Human Thought, Mathematics, and Physical Discovery (Revised)

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How can connections drawn by humans provide a sound basis for the discovery of natural phenomena, and especially physical laws? How, of all connections whatsoever, those that are drawn by us are conducive to making correct guesses about the physical world? Why, in particular, using mathematics in physics is so fruitful? How, more generally, do we explain the "'correspondence' ... between the human brain [/mind] and the physical world as a whole"? (Steiner 1998: 176) These are some of the ways Mark Steiner formulates the classical philosophical question: How do humans, who can only "see" things through the prism of their own thought, attain knowledge of the world?

Speaking about physical discovery, Steiner cites Charles Peirce. In fact, he thinks that some of "Peirce's words are so apt" (*ibid.:* 74) that he cites them twice, both at the beginning and at the end of the chapter "Mathematics, Analogies, and Discovery in Physics" (1998). The twice-cited passage is 7.680 of Peirce (1958). Steiner begins by citing the immediately preceding passages:

> But just so when we experience a long series of systematically connected phenomena, suddenly the idea of a mode of connection, of the system, springs up in our minds, is forced upon us, and there is no warrant for it and no apparent explanation of how we were led so to view it. You may say that we put this and that together; but what brought those ideas out of the depths of consciousness? On this idea, which springs out upon experience of part of the system, we immediately build expectations of what is to come and assume the attitude of watching for them. [Peirce 1958: 7.678; cited in Steiner 1998: 49]
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> It is in this way that science is built up; and science would be impossible if man did not possess a tendency to conjecture rightly. [*Ibid.:* 7.679; cited in Steiner, *op.cit.*]

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* I would like to thank Yemima Ben Menachem and Carl Posy for very helpful comments.

He then proceeds to the main passage:

> It is idle to say that the doctrine of chances would account for man's ultimately guessing right. For if there were only a limited number *n* of hypotheses that man could form, so that 1/*n* would be the chance of the first hypothesis being right, still it would be a remarkable fact that man only could form *n* hypotheses, including in the number the hypothesis that future experimentation would confirm. Why should man's *n* hypotheses include the right one? The doctrine of chance could never account for that until it was in possession of statistics of the hypotheses that are inconceivable by man. But even that is not the real state of things. It is hard to say how many hypotheses a physicist could conceive to account for a phenomenon in his laboratory. He might suppose that the conjunctions of the planets had something to do with it, or some relation between the phases of variability of the stars in α Centauri or the fact of the Dowager empress having blown her nose 1 day 2 hours 34 minutes and 56 seconds after an inhabitant of Mars had died. [*Ibid.:* 7.680, cited in *op.cit.:* 49, 74]

Summing up Peirce's view, Steiner says:

> Peirce noted [that] abduction (guessing) would be futile if the human race had not an inborn talent for hitting on the truth[.] ... Peirce argued that the success of science to date could not be explained by chance[.] ... Peirce, therefore, looked for the explanation of this pre-established harmony between the connection of thoughts and the connection of events. [Steiner 1998: 49]

What cognitive strategies do humans use to come up with fruitful "guesses" in science, especially in scientific areas that are not open to sensory perception? One significant part of the answer, according to Steiner, is: *analogy*.

> How did physicists discover successful theories concerning objects remote from perception and from processes which could have participated in Natural Selection? My answer: by analogy. Having no choice, physicists attempted to frame theories "similar" to the ones they were supposed to replace. [*Ibid.:* 52-3]

This, however, raises the question: "similar" in what respect? Steiner's answer: similar from the point of view of "taxonomy – ... scheme of classifying" (*ibid.:* 53). Taxonomical analogy of what kind? Not only, or even primarily, physical. Physical analogy, Steiner points out, failed in many areas, for example, atomic theory. There, "[t]he ... trouble was that the laws ... of the atom ... were proving *not* to be analogous to those of bodies" (*ibid.:* 54), objects of earlier, macroscopic, physical theories. This meant that scientists had to rely "on nonphysical analogies" (*ibid.*). What non-physical analogies did they rely on? *Mathematical analogies,* analogies based on mathematical taxonomy: "Mathematics ... provided the framework for guessing the laws of the atomic world, by providing its own classificatory schemes" (*ibid.:* 4).

Steiner distinguishes "two kinds of analogy, or taxonomy, that recur in the reasoning of the great inquirers": "Pythagorean" and "formalist". "[A] 'Pythagorean' analogy or taxonomy at time *t* ... [is] a mathematical analogy between physical laws (or other descriptions) not paraphrasable at *t* into nonmathematical language". "[A] 'formalist' analogy or taxonomy ... [is] based on the syntax or even orthography of the *language* or *notation* of physical theories" (*ibid.:* 54). Formalism, however, is also ultimately mathematical. So:

> The strategy physicists pursued... to guess at the laws of nature, was a *Pythagorean strategy*: they used the relations between the structures ... of mathematics to *frame analogies* and *guess according to those analogies*. The strategy succeeded. This does not mean that every guess, or even a large percentage of the guesses, was correct – that never happens on any framework for guessing. What succeeded was the global strategy. ... [E]mpirical information was brought to bear on new cases through the medium of mathematical classification. This is just to make a *Galilean point: in formulating conjectures, the working physicist is gripped by the conviction (explicit or implicit) that the ultimate language of the universe is that of the mathematician.* [*Ibid.:* 4-5, italics adjusted to the present discussion]

In the end, what Pythagorianism comes down to, for Steiner, is "the teaching that the ultimate 'natural kinds' in science are those of pure mathematics" (*ibid.:* 60). In guessing laws, "the investigator has to make up the options", and this requires setting restrictions on the options. This is where a "scheme of analogies" comes into play. It "restricts attention to a certain range of options" (*ibid.:* 74). Since Pythagorean analogies have been very successful, we continue to be guided by such analogies.

