The ABC of War: Advantage, Bargaining and Confrontation.

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October 2002

Abstract

A model of war as a Markov game is proposed where players alternate stages of bargaining with stages of confrontation, aiming to attain advantage-the ability to commit credibly to claims for favorable agreements. Long confrontations arise as a stationary equilibrium phenomenon in environments with uncertainty and incomplete information: war occurs when reality disappoints initial (rational) optimism; and persist longer when both agents are optimists and reality proves both wrong.

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1. Introduction

Faced with the challenge of sharing a joint surplus, agents discuss proposals to find an agreement: they *bargain*. Sometimes, surpassing civilized dialogue, force is used: bargainers resort to *confrontation* aiming to attain *advantage* - a situation from which favorable agreements prevail. Wars are the prime example of bargaining processes where confrontation to attain advantage plays an essential role: 'unfair advantages are a characteristic of war. In every war, it seems, at least one of the nations agrees to fight because it believes that it is stronger than the enemy, because it believes that it possesses an unfair advantage (...). Likewise in each period of peace larger nations peacefully exercise power in preserving their own interests simply because they possess that 'unfair advantage'. (Blainey 1988 p. 169).

In this paper we propose a stylized analysis of the strategic aspects underlying war and peace based on a simple dynamic model of bargaining and confrontation between two agents. Our analysis builds on three main ideas:

1. Advantage results from a credible commitment to an extreme position; that is, from the ability to put forward a *claim* for a large share[1] and maintain it over time.

[1]Troughout the paper, anytime we refer to a *claim* we mean a *claim for a large share*.

2. To put forward a claim (or to dismiss that of an opponent) agents must engage in confrontation.

3. Confrontation is an uncertain enterprise that entails a substantial loss of control over the outcomes: On the one hand, success to put forward (down) a claim is never assured; on the other, claims entail a binding commitment until some (randomly determined) future period.

It has long been recognized that unilateral commitment awards bargaining power; and that attempts to attain commitment or to dismiss that of opponents' are a fundamental source of conflict (Schelling 1960, 1966). The essence our contribution is simply to argue that, since the ability to exercise commitment is linked to the use of force, contests to attain advantage entail uncertainties that may fuel prolonged episodes of confrontation.

Our assumption that claims represent a commitment - i.e. that established claims are not freely withdrawn and always persist with some probability - is important. In our context this assumption is plausible at least for two reasons. First, because the need to maintain a reputation is a powerful source of commitment, assuring that established claims have (some degree of) persistence even if they turn out to be disadvantageous. Armies are trained to defend honor (O'Neil 1999), that is, to abide by their reputation that claims are maintained even when they are hopeless. Second, frictions in the process of decision making lead to claim persistence. This is specially important when each side is composed of a large group of individuals: claims are often established thanks to the efforts of the more radical members of the group, which are likely remain in control for a while. Indeed 'while it is hard for a government, particularly a responsible government, to appear as irrational whenever such appearance is expedient, it is equally hard for a government, particularly a responsible one, to *guarantee* its own moderation in every circumstance.'(Schelling 1966, p. 41).

We consider the following Markov game taking place over time. In each period, a *state of the game* is (randomly) realized determining what player (if any) sustains a claim for the period and the bargaining role -proposer or responder - of each player.[2] The proposer offers a partition of the surplus and the responder accepts or rejects. The effect of rejection is confrontation: the surplus shrinks by a fixed rate and a new state of the game (randomly) occurs in the following period. Acceptance terminates the game. If an agreement that disregards an the established claim of a the player is reached, she is worse off than under perpetual disagreement; otherwise payoffs coincide with the agreed shares.

[2] An interpretation is that the two parties are at odds on how to share some territory, that the territory contains a landmark of special value (a river, a city, a gold mine, etc.), and that putting forward a claim is to attain control over the landmark.

We take off analyzing the game under complete information, taking as fixed the value of claims and the strength of the players - that is, the transition probabilities form one state into another. The main insight that we obtain from this exercise is that in equilibrium claims lead to agreement (and thus to advantage) if and only if they persist with sufficient probability. When a player establishes a claim, the opponent might agree to a settlement that respects it, or she may fight to dismiss it. If a claim is very extreme (relative to its persistence), the opponent prefers confrontation to acquiescence and thus raising that claim hurts more than helps. Hence, from a balanced state - when neither player holds claims - in equilibrium agreement prevails immediately. The (potential) effects of claims are taken into account in the agreed shares, but players do not resort to confrontation. Yet, if at the initial state one player holds a claim that is not sufficiently persistent (or if such claim is attained off the equilibrium path) then disagreement prevails until that claim is dismissed.

In the second part of the paper we relax the assumption that transitions between states are governed by probabilities that are constant and well known and we introduce asymmetric information. In these environments 'War is a dispute about the measurement of power'(Blainey 1988, p. 114.), that occurs with positive probability even when bargaining starts without an initial claim, and even when agents are fully symmetric ex-ante[3]. The equilibrium is unique, but the range of possible equilibrium trajectories is very rich: the agreement may be a fair split or may recognize advantage; agreements may be immediate, or they may follow a string of confrontations; useless confrontations where no claim is established are possible.

The intuition for these results is the following. We assume that the persistence of a claim is observed only after it is established, and that both players are ex-ante uncertain about their own and the opponent's capacity to sustain claims. Players have *private information about the distribution* of their own ability to maintain claims. *Optimist (pessimist) types* have private information supporting high (low) expectations of persistence for their own claims; and consequently expect high (low) payoffs from establishing a claim. Before any claim is established, the proposer can extract a higher share if she faces a pessimist type - because the responder expects low returns from confrontation. Suppose that the (ex-ante) probability that the opponent is a pessimist type is high. Then, in equilibrium, the proposer offers a low share - one that is acceptable only if the responder is a pessimist type. But, if the responder turns out to be an optimist type, she rejects. Once confrontation starts, war continues with positive probability because a claim

[3]That is a singular result in relation to the literature. Usually the disatisfaction by one agent - i.e. some ex-ante asymetry - is a necessary condition to sustain confrontation in equilibrium.

may be put forward that has low persistence.

