# EXTENSIVE FORM GAMES

*Show all steps and calculations in your answers.*

**QUESTION 1.** GIBBONS (1988). Consider the following game with three players. Player 1 chooses $A$ or $B$, and if he chooses $A$, the game ends with payoffs $(6, 0, 6)$. If he chooses $B$, player 2 chooses $C$ or $D$, and if she chooses $C$, the game ends with payoffs $(8, 6, 8)$. If she chooses $D$, players 1 and 3 play the following simultaneous-move coordination game:

| | | Player 3 E | Player 3 F |
|---|---|---|---|
| Player 1 | G | 7, 10, 7 | 0, 0, 0 |
| | H | 0, 0, 0 | 7, 10, 7 |

(a) Draw the extensive-form representation of this game.

(b) Prove that in every subgame perfect equilibrium, player 1 chooses $B$ at the outset.

(c) Define a hypothetical situation, in which players 1 and 2 both predict Nash equilibria in subgames, and doing so rationalizes player 1 choosing $A$ at the outset. Why is this *not* an SPE?

(d) Find a Nash equilibrium in which player 1 chooses $A$ at the outset. Why is this *not* an SPE?

**QUESTION 2.** Consider the following crisis bargaining game. Players must divide a benefit of size 1, and if they fail to agree to a division, fight a costly and risky war. Players are risk-neutral, so the payoff from obtaining a benefit of size $x$ is just $u_i(x) = x$ for each player. The sequence is as follows. Player $A$ makes a proposal $x \in [0, 1]$. Player $B$ can either agree or start a war. If she starts a war, the game ends with payoffs $(w_A^1, w_B^1)$. If she agrees, player $A$ makes another proposal, $y \in [0, 1]$. Player $B$ can either agree to that or start a war. If she starts a war, the game ends with payoffs $(w_A^2, w_B^2)$. If she agrees, the game ends with payoffs $(y, 1 - y)$. The expected war payoffs, $w_i^k$, where $k \in \{1, 2\}$ denotes the period in which war occurs and $i \in \{A, B\}$ denotes the player, can be computed as follows.

War is a lottery that player $A$ wins with probability $p_A^k$, and loses with probability $1 - p_A^k$. The victorious player obtains the entire benefit and the loser gets nothing. Each player pays a cost $c = 1/10$ if fighting occurs. Assume that initially players are equal in power, so $p_A^1 = 1/2$, but that power then shifts in favor of player $A$, so that $p_A^2 = 3/5$.

(a) Draw the extensive-form representation of this game.

(b) Assume that player $A$ is *exogenously* committed to being unable to revise his proposal if $B$ agrees to the initial one (i.e., $y = x$). Find the subgame perfect equilibria.

(c) Assume that player $A$ is free to revise (or not) his initial proposal even if $B$ agrees to it. Find the subgame perfect equilibria.

(d) Interpret the findings.

**QUESTION 3.** Two people take turns removing stones from a pile of $n$ stones. Each person may, on each of his turns, remove either one stone or two stones. The person who takes the last stone is the winner and gets \$1 from the other person. Player 1 gets to move first. Who is the winner in the SPE for an arbitrary $n$? Show all your work. (Hint: try solving the game for several small values of $n$ and then prove the general result by induction.)

**QUESTION 4.** There are two players, a buyer and a seller. The buyer's value for the object is $v > 0$. Initially, the buyer chooses an investment level $I$ that can be either high, $I_H$, or low, $I_L$, with $I_H > I_L$. This increases the buyer's value of the object to $v + I$ but costs $I^2$. The seller does not observe the investment level and offers the object at a price $p$. If the buyer accepts, his payoff is $v + I - p - I^2$, and the seller's payoff is $p$. If the buyer rejects, his payoff is $-I^2$, and the seller's payoff is 0. Find the subgame perfect equilibria.

**QUESTION 5.** Consider the following two-player game. Player 1 chooses whether to play the game, $P$, or not, $N$. If he chooses not to play, the game ends with payoffs $(1, 1)$. If he chooses to play, each player simultaneously announces a non-negative integer and his payoff is the product of these integers. Formulate this as an extensive form game and find its subgame perfect equilibria. (Hint: the game does have at least one SPE.)

**QUESTION 6.** (GARDNER 2003). Consider the following crisis escalation game. Player 1 begins by choosing whether to escalate ($e$) or ignore a provocation ($\sim e$).

If he ignores the provocation, the game ends with the status quo payoffs $(0, 0)$. If he escalates, player 2 can resist ($r$) or back down ($\sim r$). If she backs down, the game ends with her loss of face and payoffs $(10, -10)$. If she resists, a nuclear confrontation ensues. In this confrontation, the players simultaneously choose either to attack ($a$) or not ($\sim a$). If either one chooses to attack, an all-out nuclear war occurs in which most of the world is destroyed (assume no mineshaft gap), and the payoffs are $(-100, -100)$. If both choose not to attack, the crisis ends and they pay some (small) mobilization costs, so the payoffs are $(-5, -5)$.

(a) Write the extensive form of this game.

(b) Find all pure-strategy Nash equilibria.

(c) Find all subgame-perfect equilibria (in pure and mixed strategies).

**QUESTION 7.** The stage game in Figure 1 is repeated twice. Players observe the outcome of the first play and the payoffs are the discounted sum of the payoffs in each stage. That is, let $u_i^t$ be player $i$'s payoff in stage $t$. The payoff for player $i$ is then $u_i^1 + \delta u_i^2$, where $\delta \in (0, 1]$ is the common discount factor.

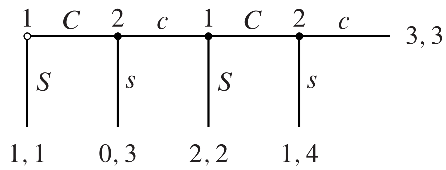
| | | Player 2 | | |
|---|---|---|---|---|
| | | $L$ | $C$ | $R$ |
| | $U$ | $10, 10$ | $2, 12$ | $0, 13$ |
| Player 1 | $M$ | $12, 2$ | $5, 5$ | $0, 0$ |
| | $D$ | $13, 0$ | $0, 0$ | $1, 1$ |

Figure 1: The Stage Game.

(a) Find all pure-strategy subgame-perfect equilibria assuming no discounting ($\delta = 1$).

(b) For each of the equilibria you found in (a), find the smallest discount factor that supports it.

**QUESTION 8.** Two players play two games sequentially. They observe the outcome from the first game, and their payoffs are the time-discounted payoffs from each of the games. That is, let $u_i^n$ be player $i$'s payoff from game $n$. Player $i$'s total payoff is then $u_i^1 + \delta u_i^2$ where $\delta \in [0, 1]$ is the common discount factor. The games are given in Figure 2.




| | $L$ | $R$ |
|---|---|---|
| $U$ | 1, 1 | 0, 0 |
| $D$ | 0, 0 | 3, 3 |

Figure 2: The Two Stage Games.

(a) What are the Nash equilibria of each stage game?

(b) How many pure strategies does each player have in the multistage game?

(c) Find all pure-strategy subgame-perfect equilibria when $\delta = 0$.

(d) Find a subgame-perfect equilibrium for the multistage game in which players receive the payoffs $(2, 2)$ in the first stage when $\delta = 1$.

(e) What is the smallest discount factor that can support the subgame-perfect equilibrium you found in (d)?

(f) For values of $\delta$ greater than the one you found in (e), are there other outcomes of the first-stage game that can be supported in a subgame-perfect equilibrium?