One central Pythagorean principle of physical discovery is *symmetry.* What is a symmetry? "According to the modern definition, an object has a symmetry if it is invariant under a group of 'transformations'" (*ibid.:* 84). One type of transformation, important in relativity theory, is Lorentz transformation: a "formula linking the coordinates (*x,y,z,t*) of an event in one frame with those (*x',y',z',t'*) of the *same* event in another, in order that the speed of light should remain constant" (*ibid.:* 108). From the point of view of the Pythagorean, "[a] major function of a law... is to pick out a mathematical structure, or symmetry, that can be used to describe nature. ... Einstein argued that every law of nature must be invariant under a Lorentz transformation: a law of nature, $f(x,y,z,t)=0$, must obey the condition: $\forall x,y,z,t: f(x,y,z,t) \equiv f(x',y',z',t')$" (*ibid.:* 108-9).

In this paper I focus on two questions related to Steiner's concerns: (I) How can connections drawn by humans provide a basis for discoveries about nature? (II) What special features of mathematics equip it for its role in physics?

I. How Can Connections Drawn by Humans Provide a Basis for Scientific Discoveries?

Scientific discovery relies on humans' ability to draw connections that lead to genuine discoveries about the natural world. How can connections drawn by humans provide a sound basis for the discovery of natural phenomena, and in particular, laws of nature? How, of all connections whatsoever, those that are drawn by us are conducive to getting the world right? This, as Peirce and Steiner rightly note, cannot be attributed to mere chance. Nor can it be explained by evolutionary adaptation, given that much of human knowledge, and in particular, theoretical scientific knowledge, goes far beyond those aspects of our life that are shaped by evolutionary needs. "[I]f we are to acquire an ability to guess correctly at the laws of atomic physics", Steiner says, "we must go beyond natural selection" (*ibid.:* 52). But how do we explain the "'correspondence' ... between the human brain [/mind] and the physical world as a whole"? (cited above)

In reflecting on this question one can either focus on a particular aspect (or a few particular aspects) of this correspondence and study its (their) manifestations in depth or focus on the breadth of this correspondence and identify multiple elements that contribute to it and some of their interconnections. In his 1998 book Steiner takes the first route. The aspect he focuses on is *mathematical analogy*, especially in the domain of "atomic and subatomic laws" (*ibid.:*3). He shows how time and again "[m]athematics ... provided the framework for guessing the laws of the atomic world, by providing its own classificatory schemes" (*ibid.:* 4). "[E]mpirical information was brought to bear on new cases through the medium of mathematical classification" (*ibid.:* 5). Steiner examines the use of this strategy in "[t]he history of the wave idea in physics" (*ibid.:* 77), the "development of relativistic quantum mechanics" (*ibid.:* 82), "field equations of General Relativity" (*ibid.:* 94), "the program in physics known as 'gauge field theories', inaugurated by Yang and Mills" (*ibid.:* 168), and more. Among the heuristics he identifies based on this approach are: "a mathematical possibility will be realized by nature" (*ibid.:* 82); the "formalism [is] 'wiser than we are'" (*ibid.:* 84, reference to Hertz), "a mathematical isomorphism betokens physical equivalence" (*ibid.:* 88), "mathematical equivalence = physical equivalence" (*ibid.:* 90), and "[o]ne formulates equations by analogy to the mathematical form of other equations, even if little or no physical motivation exists for the analogy" (*ibid.:* 94). He points out the "remarkable discoveries" that "symmetry arguments" have led to "in elementary particle physics" (*ibid.*: 84), he notes (following Emmy Noether 1918) that the "importance of symmetries in physics lies in their relationship to laws of conservation" (*ibid.*), that "theorems of group theory, and nothing more, allow detailed numerical predictions which appear to come out of thin air" (*ibid.:* 90), and that the appeal to mathematics is built into our interest in laws: "[a] major function of a law, for the Pythagorean, is to pick out a mathematical structure, or symmetry, that can be used to describe nature" (*ibid.:* 108).

In what follows, I will focus on the second aspect of the mind-world correspondence mentioned above, centered on the multiplicity of elements contributing to it. Beginning with Kant, I will proceed to Quine, and then present my own thoughts, which are influenced by, but also diverge from, both. In so doing, I will meet Steiner on the level of the classical epistemic question which concerns both of us.

A. Kant

A well-known answer to the classical mind-world question was given by Immanuel Kant (1781/7).

(a) Epistemic Revolution. Seeking to establish the possibility of human knowledge of the world in light of (i) skeptical challenges (mainly David Hume's) and (ii) problems arising from earlier attempts to establish its possibility (e.g., Gottfried Leibniz's), Kant launched an epistemic revolution. This revolution was supposed to establish the possibility of human knowledge in general, and in particular scientific knowledge of the kind challenged by Hume: knowledge of causal relations and laws of nature. The problem, as Hume pointed out, was that causal relations are necessary[1] and laws of nature are both necessary and universal, but neither necessity nor universality can be based on sensory experience. To achieve his goal, Kant developed a new philosophical methodology (Copernican revolution) which assigned a crucial role to the *structure of human cognition* in establishing the possibility of *human knowledge* of causal relations and natural laws. The key point is that knowledge requires representation of the world, but representation depends on the structure of human cognition. If this structure imposes lawfulness and necessity on the represented world, then the world is known to us as governed by laws and necessary relations, just as physical science represents it. Our knowledge of the world depends on the structure of human cognition. Not just any type of cognition would generate veridical knowledge of the world. The key to understanding scientific knowledge, then, is understanding the structure of human cognition and its access to the world.