The roots of armed confrontations and their relationship to game theoretic models of bargaining have received substantial attention in literature. Uncertainty and disagreements on relative strength have long been acknowledged as fundamental causes of war. Among rational agents, disagreement over uncertain outcomes must be the result of incomplete information.[4] Indeed, the fundamental role of asymmetric information in generating inefficiencies in bargaining is well known[5]: When agents enjoy private information (for example about their costs of delay, or the value of their outside options) in equilibrium agreements are delayed because rejecting proposals is the only credible way to transmit information. The role of asymmetric information in prompting disagreement in negotiations *prior* to fighting - when war is an outside option - is discussed in Banks 1990, Fearon 1995 and Powell 1996, 1999. Wagner 2000 argues that because wars are the processes by which parties learn each others real forces and costs - thus opening the door to agreements that are impossible without war - the focus of analysis ought to be on bargaining *while* fighting. His discussion of the process by which wars start, develop and end is persuasive: wars commence because inconsistent expectations on the consequences of (limited) fighting initially prevent the existence of agree-

[4]Fearon 1995 argues this point in the present conext.

[5]See Ausubel, Cramton and Denekere 2001 for a survey.

ments that both parties prefer to confrontation; as fighting proceeds expectations are adjusted and mutual gains from agreement arise. Precise discussions of these ideas requires that the strategic and informational details of the game that parties play are spelled out; so that arguments relay on rigorous equilibrium characterizations. This is done in Filson and Werner 2002 and Powell 2001 as well as in the present paper.

Filson and Werner 2002 and Powell 2001 offer important insights on wars as a strategic process of information transmission that takes place at two levels: by the revelation power of proposals and by the hard evidence of failure or success in the use of force. Those insights, however, come at the cost of equilibrium characterizations that are extremely involved[6], hindering transparent discussions of comparative statics.

In contrast with the literature, our main concern is to understand the strategic trade-off between acquiescence and confrontation in a dynamic contest for advantage. In our setup private information may be present and has great impact; but information transmission is not the unique force that sustains confrontation in equilibrium. This allows a simple treatment of informational issues that assures straightforward characterizations of equilibria. Since clear cut results covering a

[6]This is so in spite of drastic simplifications. In Filson and Werner 2002 the equilibrium characterization is restricted to games with two types and up to two periods of confrontation.

broad class of environments are easily obtained, our model provides a transparent theoretical benchmark to test hypothesis on the occurrence, persistence and resolution of war.

The paper is organized as follows. Section 2 presents the bargaining model. Equilibria under complete information and constant strength are discussed in section 3. In section 4 equilibria when strength varies and is subject to incomplete information are analyzed.

2. The game

Two players $i = 1, 2$ jointly generate one unit of surplus, provided that they agree on how to share it. Bargaining and confrontation takes place over time. At each t, $t = 1, 2, ...$, one of the players proposes an agreement, a pair (x_1, x_2) that the other must accept or reject. The identity of the proposer and the claims held at each t depend on the *state of the game at t*. There are four possible states, that we denote by β_1, β_2, α_1 and α_2. When the game is in a *balanced state* β_i no player holds a claim and the proposer is player i. In state α_i, *player i* holds a claim a_i, $0 < a_i \leq 1$ and player j acts as the proposer.[7] Although situations in which both players hold (incompatible) claims do occur in real conflicts[8], this

[7]The assumption that j is the proposer in state α_i can be relaxed as well.
[8]Think for example of 'claims' over Jerusalem.

possibility is ruled out for simplicity and to avoid results that build on the 'purely random occurrence of war'. Nothing substantial depends on this simplification.

When a proposal meets with rejection, one period of confrontation follows that has two effects: First, a positive portion of the surplus $(1-\delta)$ is destructed. Second, the state of the game may change. From t to $t+1$ states evolve according to the following transition probability matrix:

$_t\backslash^{t+1}$	β_1	β_2	α_1	α_2
β_1	$\frac{p_{2t}}{2}$	$\frac{p_{2t}}{2}$	0	$1-p_{2t}$
β_2	$\frac{p_{1t}}{2}$	$\frac{p_{1t}}{2}$	$1-p_{1t}$	0
α_1	$\frac{1-q_{1t}}{2}$	$\frac{1-q_{1t}}{2}$	q_{1t}	0
α_2	$\frac{1-q_{2t}}{2}$	$\frac{1-q_{2t}}{2}$	0	q_{2t}

The parameters q_{it} and $1-p_{it}$ measure i's strength respectively at defense and attack : q_{it} is the probability that player i will maintain her claim into period $t+1$ (i.e. the persistence of player i's claim at t) while $1-p_{it}$ is the probability that player i establishes a claim when she rejects a proposal in state β_j.[9]

Perpetual confrontation pays 0. Acceptance of a proposal (x_1, x_1) at t ends the game with agreement at t: if the state is α_i and $x_i < a_i$ the payoff to player

[9]The assumptions that confrontation in state β_i cannot lead to α_j, and that α_j cannot directy follow from α_i are made for simplicity.

i is $c < 0$; otherwise she obtains $\delta^t x_i$.

A *history of the game at* t is a sequence indicating the states of the game from 0 to t, the rejected proposal from 0 to $t-1$, and possibly a sanding proposal. A *strategy* for player i, denoted σ_i, selects the action of player i at each history in which she must move. A strategy profile is a *subgame perfect equilibrium* if, at every history, it is a best response to itself. A strategy profile is *stationary* if actions depend only on the state of the game and the current offer. We use the term *equilibrium* to refer to a subgame perfect equilibrium in stationary strategies.

The present game is closely related to the bargaining games studied in Merlo and Wilson 1995 where the set of admissible agreements (cakes) and the bargaining protocol at each time follow a Markov process. Unfortunately our analysis cannot build on theirs since their characterization of equilibria requires that the efficiency frontier of all cakes has no flat intervals, an assumption that fails in our game precisely because the set of (relevant) agreements in claim states is a flat truncation of the set of admissible agreements in balanced states. Consequently we provide our own characterization of equilibria.

3. Equilibria when strengh is known

Assume that the transition probabilities are known and remain constant $q_{it} = q_i, p_{it} = p_i$ for all t, and $0 < q_{it}, p_{it} < 1, i = 1, 2$ so that no state of the game is absorbing.[10]

Since agreements that ignore standing claims yield payoffs that are worse than perpetual disagreement, in equilibrium all agreements respect claims. Moreover, outcomes and payoffs depend on whether claims yield agreement (and thus advantage) or confrontation. We will show that agreement necessarily prevails in balanced states. Agreement may prevail under both claims, in one but not in the other, or disagreement may follow either claim. We will prove that only the first and the third scenario exclude each other.[11]

We say that *i's claim a_i is relevant* at a given equilibrium σ, if it constraints to agreements that pay i at least as much as one period of confrontation; that is $a_i \geq \delta \left(q_i v_i(\alpha_i) + (1 - q_i) v_i \right)$, where v_i denotes player i's average payoffs in balanced states and $v_i(\alpha_i)$ denotes her expected payoffs in state α_i. We assume

[10]Thus total defeat of one party over the other is assumed away. If it were not, the game would be one with fixed outside options (see Muthoo, 1999).

