Math 285

Def A stochastic process is a collection of random variables \((X_t)_{t \in \mathbb{T}}\), where \(\mathbb{T}\) is a set of times. If \(\mathbb{T} = \{0, 1, 2, \ldots\}\), we call \((X_t)_{t \in \mathbb{T}}\) a discrete time SP. If \(\mathbb{T} \subseteq \mathbb{R}\), then we call \((X_t)_{t \in \mathbb{T}}\) a continuous time SP.

Discrete Time Markov Chains

Def Let \(S\) be a finite or countable set. A SP \((X_n)_{n=0}^\infty\) is called a discrete-time Markov chain with state space \(S\) if each \(X_n\) takes values in \(S\) and if for all \(n \in \mathbb{N}\) and all \(i_0, i_1, \ldots, i_{n-1} \in S\) such that \(P(X_0 = i_0, \ldots, X_{n-1} = i_{n-1}) > 0\), we have

\[
P(X_n = i_n | X_0 = i_0, \ldots, X_{n-1} = i_{n-1}) = P(X_n = i_n | X_{n-1} = i_{n-1})
\]

We say \((X_n)_{n=0}^\infty\) is a time-homogeneous Markov chain if there is a function \(P : S \times S \to [0, 1]\), s.t.

\[
P(X_n = i_n | X_0 = i_0, \ldots, X_{n-1} = i_{n-1}) = P(i_{n-1}, i_n).
\]

This is a transition probability

\[
P(i, j) = P(i, j) = \begin{cases} 1 & \text{if } i = j \in S \text{ and } \sum_{j \in S} P(i, j) = 1 \end{cases}
\]

For all \(i \in S\), we have \(\sum_{j \in S} P(i, j) = 1\).

Note: If \((X_n)_{n=0}^\infty\) is a time-homogeneous MC with transition probabilities \(P(i, j)\), then

\[
P(X_0 = i_0, \ldots, X_n = i_n) = P(X_0 = i_0)P(X_1 = i_1 | X_0 = i_0)P(X_2 = i_2 | X_2 = i_1) \cdots P(X_n = i_n | X_{n-1} = i_{n-1})
\]
\[= P(X_t = i) P(i, i) P(i, i) \cdots P(i, i, i)\]

**Example** (Simple random walk on \(Z\)): We have \(S = \mathbb{Z}\), and \(P(i, i-1) = P(i, i+1) = \frac{1}{2}\) for \(i \in \mathbb{Z}\).

**Example** (Ehrenfest Urn): We have \(N\) balls and 2 urns. At each time, choose a ball at random and move it to the other urn. Let \(X_n\) be the number of balls in the first urn after \(n\) draws. Then \((X_n)_{n \geq 0}\) is a MC with state space \(S = \{0, 1, \ldots, N\}\) and transition probabilities \(P(i, i-1) = \frac{i}{N}\) and \(P(i, i+1) = \frac{N-i}{N}\) for \(i \in S\).

**Example** (Wright-Fisher Model): A population of fixed size \(N\) can have genes of type \(A\) or \(a\). The population in generation \(n+1\) is obtained by sampling with replacement from the population in generation \(n\). Let \(X_n\) be the number of \(A\) genes in generation \(n\), then \((X_n)_{n \geq 0}\) is a MC with state space \(S = \{0, 1, \ldots, N\}\) and transition probabilities
\[
P(i, j) = \binom{N}{j} \left(\frac{i}{N}\right)^j \left(\frac{N-i}{N}\right)^{N-j}
\]

**Problem:** Let \(Y_n\) be the outcome of the \(n\)th roll of a die. Let \(X_0 = 0\), and let \(X_n = Y_1 + \cdots + Y_n\) for \(n \in \mathbb{N}\). Show \((X_n)_{n \geq 0}\) is a MC.
Solution: Let \( S = \{0, 1, 2, \ldots\} \) for \( i, j \in S \), let
\[
P(i, j) = \begin{cases} \frac{1}{6} & \text{if } 1 \leq j - i \leq 6 \\ 0 & \text{otherwise} \end{cases}
\]

Suppose \( i_0, i_1, \ldots, i_n \in S \) and \( P(X_0 = i_0, \ldots, X_{n-1} = i_{n-1}) > 0 \)
then
\[
P(X_n = i_n | X_0 = i_0, \ldots, X_{n-1} = i_{n-1}) = \frac{P(X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i_n)}{P(X_0 = i_0, \ldots, X_{n-1} = i_{n-1})}
\]

\[
= \frac{P(Y_n = i_{n-1} = i_n, \ldots, Y_0 = i_0)}{P(Y_n = i_{n-1} = i_n, \ldots, Y_0 = i_0)}
\]

\[
= P(Y_n = i_{n-1})
\]

\[
= P(\text{in-1}, \text{in})
\]

Thus, \( (X_n)_{n=0}^\infty \) is a time-homogeneous MC with state space \( S \) and transition probabilities \( p(i, j) \).