(b) The Structure of Human Cognition. The two ultimate sources of cognition, according to Kant, are *mind* and *world*, and the two basic modes of cognition are "receptivity" and "spontaneity" (A50/B74), where "spontaneity" stands for cognitive operations on "received"

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[1] "[I]t is commonly supposed that there is a necessary connection betwixt the cause and effect" (Hume 1740: 193); "We suppose that there is some connection between [cause and effect], some power in the one by which it infallibly produces the other, and operates with the greatest certainty and strongest necessity." (Hume 1751: 85-6)

input: In a simple act of cognition, the mind *receives* stimuli from some source external to it, and it *processes* these stimuli using its own internal resources. The external element is *world* or *reality*; the internal element – *mind*. The mind has three faculties: (i) *sensibility or intuition* (which itself has two components: sensory intuition and pure intuition), (ii) *imagination* and (iii) *understanding* (Kant's correlate of what I call here "intellect").

Roughly, and without aiming at a comprehensive description, we may sketch the stages involved in generating scientific cognitions as follows:[2] First, external input is received through our sensory apparatus. Next, this input is "shaped" by our (sensory and pure) intuition. The results then undergo a chain of syntheses, starting with syntheses generated by our imagination and continuing with syntheses produced by our faculty of concepts – understanding. In each stage we achieve some level of *representation of the world*, and representations generated in lower stages can be further synthesized in higher stages. Representations of full-fledged *objects* are achieved only at relatively late stages. I.e., it is only after our sensory input is shaped by our forms of intuition and undergoes repeated syntheses by imagination and the understanding (intellect)[3] that full-fledged *objects* emerge. Full-fledged objects thus reside at a level of reality which is significantly shaped by the way our mind operates. But they are anchored in another, deeper, level of reality, which is independent of us.

The highest level of representation is that of the so-called *categories.* The categories are the highest forms of synthesis. They are related to the *fixed* logical forms of judgments (sentences) and are themselves *fixed* and *unchanging*. The categories alone do not provide knowledge of the world, but in conjunction with lower, yet still very high, forms of synthesis –

[2] Given the multiplicity and diversity of interpretations of Kant, the present sketch cannot agree with all existent interpretations. It reflects my own understanding, which has points in common with some well-known interpretations but does not fully coincide with any.

[3] Kant distinguishes between "*figurative* synthesis ... and ... *synthesis intellectualis"* (Kant 1781/7: B151). The former is the ground for the applicability of mathematics. (I would like to thank Carl Posy for this comment.)

so-called *schemata* – they generate *principles* applicable to *nature.* Thus, the *hypothetical* form of judgment gives rise to the category of *cause and effect,* which, being schematized, generates the principle of *causality*, applicable to natural objects (arrived at by various combinations of receptivity and spontaneity).

Our cognition of nature takes many forms: from cognition of individual objects and their contingent properties (relations) to highly general and abstract cognitions, cognitions of universal and necessary laws of nature on various levels. An example of a cognition of the former kind is a cognition of a certain book being heavy or of one book lying next to another. An example of a cognition of the latter kind is cognition of general-and-necessary principles like the principle of causality: "All alterations take place in conformity with the law of the connection of cause and effect" (B232). Underneath these principles we find the laws of nature, such as the law of universal gravity.

If Kant's account of the structure of human cognition is right, it show that, and explains why and how, our scientific cognition of the world is *not random*. Our cognition proceeds, step by step, in a regimented process of (Kantian) receptivity and spontaneity, one that connects us to the world through our sensory organs and, employing highly structured syntheses by our imagination and understanding, produces a blueprint of the world as exhibiting causal connections and governed by laws.

This achievement, however, comes at a price:

(c) Science is Limited to Appearances. Kant's account requires that we divide our concept of *world* into two: the world as *it is in itself*, and the world as *it appears to us to be*. Reality has two levels: (i) reality as it is – "thing in itself" (Bxx), "noumenon" (B307), "the *unconditioned*" (Bxx), "absolute reality" (A35/B52), and (ii) experienced reality – "appearance" (Bxx), "phenomenon" (A183/B227). It is a central tenet of Kant's epistemology that "[w]hat the objects may be in themselves would never become known to us" (A43/B60). The level on which they are accessible to us is the level of *appearance*. It is this level of reality or world that scientists refer

to as “nature” and that science provides us with knowledge of. “[N]ature”, in Kant’s words, is “the sum of all appearances” (B163). This does not make the level of thing-in-itself dispensable, since without it “we should be landed in the absurd conclusion that there can be appearance without anything that appears” (Bxxvi-vii). Still, scientific knowledge is knowledge of appearances.

This is problematic since what humans seek to know is the *world as it is in itself*, not an image of the world generated by our mental apparatus. Some Kant scholars tried to lessen the blow by construing the world of appearance as closer to the world as it is in itself than it seems to be. Henry Allison (1983), Lucy Allais (2015), and others[4] proposed a "dual aspect" view of the Kantian world – epistemic in Allison's case, metaphysical in Allais's. According to this view, noumenon and phenomenon are two aspects of one and the same, real, world, and science studies this real world, albeit from a particular perspective. Alternatively, science studies a particular aspect of this world. Be that as it may, it is quite clear that by explaining, constructively, the structure of the connection between the human mind and the world, Kant makes significant progress toward showing that, why, and how, this connection is systematic rather than random. As for explaining successful scientific guessing, Kant offers a grounding for the existence of a causal nexus underlying such guessing, but does not explain how we guess correctly specific causal connections.

B. Quine.

A different approach to the mind-world problem is suggested, albeit indirectly, by W.V. Quine. We arrive at this approach by a change in gestalt vis-a-vis some of Quine's holistic ideas, including the Duhem-Quine thesis.