[11]Consequently, the uniqueness of stationary equilibria - a standard feature in bargaining games of alternating proposals - is not assured.

that

$$a_i \geq \max\left\{\frac{(1-q_i)\delta}{1-\delta q_i}, \frac{1}{2}\right\}, i = 1, 2, \tag{3.1}$$

a condition assuring that the *claims of both players are always relevant.*[12]

We start with two preliminary results that link the prevalence of agreement at the different states. Lemma 3.1 is proved in the Appendix, Lemma 3.2 follows immediately from Lemma 3.1.

Lemma 3.1. *In any equilibrium, if agreement prevails in state α_i, then in state β_j an agreement is reached as well.*

Lemma 3.2. *In any equilibrium, if agreement prevails in states α_1 and α_2, then an agreement is reached in states β_1 and β_2.*

The implication of the preceding Lemmas is that disagreements cannot occur only in the balanced states. If there is confrontation in equilibrium, that must occur in a claim state. We must thus ask: Are disagreements possible in claim states, or do claims always lead to advantage by acquiescence of the opponent? Our answer closely parallels that of Clausewitz 1976, p. 77: "If the enemy is to be coerced you must put him in a situation that is even more unpleasant than

[12] $a_i \geq \frac{(1-q_i)\delta}{1-\delta q_i}$ implies that $a_i \geq \delta(qa_i + (1-q_i)v_i)$ for all $v_i \leq 1$, that is, a payoff a_i dominates the payoff from continuation even if the payoff in the balanced state is 1.

the sacrifice that you call on him to make. The hardships of that situation must not of course be merely transient. Otherwise the enemy would not give in but would wait for things to improve"

In other words, in a claim state, an agreement is impossible when the share that is left to the opponent is too small. Indeed, a claim a_i cannot be imposed when the share left to the opponent, $1-a_i$, is bounded above by the expected gains of an additional period of disagreement, that is when

$$1-a_i<\delta\left(q_i v_j(\alpha_i)+\left(1-q_i\right)v_j\right), \tag{3.2}$$

where $v_j(\alpha_i)$ denotes j's expected payoffs in state α_i. If (3.2) holds, the expected payoff $v_j(\alpha_i)$ must solve $v_j(\alpha_i)=\delta\left(q_i v_j(\alpha_i)+\left(1-q_i\right)v_j\right)$, while the expected payoff for player i must solve $v_i(\alpha_i)=\delta\left(q_i v_i(\alpha_i)+\left(1-q_i\right)v_i\right)$; therefore $v_j(\alpha_i)=\frac{(1-q_i)\delta v_j}{(1-\delta q_i)}$ and $v_i(\alpha_i)=\frac{(1-q_i)\delta v_i}{(1-\delta q_i)}$. Substituting the expected payoffs in (3.2), we obtain that the necessary and sufficient condition for agreement at α_i is

$$1-a_i\geq\frac{(1-q_i)\delta v_j}{1-\delta q_i}. \tag{3.3}$$

Writing

$$\phi_i\left(v_j\right)\equiv\frac{1-a_i-\delta v_j}{\delta\left(1-a_i-v_j\right)}, \tag{3.4}$$

condition (3.3) is equivalent to $q_i \geq \phi_i(v_j)$. That is, the claim *must not be merely transient*, it must persist at least with probability $\phi_i(v_j)$. This completes the proof the following Lemma.

Lemma 3.3. *In equilibrium, in state α_i, agreement prevails if and only if $q_i \geq \phi_i(v_j)$; and in this case the agreement is $(a_i, 1 - a_i)$.*

Remark 1. *Note that $\phi_i(v_j)$ is strictly increasing in v_j and satisfies $0 < \phi_i(v_j) < 1$, if and only if $\delta v_j \geq 1 - a_i$.*

We are now ready to prove (see the Appendix for a detailed argument) that disagreement prevails only in claim states.

Proposition 3.4. Confrontation only in claim states. *In equilibrium agreements are immediate unless state α_i occurs and $q_i \leq \phi_i(v_j)$.*

We now proceed to the complete characterization of equilibria. This is done in two steps: First, we evaluate the expected payoffs at the different potential equilibrium profiles. Second, we explore necessary and sufficient conditions to sustain each of the potential profiles as an equilibrium.

To evaluate expected payoffs, observe that Proposition 3.4 implies that any equilibrium profile must be one of the following three:

1. A *confrontation* profile, denoted σ^c, where agreement is reached only in states β_1 and β_2.

2. A *peaceful* profile, denoted σ^p, where agreement is reached at all states.

3. An $i-$*advantage profile,* denoted σ^{ai}, where agreement is reached in states β_1, β_2 and α_i but not in α_j.

Computing the values of average payoffs at balanced states for each of these strategy profiles, denoted v_i^c, v_i^p, v_i^{ai} and v_i^{aj}, is immediate: writing $\rho_i \equiv 1-\delta p_i - (1-p_i)\frac{(1-q_i)\delta^2}{1-\delta q_i}$ and $\lambda_i \equiv 1-\delta(p_i + a_i(1-p_i))$ to simplify the exposition, the values follow from straightforward algebra.

Lemma 3.5. *The expected payoffs of player i in a balanced state under the four categories of (potential) equilibrium profiles are as follows:*

v_i^p	v_i^{ai}	v_i^{aj}	v_i^c
$\frac{1-\delta p_j+\delta a_i(1-p_i)-\delta a_j(1-p_j)}{2-\delta(p_i+p_j)}$	$\frac{\rho_j}{\rho_j+\lambda_i}$	$\frac{\lambda_j}{\rho_i+\lambda_j}$	$\frac{\rho_j}{\rho_2+\rho_1}$

Remark 2. *Note that* $v_i^{ai} > \max\{v_i^c, v_i^p\} > \min\{v_i^c, v_i^p\} > v_i^{aj}$.

For each configuration of parameters $(a_{1,}a_2, p_1, p_2)$ define the sets

$$Q^c = \{(q_1, q_2) \mid (q_1, q_2) \ll (\phi_1(v_2^c), \phi_2(v_1^c))\},$$

$$Q^p = \{(q_1, q_2) \mid (q_1, q_2) \gg (\phi_1(v_2^p), \phi_2(v_1^p))\},$$

$$Q^{ai} = \left\{(q_1, q_2) \mid q_i > \phi_i\left(v_j^{ai}\right), q_j \leq \phi_j\left(v_i^{ai}\right)\right\}.$$

Necessary and sufficient conditions to sustain each of the potential profiles as an equilibrium are now immediate. We state them in the following Proposition.

