There are many (equational) varieties between Heyting and Boolean algebras.

- However, the completeness process only puts us in the ***same*** variety in the two extreme cases.

- But, it ***does work*** for the extension to ***modal*** S4 Heyting and Boolean algebras (to be explained next).

What Happened to Gödel?

The usual {0,1}-valued completeness theorem follows from the Boolean version for *countable languages via* the

Rasiowa-Sikorski Lemma: Ultrafilters can be found preserving any given *countable* list of sups and infs in a Boolean algebra.

Hence, the MacNeille completion is not needed for {0,1}-valued completeness.

What is a Lewis (S4) Algebra?

A complete Boolean algebra plus a "necessity" operator satisfying:

$$\Box 1 = 1$$

$$\Box\Box p = \Box p \leq p$$

$$\Box(p \wedge q) = \Box p \wedge \Box q$$

Note: The second two laws can be combined:

$$\Box p = \bigvee\{q \mid q = \Box q \leq p\}.$$

"Possibility" is defined as $\Diamond p = \neg\Box\neg p$.

Some Abbreviations

Ha = Heyting Algebra

cHa = Complete Heyting Algebra

Ba = Boolean Algebra

cBa = Complete Boolean Algebra

La = Lewis Algebra

cLa = Complete Lewis Algebra

Note: For semantics don't forget to add:

$$[\![\Box\Phi]\!] = \Box[\![\Phi]\!]$$

What is a Frame?

Definition. A ***frame*** is any complete lattice which is $(\wedge\bigvee)$-***distributive***.

Theorem. In a cLa, the □-stable elements form a ***subframe***.

Theorem. In a cBa, ***any*** subframe creates a cLa.

Hint: We can define: $\Box p = \bigvee\{q \in \mathbf{H} \mid q \leq p\}$, where **H** is the subframe. Such structures can be regarded as **abstract** topological spaces.

An Important Theorem

Theorem. ***Every*** frame can be made into a cHa.

Define: $q \to r = \bigvee\{p \mid p \wedge q \leq r\}$.

Corollary. In a cHa every subframe can be regarded as a cHa (but ***not*** with the same $\to$).

Note: $\neg p = p \to 0$.

Boole vs. Heyting vs. Lewis

Theorem (old). For every cBa **B**, the cHa **H** of all ideals of **B** is such that **B** $\cong \{\neg\neg p \mid p \in$ **H**$\}$.

Theorem (new?). For every cLa **L**, the cHa **H** of all ideals of **L** is such that **L** $\cong \{\neg\neg p \mid p \in$ **H**$\}$ and $\Box_L p = \neg\neg\Box_H\, p$, where we define

$\Box_H\, p = \{q \in \mathbf{L} \mid \exists r \in p[q \leq \Box_L\, r]\}$.

Theorem (old). For every cHa **H**, there is a (non-canonical) cLa **L** such that

H $\cong \{\Box p \mid p \in$ **L**$\}$.

Exercise

Question. What are Lewis **(S5)** algebras?

Answer. A cBa and a ***complete Boolean subalgebra***.

Hint: The extra (S5) axiom amounts to

$$\Diamond p = \Box \Diamond p.$$

By way of example, think of a powerset and the subalgebra of sets invariant under an equivalence relation.

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