The Duhem-Quine thesis is commonly understood as a negative thesis, centered on the *underdetermination* of scientific theories by the available evidence. Kyle Stanford (2009/17)

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[4] An earlier proponent was Paton (1936).

presents the common interpretation of this thesis as follows:

> [T]he traditional locus classicus for underdetermination in science is the work of Pierre Duhem... at the turn of the 20th century. In *The Aim and Structure of Physical Theory*, Duhem formulated various problems of scientific underdetermination in an especially perspicuous and compelling way, although he himself argued that these problems posed serious challenges only to our efforts to confirm theories in physics. In the middle of the 20th Century, W. V. O. Quine suggested that such challenges applied not only to the confirmation of all types of scientific theories, but to all knowledge claims whatsoever.
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> Holist underdetermination ... arises whenever our inability to test hypotheses in isolation leaves us underdetermined in our *response* to a failed prediction or some other piece of disconfirming evidence. That is, because hypotheses have empirical implications or consequences only when *conjoined* with other hypotheses and/or background beliefs about the world, a failed prediction or falsified empirical consequence typically leaves open to us the possibility of blaming and abandoning one of these background beliefs and/or 'auxiliary' hypotheses rather than the hypothesis we set out to test in the first place. [Stanford 2009/17]

Negative conclusion:

> When the world does not live up to our theory-grounded expectations, we must give up *something*, but because no hypothesis is ever tested in isolation, no experiment ever tells us precisely which belief it is that we must revise or give up as mistaken[.] [*Ibid.*]

By the same token, no experiment, no evidence, ever tells us what belief we must accept as true.

This negative face of the Duhem-Quine thesis, however, can be turned on its head. The same world, the same facet of the world, can be correctly described by many combinations of hypotheses and background assumptions. This increases the likelihood that some true descriptions of the world (some combinations of hypotheses and background assumptions that correctly describe the world) comport with the structure of human cognition. There is no unique language of nature, no unique way to attain knowledge of the natural world; rather, there are numerous ways of reaching the natural world cognitively, and some of these may be open to beings with our cognitive make up. If one correct theoretical account of the world is closed to us, another may be open. If the structure of human cognition prevents us from correctly theorizing about the world in one way, it may equip us with another, equally correct, way of theorizing about it. This is the positive face of the Duhem-Quine thesis.

Other glimpses of the positive face of this thesis can be found in many of Quine's writings, although Quine himself is not concerned with the question that we are asking. Quine's claim that there are multiple "*empirically* equivalent systems of the world" (Quine 1975: 313, my italics) is another version of the underdetermination thesis that, turning Quine on his head, can be read positively. If T is an empirically correct theory of the world that can be constructed by beings with a cognitive apparatus that differs from ours (in certain relevant ways) but cannot be constructed by us (given the makeup of our cognitive apparatus), there may very well be other, equally *empirically correct* theories of the world that can be constructed by us (by beings with our cognitive makeup).

Quine's statement that "to call a posit a posit is not to patronize it" (Quine 1960: 22) can be viewed as suggesting that humans may be able to reach the world cognitively through posits; the fact that posits are created by us does not mean that they cannot serve as fruitful tools of discovery and theorizing.[5] Quine's claim that "if you take the total scattered portion of the spatiotemporal world that is made up of rabbits, and that which is made up of undetached rabbit parts, and that which is made up of rabbit stages, you come out with the same scattered portion of the world each of the three time" (Quine 1968: 32) is a version of his underdetermination thesis that may be read as saying that there are multiple ways of slicing the world, so that multiple slicings can produce a correct understanding of the world. In other words, it is possible to change the units in terms of which the world is correctly described, increasing the likelihood of correct descriptions using humanly conceivable units. Quine's observation that "[t]here are Frege's

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[5] Explanation: To patronize posits, epistemically, means to treat them as epistemically inferior – knowledge involving posits is not knowledge of the world as it is. What does not-patronizing posits mean? Leaving aside what Quine himself meant by this, I would like to suggest that one way of not patronizing posits is viewing them as *tools* for reaching the world (ones that we need due to our cognitive shortcomings). In subsection C below, under "*d. Truth as Non-Naive Correspondence*", I give an example of such a use of posits as tools.

version [of the natural numbers], Zermelo's and von Neumann's [versions], and countless further alternative [versions]", and they are "all mutually compatible and equally correct" (*ibid.:* 43) can be viewed as suggesting that there are multiple ways of correctly representing the *same formal structures* in the world. This example is especially forceful as far as the possibility of *correct* humanly accessible mathematical descriptions of the world are concerned. And Quine's claim that "[o]ne ontology is always reducible to another when we are given a proxy function *f* that is one-to-one" (*ibid.:* 57) may be understood as suggesting that one fruitful tool for moving from one representation of the world to another, formally equivalent, representation, is the 1-1 and onto function, which is cognitively accessible to humans (and is often used in symmetries/ invariances).

These observations increase the likelihood of correct theories of the world constructed by humans beyond mere chance.[6] The world is, in a sense, open to diverse kinds of theorizing, and these may include human theorizing.[7] It is noteworthy that these Quine-inspired observations do not require a Kantian bifurcation of *world* into thing-in-itself and appearance.

Putting together the Quine-inspired idea of multiple correct descriptions of the world together with the critical-constructive approach to the structure of human cognition traceable to Kant, we make significant progress toward meeting the challenge of explaining the possibility of correspondence between the human mind and the physical world beyond mere chance, (including those regions of the world that are inaccessible, or only minimally accessible, to sensory perception).