Proposition 3.6. *i) A peaceful equilibrium exists if and only if* $(q_1, q_2) \in Q^p$*; ii) A confrontation equilibrium exits if and only if* $(q_1, q_2) \in Q^c$*; iii) An* $i-$*advantage equilibrium exits if and only if* $(q_1, q_2) \in Q^{ai}$.

The equilibrium profile(s) that prevails at each parameter configuration depends on the specific geometry of Q^c,Q^p, $Q^{a1}$and Q^{a2}. Figure 1 displays this geometry for a generic configuration p_1, p_2 , a_1, and a_2.

insert Figure 1

The general features displayed in Figure 1, namely that $Q^{ai} \cap Q^p \neq \emptyset$, $Q^{ai} \cap Q^c \neq \emptyset$ and that $q \notin Q^p \cup Q^c \Leftrightarrow q \in Q^{a1} \cup Q^{a2}$, are discussed in detail in the proof of Proposition 3.7 that is relegated to the Appendix.

Proposition 3.7. *An equilibrium always exists. A peaceful equilibrium excludes the existence of a confrontation equilibrium, and vice-versa. Equilibria in* $1-$ *advantage strategies and* $2-$*advantage strategies may coexist; and these may coexist with either a peaceful equilibrium or a confrontation equilibrium as well.*

Remark 3. *The potential multiplicity of stationary equilibria opens the door to subgame perfect equilibria in which confrontation occurs in the balanced state, provided that non-stationary strategies are allowed. Since this is a standard result (see Muthoo 1999) we do not elaborate it further.*

We may now summarize the findings of this section. The prospect of advantage alters the distribution of the surplus because it may increase the expected gains of the agent engaging into confrontation. But this force is limited: when claims are excessive, in equilibrium, they lead only to disagreement and thus players prefer to avoid them when they are in a balanced state. As long as the game starts in a balanced state, an agreement prevails immediately and claims are never raised. Still, if the initial state is a claim state and persistence is relatively low, there is confrontation until the claim is dismissed and a balanced state occurs. In a nutshell, under complete information, from a balanced state 'one would never need to use the physical impact of the fighting forces - comparing figures and their strength would be enough.'(Clausewitz 1976, p. 76.).

The fact is, however, that in real conflicts parties do resort to confrontation attempting to raise claims and expecting that their claims will quickly yield a favorable negotiated settlement. The lesson of history, moreover, is one of many wars where claims met resistance and were reversed. In the summer of 480 BC Xerxes seemed to be about to impose his advantage over Athens: the Persian alliance with Carthage assured control over the Greek colonies in Sicily and many of the smaller Greek states were eager to settle peacefully. Still Athens refused to yield and fortunes were reversed at Salamis. Napoleon's Russian campaign of 1812 successfully reached Moscow, yet it failed 'because the Russian government kept its nerve and the people remained loyal and steadfast' (Clausewitz 1976, p. 628). In the summer of 1940 Hitler was celebrating victory and awaiting Churchill to sue for peace; he did not and events took a very different course. Argentina's invasion of the Falkland in April of 1982 and Iraq's invasion of Kuwait in 1990 also fall under this category. Historical examples - as Korea or Vietnam - where parties alternated claims and neither managed to impose them are not unusual either.

Why would a rational government ever raise claims that might eventually prove so disadvantageous? We answer this question in the next section, as we extend the model to account for uncertainty and asymmetric information.

4. Uncertainty and Asymmetric Information

According to Blainey a common trait of wars is that the two parties "were persuaded to fight because most of their leaders were excessively optimistic and impatient men, and persuade to cease fighting because those leaders, having failed, were replaced by more cautious men." (Blainey 1988 p. 123).

The analysis that follows displays a formal set up in which Blainey's description holds precisely. We extend our model to address situations with uncertainty and asymmetric information and show that in these circumstances, confrontations might arise (and persist) along the equilibrium path even if the initial state is balanced.

Our basic assumption is that the probabilities by which players defend their claims are uncertain and its real value is realized only after claims are established. When a player receives an offer in a balanced state, as she considers whether to reject, she has private information on the persistence of her claim but remains uncertain about its precise value until the claim is established. A strong player is one that can be an optimist and expect that, once it is established, her claim will be highly persistent. A weak player, expecting a low persistence, must be a pessimist.

Suppose that the proposer makes a *separating offer* - one that is accepted by a weak/pessimist opponent but rejected by a strong/optimist opponent. A separating offer leads to confrontation if the responder turns out to be an optimist. Running the risk of confrontation may well be (ex-ante) optimal (vis à vis to a *pooling offer* that is accepted by both types) when the probability of facing a pessimist opponent is high enough. Hence separating offers opening the door to confrontation are an equilibrium phenomenon when the proposer is an optimist - in her (ex-ante) beliefs about the type of her opponent - and confrontation indeed occurs when the responder is a strong/optimist type. This confrontation may lead to claim being established; and in this case its persistence is learned through the hard lesson of war. If it is high, the (initial) proposer must sober up and yield. But it may turn out to be low. Then both realize that they were excessively optimist and that confrontation will surely continue for as long as the claim is maintained..

Formally, we assume that a_1, a_2, p_1 and p_2 are known, while q_1 and q_2 are a random variables whose value is realized only after the respective claim is established.. The value of q_i is drawn from distribution F^s or F^w depending on whether i's type is strong or weak; where F^s (first order) stochastically dominates F^w . That is, $F^s(q) \leq F^w(q)$ for all $q \in [0, 1]$. We assume that F^w and F^s have positive densities f^w and f^s on $[0,1]$.[13]In balanced states, the probability that player i is weak in period t is $w_{it}, 0 < w_{it} < 1$, and this is common knowledge.

Moreover, each period that the game is in a balanced state, players privately observe their type for that period. After a player puts forward a claim, the probability to defend it is publicly observed and it remains constant over time as long as the claim is maintained. On the other hand, if the game returns to a balanced state, future realizations of q_i are drawn independently.[14]

A *system of beliefs* for player i, Π_i, maps histories into probability distributions over the types of player j. A *perfect Bayesian equilibrium* (PBE) is an assessment (σ, Π) such that, σ is a pair of strategies that are best response to each other at each history, and Π is a belief profile consistent with Bayes' rule. At a *Markov strategy* actions depend only on the state, the current beliefs and the current offer.

We consider a symmetric and stationary configuration of the game, that is $a_i = a > \frac{1}{2}$, $p_i = p$ and $w_t^i = w$ for all i and t and characterize the *symmetric Markov equilibrium*, a profile of Markov strategies that is a PBE and such that $\sigma_1 = \sigma_2$. For the remainder of the section we use the term equilibrium to refer to

[13]This assumptions are made for simplicity. The results are roubust to other formulations where types are distributions ranked by first order stochastic dominance. The case that the p_i are uncertain is also easy to handle.

[14]Remark 5 discusses the possibility of relaxing these assumptions.

symmetric Markov equilibrium.[15]

Fix an equilibrium and let v denote the ex-ante (before types are drawn) expected gains of players in balanced states.[16] Recall that if player i puts forward a claim but the probability that she maintains it one more period is sufficiently small disagreement prevails. Indeed, by the same argument used to prove Lemma 3.3 we know that if state α_i occurs the game stays in disagreement as long as $q_i \leq \widehat{q}$, where $\widehat{q} = \phi(v)$ (recall that $\phi(v)$ is determined by equation (3.4)). Otherwise, for $q_i > \widehat{q}$, agreement $(x_i, 1 - x_i) = (a, 1 - a)$ prevails.[17]

We will see that only if the probability that players are weak is not too high the equilibrium is pooling - the proposer's offer is surely accepted regardless of the responder's type - and agreement prevails for sure in balanced states. Otherwise, the equilibrium is separating - the proposer's offer is acceptable only if the responder is weak. Play along the separating equilibrium is as follows. The proposer makes an offer that leaves a weak responder indifferent between acceptance and rejection (and that a strong responder strictly prefers to reject). Upon rejection, with probability p, the responder's claim is put forward and the

[15]Observe that an equilibrium is fully characterized by specifying proposals and acceptance thresholds for each type at each state. The system of beliefs does not need to be specified beyond Bayes Rule: At any off the equilibrium history that is not terminal, either a claim is attained and q is fully revealed; or the game remains in the balanced state and new types are drawn, in which case players must believe that her opponent is weak with probability w.

[16]Observe that if rejections occur with positive probability, $v < \frac{1}{2}$.

[17]Note that $0 < \widehat{q} < 1$ only if $a + \delta v > 1$.

realized value of q is observed by both parties. Immediate agreement $(a, 1-a)$ follows if and only if q satisfies $q \geq \overline{q}$; otherwise confrontation follows until the claim is dismissed. At a new balanced state, the responder observes her new type, a new separating offer is made, and so on.

Given an equilibrium (and its associated expected value v) the *responder rejection values at v,* denoted $V^w(v)$ and $V^s(v)$, are the expected gains upon rejection in state β_j, respectively for a weak and a strong responder. Note that $V^w(v) = p\delta v + (1-p)\delta \left[E_w \left[u(q) \mid q \leq \widehat{q}\right] + (1 - F^w(\widehat{q}))a\right]$ and $V^s(v) = p\delta v + (1-p)\delta \left[E_s \left[u(q) \mid q \leq \widehat{q}\right] + (1 - F^s(\widehat{q}))a\right]$, where $E_a \left[u(q) \mid q \leq \widehat{q}\right] = \delta v \int_0^{\widehat{q}} \frac{1-q}{1-\delta q} f^a(q)dq$. Writing $\int_0^{\widehat{q}} \frac{1-q}{1-\delta q} f^a(q)dq \equiv \Phi^a$ to simplify notation the responder rejection values are

$$V^w(v) = p\delta v + (1-p)\delta \left[\delta v \Phi^w + (1 - F^w(\widehat{q}))a\right], \tag{4.1}$$

$$V^s(v) = p\delta v + (1-p)\delta \left[\delta v \Phi^s + (1 - F^s(\widehat{q}))a\right]. \tag{4.2}$$

Similarly we may define the *proposer rejection value* given beliefs Π*,* denoted $U_\Pi(v)$, as the continuation value of the proposer upon a rejection:

$$U_\Pi(v) = p\delta v + (1-p)\delta \left[\delta v \Phi^\Pi + (1 - F^\Pi(\widehat{q}))(1-a)\right]. \tag{4.3}$$

It is a matter of simple algebra to check that the rejection value of a strong responder is greater that of a weak responder, i.e. that $V^s(v) - V^w(v) > 0$.[18] Our next step is to point out that at a pooling equilibrium agreement prevails surely in the balanced states. The following is proved in the Appendix:

Lemma 4.1. *A pooling equilibrium in which disagreement prevails for sure in a balanced state cannot exist.*

Consequently, an equilibrium must be either a pooling equilibrium where both types accept, or separating equilibrium where the responder accepts if and only if she is weak.

The necessary and sufficient conditions that pooling or separating profiles must met to be an equilibrium are closely related to thresholds on the probability that the responder is of weak type. The following definition will simplify the exposition: the *optimism threshold* at v, denoted w_v, is the ratio

$$w_v \equiv \frac{1 - V^s(v) - U_{f^s}(v)}{1 - V^w(v) - U_{f^s}(v)}. \tag{4.4}$$

Consider a pooling equilibrium. Since in states β_i the initial proposal is surely

[18] $V^s(v) - V^w(v) = (1-p)\delta\left[\delta v\left(\Phi^s - \Phi^v\right) + a\left(F^w(\widehat{q}) - F^s(\widehat{q})\right)\right]$and observe that the right hand side is positive if and only if $a \geq \delta v \frac{\Phi^w - \Phi^s}{F^w(\widehat{q}) - F^s(\widehat{q})}$, an inequality that holds, since $a \geq \frac{1}{2}$, $\delta v < \frac{1}{2}$ and $\frac{\Phi^w - \Phi^s}{F^w(\widehat{q}) - F^s(\widehat{q})} < 1$.

accepted the complete symmetry of the environment implies that $v = \frac{1}{2}$. If state α_i occurs (off the equilibrium path) disagreement prevails for $q < \phi(\frac{1}{2}) \equiv q_{\frac{1}{2}}$. Hence the responder rejection values are uniquely given as $V^w(\frac{1}{2})$ and $V^s(\frac{1}{2})$. On the other hand, since a rejection reveals that the responder is strong, the proposer's rejection value is uniquely given as $U_{f^s}(\frac{1}{2})$. In a pooling equilibrium the proposal in state β_i must be acceptable to both types of responder, implying that the strong responder rejection value is offered, i.e. $y^* = V^s(\frac{1}{2})$. On the other hand the proposer must prefer to offer $y^* = V^s(\frac{1}{2})$ and obtain a sure acceptance rather than proposing a lower offer, $y' < y^*$, obtaining acceptance only if the responder is weak. The least that must be offered to obtain a positive probability of acceptance is $y' = V^w(\frac{1}{2})$. Therefore it is necessary that $1-V^s(\frac{1}{2}) \geq w(1 - V^w(\frac{1}{2})) + (1 - w)U_{f^s}(\frac{1}{2})$. Hence, a pooling proposal is optimal if an only if

$$w \leq w_{\frac{1}{2}} \tag{4.5}$$

that is, if the probability that the opponent is weak does not exceed the optimism threshold at $v = \frac{1}{2}$.