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[6] Starting with the observation that empirically adequate scientific theories are more likely to be true than empirically inadequate theories, I draw a connection between empirical adequacy and likelihood of truth. If there are multiple ways to construct equally empirically adequate theories, this increases the number of theories that are candidates for true theories.

[7] Here I go beyond Quine. But in this paper I allow myself to extend what Quine said beyond the context he limited himself to (when I think that what he said holds in a broader context), and this enables me to draw new conclusions.

C. Further Principles

Glimpsing further, let me suggest a few additional principles that have a bearing on the mind-world correspondence. Each of these principles describes an aspect of human cognition that increases the likelihood of hitting upon, or making progress toward, a correct theory of the world.

a. Epistemic Freedom. The acquisition of knowledge, especially theoretical knowledge, is for the most part an active rather than a passive process. It is not an automatic reaction to stimuli. It is a dynamic, intentional act, initiated by us – an act of epistemic freedom. More precisely, it is an act of *directed* freedom. An act directed toward knowledge, not chosen on a whim. (Consider, as an example, Galileo Galilei's design of his inclined plane experiment.)[8] Directed freedom is manifested in all parts of scientific research: asking questions, identifying problems, designing research programs and experiments, making calculations, drawing analogies, constructing theoretical models, applying these models, collecting evidence, drawing conclusions, formulating theories, critically examining theories, discarding, affirming, or revising theories and models, deciding where to go from here, raising new questions, and so on. Humans' ability to exercise directed freedom considerably enhances, though does not guarantee, the likelihood of mind-world correspondence.

The question, however, arises: What cognitive capacities are available to us in planning

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[8] "The experiment made to ascertain whether two bodies, differing greatly in weight will fall from a given height with the same speed offers some difficulty; because, if the height is considerable, the retarding effect of the medium, which must be penetrated and thrust aside by the falling body, will be greater in the case of the small momentum of the very light body than in the case of the great force of the heavy body; so that, in a long distance, the light body will be left behind; if the height be small, one may well doubt whether there is any difference; and if there be a difference it will be inappreciable. It occurred to me therefore to repeat many times the fall through a small height in such a way that I might accumulate all those small intervals of time that elapse between the arrival of the heavy and light bodies respectively at their common terminus, so that this sum makes an interval of time which is not only observable, but easily observable. In order to employ the slowest speeds possible and thus reduce the change which the resisting medium produces upon the simple effect of gravity it occurred to me to allow the bodies to fall along a plane slightly inclined to the horizontal. For in such a plane, just as well as in a vertical plane, one may discover how bodies of different weight behave". (Galilei 1638: 84)

and implementing our epistemic initiatives? The traditional answer to this question postulates a bifurcation of our cognitive resources into sensory and intellectual. But while many disciplines study the sensory routes from mind to world, few study the intellectual routes. Traditionally, the main theory of intellectual access to the world has been the theory of "pure", "mathematical", or "rational" intuition, where such intuition is modeled, to a greater or lesser degree, after sensory perception.[9] But pure intuition does not cover all the intellectual activities involved in the acquisition of knowledge: it leaves out mathematical calculation, logical and other kinds of derivation (probabilistic, inductive, etc.), Kantian synthesis, analogy drawing, and much more. Most contemporary accounts of these activities do not focus on their connection to the world. Many assume, especially with regard to logic, but often with respect to mathematics as well, that there is no such connection. Steiner's emphasis on the role of mathematical analogies in zeroing in on the world counters this assumption.[10] To understand the diverse contributions of intellect to scientific discovery, a broader category of intellectual activity than pure intuition is needed.

b. Figuring Out. One thing that mathematical calculation, deductive, inductive, and abductive inference, Kantian synthesis, making analogies, and other intellectual activities have in common is *drawing connections*. Drawing connections is central to the pursuit of knowledge. This was Peirce's point. But not all connections are directed toward knowledge. Some

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[9] "Pure intuition" is often associated with Kant (1781/7), "mathematical intuition" with Kurt Gödel (1947), and "rational intuition" with Charles Parsons (2008). Steiner's view of mathematical intuition, as described in his 1975 book *Mathematical Knowledge*, is mixed: on the one hand, he "remov[es] some a priori objections to the existence of such a faculty, and not[es] the number of impressive philosophers [and mathematicians] who accept mathematical intuition as a fact" (*ibid.:* 20, see also 136-7). On the other hand he believes that "Platonist accounts of this supposed faculty have been sorely lacking" (*ibid.*). Turning to "a speculative discussion of the prospects for an *empirical* theory of mathematical intuition", he concludes that on this approach "[i]ntuition emerges with fewer pretensions, but less shrouded in mystery" (*ibid.:* 20-1).

[10] For two exceptions to the view that logic is not connected to the world see Penelope's Maddy's naturalist account of logic's connection to the world in her 2007 book and elsewhere, and my non-naturalist (but also non-Platonist, non-apriorist) account of this connection in Sher 1991, 2016, and elsewhere.

connections produce works of art, others make us laugh, still others cause us to be frightened, and so on. *Figuring out is drawing connections that are oriented toward knowledge:* connections that show us how the world, or some facet of the world, is, tell us what to do in order to obtain a given goal, explain why something happens, and so on. It is the drawing of such connections that we need to focus on in order to understand how intellect operates in epistemic endeavors. A simple metaphor of figuring out is "putting two and two together". Other, related expressions include "work out", "make out", "fathom", "decipher", "solve", "think out", "think through", "get to the bottom of", "find the answer to", "unravel", "untangle", "crack".