When condition (4.5) fails, we will say that the *proposer's optimism is granted.* The motivation for this definition will become clear shortly as we discuss the necessary and sufficient conditions for a separating equilibrium.

Consider a separating strategy profile, and observe that the ex-ante expected gains in the balanced states are $v < \frac{1}{2}$, because in state β_i the proposer offers only the rejection value of the weak type, $x = V^w(v)$, and x is accepted by w but not by s. Moreover, the proposers's beliefs about q upon rejection must be f^s, and therefore the proposer rejection value is $U_{f^s}(v)$. Since players act as responder and proposer with probability $\frac{1}{2}$ each, the ex-ante expected value v must satisfy $v = \frac{1}{2}\left(wV^w(v) + (1-w)V^s(v)\right) + \frac{1}{2}\left(w(1 - V^w(v)) + (1-w)U_{f^s}(v)\right)$, that simplifies to

$$v = w\frac{1}{2} + (1-w)\frac{U_{f^s}(v) + V^s(v)}{2}. \tag{4.6}$$

Let $v^*(w)$ denote a solution to (4.6). A strategy profile where the proposer offers $V^w(v^*(w))$ in state β can be sustained as an equilibrium, if and only if the proposer does not prefer to offer $V^s(v^*(w))$, i.e. provided that

$$w \geq w_{v^*(w)}, \tag{4.7}$$

that is, w must exceed the optimism threshold at $v^*(w)$.

Our next result is proved in the Appendix. It assures when the proposer's optimism is granted, it is surely the case that (4.7) holds.

Lemma 4.2. *For each $w \in (0,1)$ there is a unique $v^*(w)$, $v^*(w) \in (1-a, \frac{\delta}{2})$, and*

$w \geq w_{v^*(w)}$ *if and only if* $w \geq w_{\frac{1}{2}}$.

Consequently, the main result of this section, that a profile satisfying the necessary and sufficient conditions for an equilibrium always exists and it is unique follows immediately.

Proposition 4.3. *There is a unique equilibrium.* **a)** *If the proposer's optimism is not granted, that is when* $w \leq w_{\frac{1}{2}}$, *the equilibrium is pooling: for all* t *in state* β_i *the proposer offers the strong rejection value* $y^* = V^s(\frac{1}{2})$ *and the responder surely accepts.* **b)** *If the proposer's optimism is granted, that is when* $w > w_{\frac{1}{2}}$, *the unique equilibrium is separating: for all* t *in state* β_i *the proposer offers the weak rejection value* $x^* = V^w(v^*(w))$ *and the responder accepts only if she is weak. If the responder is strong and state* α_j *occurs upon rejection, at* $t+1$ *agreement at* $(a_j, 1-a_j)$ *prevails if* $q_j \geq \phi(v^*(w))$; *otherwise disagreement prevails at* $t+1$ *and at all* $t+k$ *until the game returns to a balanced state; then the new proposer offers* x^* *and so on.*

Example 4.4. *Let* $\delta = .8, \quad p = \frac{1}{2}, \quad a = \frac{3}{4}, \quad w = .61,$ *and assume that the distributions of* q *are* $F^s(q) = q^2$ *for strong types and* $F^w(q) = 2q - q^2$ *for weak types.*[19] *Ex-ante payoffs, persistence thresholds, and rejection values respectively*

[19]The two densities, $f^s(q) = 2q$ and $f^w(q) = 2(1-q)$, are symmetrically skewed, strong types have a peak at $q = 1$ while the peak for weak types is $q = 0$. Average persistences are respectively $E^s(q) = \frac{2}{3}$ and $E^w(q) = \frac{1}{3}$.

for the pooling and the separating strategy profile are as follows:

	v	$\phi(v)$	$V^w(v)$	$V^s(v)$	$U_{f^s}(v)$
pooling	${}^1/_2$	${}^3/_4$	$.354$	$.504$	.316
separating	$.437$	${}^2/_3$	$.322$	$.494$	.283

And the optimism threshold is

$$w_{\frac{1}{2}} = \frac{1 - .504 - .316}{1 - .354 - .316} = .545.$$

Since $w > w_{\frac{1}{2}}$ we conclude that the proposer's optimism is granted and the unique equilibrium is the separating profile.[20] *In states β the proposer offers $x^* = .322$. If the game enters state α_i and $q \geq \frac{2}{3}$ an agreement in which player i obtains $\frac{3}{4}$ prevails; otherwise it is revealed that $q < \frac{2}{3}$ and play may stay in disagreement for a long time.*

Remark 4. *A great variety of equilibrium histories are possible along the separating equilibrium :*

1. The war that gets nowhere : *An agreement prevails at $t = k$, $k \geq 1$,without neither side ever raising a claim. Along a sequence of balanced*

[20]Note that $w_{v^*(.61)} = \frac{1-.494-.283}{1-.322-.283} = .564$, so that $w > w_{v^*(.61)}$ holds as well.

states $(\beta, ..., \beta)$*,* $k-1$ *proposals meet rejection because a high type responder is drawn at each* $t = 1, ..., k-1$*, but the responder is never successful at raising a claim of advantage; at* $t = k$ *a responder of low type is finally drawn.*

2. THE WAR THAT LEADS TO VICTORY: *An agreement* $(a, 1-a)$ *prevails at* $t = k+1$*,* $k \geq 1$ *after sequence of states* $(\beta, ..., \beta, \alpha_i)$*. The game is in a balanced state for* k *periods and* k *proposals meet rejection because a high type responder is drawn at each* $t = 1, ...k$*, but the responder does not attain a state of advantage until* $t = k+1$*; when a claim of is raised advantage prevails because* q *is high enough.*

3. THE WAR WHERE CLAIMS COME AND GO: *A war that is settled after* $t = k+n$*,* $k, n \geq 1$ *with a player putting forward a claim and the opponent challenging it. Agreement follows a sequence of state* $(\beta, ..., \beta, \alpha_i, ...\alpha_{i,}\beta, ...)$*. Initially* k *proposals meet rejection because at each* $t = 1, ...k$ *a high type responder is drawn, while no claim is established until* $t = k+1$. A *claim is not met with acquiescence because the realization of* q *is low, but the claim persists for* n *periods. Later on, the continuation history may be either of the two above, or another round of claims raised and dismissed, and so on.*

Remark 5. *The assumption that q becomes public as soon as a state of advantage is realized rules out equilibria where claims are initially challenged but eventually prevail. Such histories might occur in equilibrium under the assumption that agents learn their own q quickly while opponents must learn it by the evidence that the a claim persists.*

In balanced states war starts because the proposer's optimism is mistaken; it continues in claim states because the realized persistence disappoints the initial expectation of the claiming agent. In summary, war occurs when reality disappoints the proposer's optimism; and persist longer when both agents are optimists and reality proves both wrong.