Figuring out is something humans do at every age, every walk of life, every profession, and every discipline. Babies figure out things all the time. A computer technician figures out what is wrong with your computer and how to solve the problem. King Solomon figured out a test for determining which woman is the real mother of baby X. Galileo figured out how to lengthen the fall time of the balls. Kant figured out that Hume's skeptical challenge might be met by a "Copernican" revolution. John Couch Adams and Urbain Le Verrier figured out that there is an additional planet beyond Uranus. Gödel figured out that mathematics is incomplete and how to prove its incompleteness. Alfred Tarski figured out a way to determine whether sentence S is a logical consequence of a set of sentences Γ while avoiding the limitations of the proof-theoretic and substitutional methods. And so on.[11] Steiner's discussion of mathematical analogies in gauge field theory (1998: 168-76) involves an intricate chain of figuring out. It is quite clear that, as the examples of Galileo, Le Verrier, Gödel, Tarski, and others show, the connections drawn by humans freely, yet under constraints issued by the world itself on the one hand and our norms of truth, evidence, justification, and explanation on the other, owe their (partial) success to more than just chance or luck.

What is the structure of figuring out? The fact that figuring out takes place in an

[11] In thinking about figuring out for the purpose of this paper there is no need to draw a sharp distinction between "figuring out that" and "figuring out how".

environment of epistemic freedom suggests that there is no unique structure to all acts of figuring out. For example, Kant's structure of knowledge-inducing intuitions and syntheses is too rigid and static to do justice to the richness, plasticity, and open-endedness of figuring out. Kant characterized the cognitive activities described by his structure as *spontaneous*, but Kant's spontaneity is *unfree*. In particular, the ultimate principles of synthesis – the categories – are *fixed once and for all*. There is no room for humans to critically examine the existent categories and consider replacing them by new, potentially more efficacious, categories. Others, too, object to Kant's overly rigid conceptions of categories and concepts, favoring a more fluid conception. For example, C.I. Lewis says that "[t]he ... prejudice of an absolute human reason, universal to all men and to all time, has created an artificially exalted and impossible conception of the categories as fixed and unalterable modes of mind." (Lewis 1929: 233)

It is a basic part of epistemic freedom that in conducting a scientific investigation we appeal to a conceptual framework which is fixed in the sense of being relied upon during this investigation or some stage thereof. But the framework might change, both from one investigation to another and in the course of a given investigation. Selecting our concepts (within the limits of what is available to humans both in principle and in a given stage in the development of science) is part of the process of figuring out: figuring out how to solve a given problem, how to remove an obstacle, what needs to be done (would be fruitful to do) next, and so on.

Some approaches to cognitive change, however, also neglect epistemic freedom. For example, bio-psychological investigations of cognitive change (such as those emphasizing conditioning or stimulus-response, which are common in the behaviorist literature) leave out much of what is involved in free, intentional, goal-oriented, figuring out. The same holds for accounts based on evolutionary adaptation. This was noted even by philosophers who approach conceptual change in different ways than I do, for example, Paul Thagard, who offers a computational explanation of such change. Thus, Thagard says: "concept generation in science is generally part of the problem solving process ... . Organisms, on the other hand, do not generate mutations in order to overcome ... problems." (Thagard 1992: 154) And speaking in terms of theory change, he says: "The relevant differences between genes and theories is that theories have people trying to make them better" (Thagard 1980: 190).

This does not mean that there is no room for a systematic investigation of figuring out, either on the biological and psychological levels or on the philosophical level. Michael Friedman's dynamic conception of knowledge (Friedman 2001) is a step in this direction. Likewise, I discussed the epistemic framework of friction (constraint) and freedom, required for figuring out, in Sher (2016). An additional contributor to the scope, breadth, flexibility, and open-endedness of the mind-world connection is its holistic character (which includes, yet goes beyond, the Duhem-Quine thesis discussed above).

c. World-Oriented Holism. Holism means different things to different philosophers. Sometimes it even means different things to the same philosopher. Quine, for example, understood by holism both "wholism" or "one-unit holism" and "interconnectedness holism".[12] One-unit holism says that "the [smallest, hence only] unit of [epistemic] significance is the whole of science" (Quine 1951: 42). Interconnectedness holism says that there are many units of knowledge and they are potentially interconnected in open-ended ways. One significant difference between these two kinds of holism is their approach to units of knowledge and structure. One-unit holism implies that our body of knowledge is a large lump of undifferentiated material, having no inner structure. This undermines its viability as an account of knowledge and methodology.[13] This criticism, however, does not apply to interconnectedness holism. The latter regards our body of knowledge as an interconnected network or web, where any two units of knowledge might be (though need not be) interconnected in one way or another. As such, and in

[12] These are my terms.

[13] The linguistic correlate of this holism was harshly criticized by Dummett (1973, 1973/81) on just this ground. (See also Fodor and Lepore 1992.)

contrast to one-unit holism, it amplifies the multiplicity of distinct units of knowledge and the centrality of structure. According to this holism, it is distinct units of knowledge that stand in various relations, forming structures.[14]

The holism I have in mind as contributing to successful mind-world connections is an interconnectedness holism that goes beyond Quine's and differs in a number of ways from other conceptions of holism as well. Elsewhere I called it "foundational holism" because of its ability to provide a philosophical foundation for knowledge without falling into the traps of foundationalism. Some of its distinctive characteristics are:

1. It is world-oriented. The connections between units of knowledge that are conducive to knowledge are grounded in connections between objects and structures of objects in the world. Accordingly:

2. It is a correspondentist rather than a coherentist holism, one that focuses on relations between theories and the world rather than on relations between theories (independently of the world).

3. It regards the question of what kinds of connections exist in the world an open question.

4. It affirms the reality, in principle, of multiple cognitive routes from mind to world, exhibiting multiple patterns.

5. It is foundation*al* in the sense of requiring a grounding of our claims to knowledge in the world, but it is not foundation*alist*. Specifically, it does not require that all knowledge be grounded in the world in a strongly-ordered manner. The patterns that ground various theories in the world are open-ended, and it is up to human ingenuity to figure out new patterns.

6. It is not empiricist. It does not limit the cognitive routes from mind to world to ones based on sensory perception. This is a major difference between this holism and Quine's. Whereas Quine views "[t]he totality of so-called knowledge or beliefs... as a man-made fabric which impinges on the [world] only along the edges[, i.e., through] ... experience" (Quine 1951: 42), the present world-oriented holism allows non-purely-

[14] In being highly structured, this type of holism is immune to Friedman's (2001) objection that Quinean holism is *undiscerning*, connecting any unit of knowledge to any other, and to the same extent.

experiential routes as well.

7. It is not Platonist or apriorist either. There is just one world and the objects in it have properties[15] of many kind, including non-experiential properties, such as formal properties (see below). Knowledge of such properties need not be apriori, but may combine experiential and intellectual elements.
8. An apt metaphor for this holism is Neurath's boat,[16] viewed as a research boat on a mission to discover and explain the world. There is no Archimedean standpoint, no solid land to bring the boat to for repair and equipment. The boat is always at sea, and its sailors repair and equip it in the water, using whatever resources are available to them at the time, both natural and human-made. Progress is often made in flexible, back and forth, up and down, sideways moves, shifting our standpoint to expand perspective or maximize efficiency.

The enormous power to maneuver and the richness of options that holism brings into scientific research further add to humans' ability to draw connections conducive to knowledge, far beyond the vagaries of chance. This is reflected, among other things, in a non-traditional type of correspondence-truth at work in various fields of knowledge:

d. Truth as Non-Naive Correspondence. To the extent that science aims at knowledge of the world as it is, it aims at true knowledge of the world in the correspondence sense. But traditional correspondence theory is too naive, too simplistic, and too restrictive to be of much use for science, especially modern theoretical science with its inherent complexities. According to traditional correspondence, true sentences must copy, picture, or mirror their target in the world, or be isomorphic to it. But not only is this account unrealistic (after all, as J. L. Austin emphasized, human language is largely conventional)[17], it is also unreasonably limiting. Why should sentences provide us with true information about the world only by copying, picturing, or mirroring it, or by being isomorphic to it? To require that they do would unreasonably limit our

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[15] "Properties", here and below, abbreviates "properties or relations or functions (operators)".

[16] Which was also Quine's favorite metaphor.

[17] See Austin (1950).

ability to maneuver theoretically in the world, making the likelihood of true theories of the world considerably smaller.

Furthermore, not only is language largely conventional,[18] it is not clear that the same convention must always (in all fields and at all times) be connected to the world in the same way. Human language developed a long time ago, in response to multiple needs, and, at times, in a haphazard way. The conventional ways of connecting linguistic expressions to the world are related to old beliefs about the world, many of which no longer fit our current beliefs. Syntax, in particular, is overly rigid. Take mathematics – say, arithmetic. Arithmetic trades in numerals (primitive or not), and numerals are, syntactically, singular terms. Singular terms, according to longstanding syntactic-semantic conventions, denote individuals; hence, addition statements, "k+m=n", are true in the traditional correspondence sense *only if* there are numerical individuals in the world.

But does the use of addition in discourse about the world really require the existence of numerical individuals? Suppose there are no numerical individuals in the world, but objects in the world have cardinality properties. For example, the property *x-is-a-moon-of-Earth* has cardinality ONE.[19] Then to say that there is one moon of Earth we do not need to believe in the existence of a numerical individual, 1. It is sufficient to believe that the 1st-level property *x-is-a-moon-of-Earth* has the 2nd-level cardinality property ONE.

Suppose, however, that humans find it easier to figure out things about cardinalities (say, how to apply certain operations to cardinalities) when they think in terms of individuals and

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[18] In the Austinian sense, according to which "we are ... free to appoint *any* symbol to describe [anything]" (*ibid.:* 118-9), especially the first time we introduce an expression or a form of speech.

[19] (i) This is, of course, a Fregean approach, although Frege himself insisted on the reality of numerical individuals.

(ii) To emphasize the level of a given object, I sometimes use special fonts: *italics* for 1st-level properties, SMALL CAPS for 2nd-level properties, and CAPS for 3rd-level properties.

operators of level 1 than when they think in terms of 2nd-level properties and 3rd-level operators. It is easier for humans to deal with structures of individuals and their properties than with structures of properties of properties of individuals and their properties. Then, it makes good sense for humans to think of 3rd-level operations on 2nd-level cardinality properties in terms of 1st-level operations on numerical individuals. Employing epistemic freedom, humans change the traditional conventions of truth and reference to fit their needs. They use *indirect or complex forms of reference and correspondence*.

In the above example (in which we suppose that (i) there are no numerical individuals in the world but properties of individuals in the world have cardinality properties, and (ii) humans find it harder to figure out things about cardinalities when they think in terms of properties of properties than when they think in terms of individuals and their properties), humans may introduce an intermediate level of *posits* between language and the world. Language, which aside from a few things (e.g., new vocabulary), is difficult to change, remains as it is, and the world is, of course, as it is. But the connection between language and the world is reconceived. Reference and correspondence involve two steps rather than one: in step one, the 0-level individual constant "1" refers to the 0-level posited individual 1; in step two, the posit 1 is correlated with the 2nd-level cardinality property ONE. A similar two-step reference applies to "+": in step one, the 1st-level functional expression "+" refers to the 1st-level posited operator +; in step two, + is correlated with the 3rd-level operator DISJOINT-UNION. A similar two-step process applies to correspondence-truth: "1+2=3" is *true* if and only if *on the level of posits*, +(1,2)=3, if and only if *on the level of the world*, DISJOINT-UNION(ONE,TWO) = THREE.