The value of claims and the probabilities to establish and maintain them have been assumed exogenous. In reality, however, these probabilities depend on the degree of advantage aimed by a player; as well as on the opponent's strength and claim value. An extension of our model allowing that bargaining parameters are interrelated and endogenously determined by the strategic choice of the agents will be the object of further research.

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A. Appendix

Proof of Lemma 3.1:

Note that, with 2 as proposer, a disagreement would prevail if and only if 1 preferred to reject any share that 2 were willing to propose; that is if

$\delta(p_1 v_1 + (1-p_1)a_1) \geq 1 - \delta(p_1 v_2 + (1-p_1)(1-a_1))$,or equivalently $p_1(v_1 + v_2) + (1-p_1) \geq \frac{1}{\delta}$, but the latter inequality cannot hold since $v_1 + v_2 \leq 1$ in any equilibrium. The same argument shows that if agreement prevails in α_2, then disagreement cannot prevail at β when 1 proposes. ■

Proof of Proposition 3.4:

By Lemma 3.3, the responder payoffs in a state β are such that

$$\text{if } (q_1, q_2) \ll (\phi_1(v_2), \phi_2(v_1)), \quad v_i^r = \delta v_i \left(p_i + (1-p_i)\tfrac{(1-q_i)\delta}{1-\delta q_i}\right),$$
$$\text{if } (q_1, q_2) \gg (\phi_1(v_2), \phi_2(v_1)), \quad v_i^r = \delta(p_i v_i + (1-p_i)a_i),$$

while the proposer obtains $v_j^p = 1 - v_i^r$.

Clearly agreement must prevail in a state β when disagreement prevails in both claim states. Hence we only need to consider strategy profiles that yield agreement in state α_i but not in state α_j.

Consider first profiles that yield agreement in state α_1 but not in state α_2. When 1 proposes, 2 accepts as long as her share is at least $v_2^r = \delta v_2 \left(p_2 + (1-p_2)\frac{(1-q_2)\delta}{(1-\delta q_2)}\right)$, and thus agreement can be attained if and only if 1 prefers to offer that share over disagreement. That is,

$$\delta v_1 \left(p_2 + (1-p_2)\frac{(1-q_2)\delta}{(1-\delta q_2)}\right) \leq 1 - \delta v_2 \left(p_2 + (1-p_2)\frac{(1-q_2)\delta}{(1-\delta q_2)}\right),$$

or equivalently $v_1 + v_2 \leq \frac{1-\delta q_2}{\delta(\delta(1-q_2)+(1-\delta)p_2)}$. A condition that always holds since the second term exceeds 1. ■

Proof of Proposition 3.7:

Consider first the set Q^p since it is specially simple: Since v_i^p is independent of (q_1, q_2), so is $\phi_i(v_j)$ and consequently

$$Q^p = \{(q_1, q_2) \mid (q_1, q_2) \gg (\widehat{q}_1, \widehat{q}_2)\}, \tag{A.1}$$

where $\widehat{q}_i = \phi_i(v_j^p)$.

On the other hand, observe that Q^c can be expressed as

$$Q^c = \{(q_1, q_2) \mid q_1 \leq \varphi_1(q_2), q_2 \leq \varphi_2(q_1)\}, \tag{A.2}$$

where $y = \varphi_1(q_2)$ if and only if y solves $y = \phi_1(\frac{\rho_1(y)}{\rho_2+\rho_1(y)})$, and where $\rho_i(y)$ denotes the solution to Eq. (**??**) for $y = q_i$; and analogously for $\varphi_2(q_1)$. It is straightforward to check that the functions φ_i are decreasing.

Figure 1 displays that $Q^c \cap Q^p = \emptyset$. Indeed, since both φ_i are decreasing it is straightforward to check that $\widehat{q}_1 > \varphi_2(\widehat{q}_2)$ and $\widehat{q}_2 > \varphi_1(\widehat{q}_1)$.

With respect to Q^{ai} observe that $(v_i^{ai}, v_j^{ai}) = (\frac{\rho_j}{\rho_j+\lambda_i}, \frac{\alpha_i}{\rho_j+\lambda_i})$ depends only on

a_i and q_j. Hence

$$Q^{ai} = \left\{(q_1, q_2) \mid q_i > \psi_i(q_j), q_j \leq \overline{q}_j\right\}, \tag{A.3}$$

where $\overline{q}_j$ solves $q_j = \phi_j\left(v_i^{ai}\right) = \frac{1-a_j-\delta\frac{\rho_j}{\rho_j+\lambda_i}}{\delta\left(1-a_j-\frac{\rho_j}{\rho_j+\lambda_j}\right)}$ and $\psi_i(q_j)$ is $\phi_i\left(v_j^{ai}\right) = \frac{1-a_i-\delta\frac{\lambda_i}{\rho_j+\lambda_i}}{\delta\left(1-a_i-\frac{\lambda_i}{\rho_j+\lambda_i}\right)}$. Since $v_i^{ai} > v_i^p$ we obtain that $\overline{q}_j > \widehat{q}_j$. Moreover, since ϕ_i is increasing, $\psi_i(q_j)$ decreases in q_j and furthermore $\varphi_i(q_j) > \psi_i(q_j)$.

We have thus shown that $Q^{ai} \cap Q^p \neq \emptyset$, $Q^{ai} \cap Q^c \neq \emptyset$ and $q \notin Q^p \cup Q^c \Leftrightarrow q \in Q^{a1} \cup Q^{a2}$. Hence, an equilibrium always exists, it is generally not unique since different types of equilibria (up to three) may coexist for some parameter configurations; yet a peaceful equilibrium and an confrontation equilibrium never coexist. ■

Proof of Lemma 4.1:

Disagreement prevails for sure in state β_i only if the proposer prefers disagreement to an agreement that the weak responder accepts, that is $U_\Pi(v) > 1-V^w(v)$.

At a profile where disagreement prevails for sure in a state β_i the beliefs of the proposer upon rejection must have density $g = wf^w + (1-w)f^s$. And with these beliefs we obtain that

$$U_g(v) = p\delta v + (1-p)\delta\left[\delta v\Phi^g + (1-G(\widehat{q}))(1-a)\right],$$

and

$$U_g(v) + V^w(v) = \quad p\delta v2 + (1-p)\delta\left(\delta v(\Phi^g + \Phi^w)\right.$$
$$\left. + (1 - G(\widehat{q}))(1-a) + (1 - F^w(\widehat{q}))a\right)$$
$$< 1,$$

where the inequality holds since $v \leq \frac{1}{2}$. ■

Proof of Lemma 4.2:

Substituting $\frac{U_{f^s}(v)+V^s(v)}{2} = p\delta v + (1-p)\delta^2 v\Phi^s + (1-p)\delta\frac{1-F^s(\phi(v))}{2}$ we obtain that at a separating equilibrium equation

$$v = \frac{1}{2}\frac{w + (1-w)(1-p)\delta(1 - F^s(\phi(v)))}{(1 - (1-w)\delta\,(p + (1-p)\delta\Phi^s))}. \tag{A.4}$$

must be satisfied. Hence (v^*, q^*), must solve the system of equations

$$v = \Psi(q) = \frac{1}{2}\frac{w+(1-w)(1-p)\delta(1-F^s(q))}{\left(1-(1-w)\delta\left(p+(1-p)\delta\int_0^q \frac{1-x}{1-\delta x} f^s(x)dx\right)\right)},$$
$$q = \phi(v) = \frac{1-a-\delta v}{\delta(1-a-v)}.$$

In is straightforward to check that the system admits a unique solution (v^*, q^*) and that $v^* \in (1-a, \frac{\delta}{2})$, $q^* < q_{\frac{1}{2}}$: Note that $\phi(v)$ is continuous and strictly increasing, and is positive for $v \in (1-a, \frac{\delta}{2})$. On the other hand, $\Psi(q)$ is continuous, strictly decreasing, positive and $\Psi(0) = \frac{\delta}{2}$.

Observe that

$$\varpi(w) = \frac{1 - V^s(v^*(w)) - U_\beta(v^*(w))}{1 - V^w(v^*(w)) - U_\beta(v^*(w))}$$

is a continuous contraction mapping of the interval $[0, 1]$ into itself, that has a unique interior fixed point $c = \varpi(c)$. Since $\varpi(0) > 0$ and $\varpi(\overline{w}) < \overline{w}$, we conclude that $c < \overline{w}$ and therefore $\varpi(w) \leq w$ for all $w \geq \overline{w}$. ■

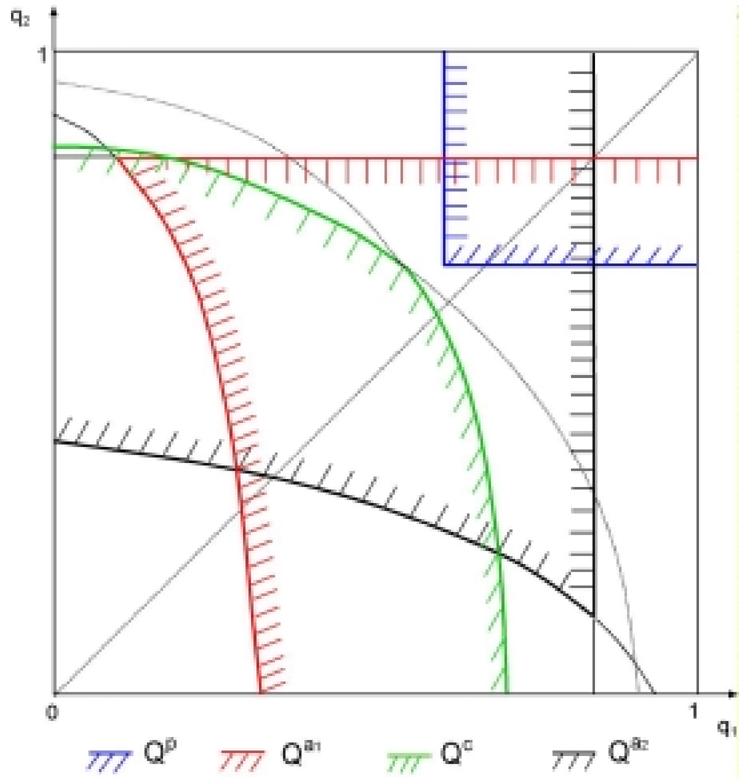

Figure 1

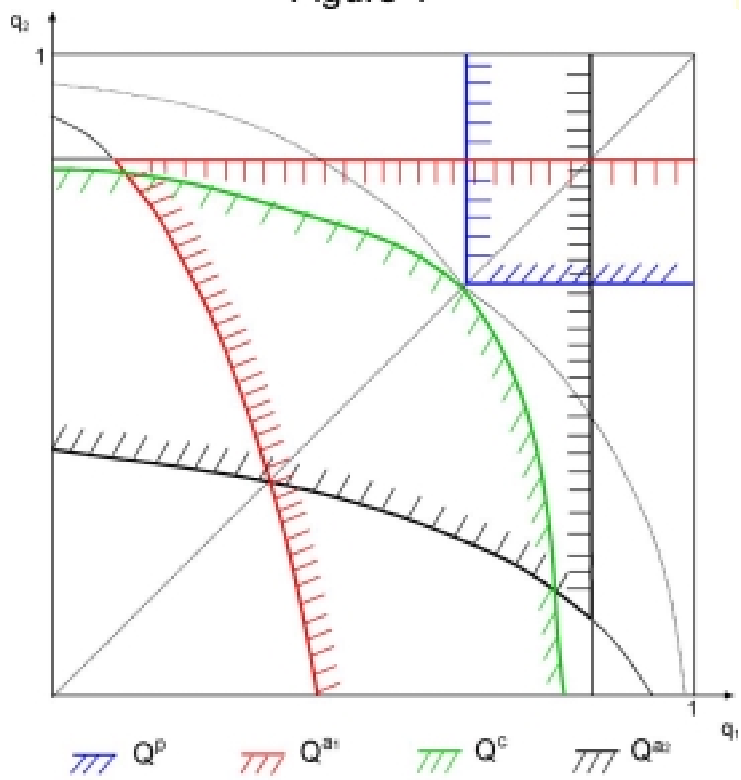

Figure 2