Note that this way of thinking about mathematics is especially suited for discussions of its applicability to physics. On the one hand, physics does not recognize mathematical individuals (which are not material); on the other hand, it does recognize the reality of mathematical features of physical objects.[20] For example, it recognizes that there is EXACTLY ONE moon of Earth, there are TWO hydrogen atoms in a molecule of water, the number/cardinality of the inner planets in our solar system plus the number/cardinality of its outer planets (the DISJOINT UNION of these cardinalities) is larger than FIVE. And so on.[21] From now on, I will assume this conception of mathematics in this paper. (As we will see in the next section, this will also help to unify mathematics with respect to one of its features that are relevant to physics.)

Indirect routes of reference and correspondence are not unique to mathematics. For other types of indirect reference and correspondence, see, e.g., Terence Horgan (2001).[22]

To conclude: epistemic freedom, figuring out, world-oriented holism, and non-native correspondence-truth, all expand our epistemic reach. They equip us with tools for charting new routes from mind to world, making the most of whatever resources are available to us at a given place and time.

Our discussion of mathematics in this section paves the way for the second question we set out to reflect on in this paper:

II. What Special Features of Mathematics Equip it for its Role in Physics?

The question of what equips mathematics to play a central role in physics has its roots in a more elementary question:

[W]hat makes arithmetic so *useful* in daily life? Why can we use it to predict

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[20] "Object", here and below encompasses individuals, properties of individuals, properties of those, and so on.

[21] I think it is reasonable to surmise that in their theories, physicists do not address the question whether numbers are mathematical individuals or cardinalities (cardinality properties), but it is reasonable for a philosopher to surmise that they would accept the above examples, regardless of whether these examples treat numbers as individuals or cardinalities.

[22] For more on non-naive, indirect, or complex reference and correspondence both in general and in mathematics, see Sher (2016, Chapter 8).

> whether I will have carfare after I buy the newspaper? Can we say – in *non*mathematical terms – what the *world* must be like in order that valid arithmetic deductions should be effective in predicting observations? [Steiner 1998: 24]

A Kantian reformulation of this question, suggested by Carl Posy, is:

> [W]hat is the "objective validity" of such logical or mathematical concepts as disjoint union, cardinal number, etc.?" [*Ibid.,* fn.]

My answer to the above questions is: the objective validity of mathematical concepts and their role in scientific discovery are partly due to the *maximal invariance* of their denotations.[23] DISJOINT UNION, CARDINALITY properties, and other mathematical properties and operators are maximally-invariant, and so are, by extension, the concepts denoting them. This explains not only mathematics' wide and reliable applicability but also its role in scientific laws.

What is maximal invariance and how does it explain mathematics' fruitfulness in science? Let us begin with *invariance simpliciter*. Invariance is a relation between two things, X and Y, where X is an event, phenomenon, fact, property, principle, law, etc., and Y is, or involves, changes or variations of one kind or another. For example, X = law of nature and Y = change of a frame of reference. *Laws of nature*, according to Einstein, *are invariant under all changes of frames of reference*. This highlights an important feature of invariance: distinguishing between what is relevant and what is not relevant to a given thing or idea. What frame of reference an observer is in is irrelevant to a law of nature. What is essential to it is independent of the particular features/parameters of a given frame. Thinking of laws of nature in terms of invariance helps us to zero in on what is essential and unchanging in them.

Symmetries, whose importance for science is considerable, are also invariances:

> Symmetries are transformations (technically one-to-one functions which map onto their codomain) that leave all relevant structure intact − the result is always exactly like the original, in all *relevant* respects. ... [Such] transformations ... leave each

[23] The term "maximal invariance" is introduced in Sher (2021). Other commonly used terms are "invariance under isomorphisms" and "invariance under bijections" (see, e.g., Per Lindström 1966, Sher 1991 and 2016, and Dennis Bonny 2008).

> individual the same in all relevant respects. ... [They] leave... [the] relevant respect *invariant*. [van Fraassen 1989: 243-4]

There are many types of invariance. If X is invariant under Y, the type of invariance involved is determined by what type of thing X is and what Y is. In this paper I focus on *property invariance.* Here X is a *property* and Y is a *1-1 and onto replacement* ***r*** *of individuals by individuals*.[24] I.e., property invariance is invariance of a property P under a (some, few, many, all) 1-1 and onto replacement(s) of individuals by individuals.[25] If P is a 1-place 1st-level property, P is invariant under **r** if and only if for any $a \in$Dom(**r**), Pa if and only if P(**r**(a)). For example, the 1st-level property *is-a-planet* is invariant under all replacements of planets (planetary individuals) by planets and non-planets by non-planets, but it is not invariant under any replacement of a planet by a non-planet and vice versa. The 2nd-level property IS-A-BIOLOGICAL-PROPERTY[26] is invariant under all replacements of biological individuals by biological individuals and of non-biological individuals by non-biological individuals, but not under any replacement of a biological individual by a non-biological individual and vice versa. (Here the idea is that a 1-1 and onto replacement of biological individuals by biological individuals induces a replacement of 1st-level biological properties by 1st-level biological properties, and hence is not detected by the 2nd-level property IS-A-BIOLOGICAL-PROPERTY. And so on.) Since "property" includes relations and

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[24] The terms "property invariance" and "replacement **r**" are also new (Sher 2021), but the procedure they describe has been used for a long time, especially in logic and mathematics. In particular, it is used in the literature on "invariance under isomorphisms" (see references above) as well as "invariance under permutations" (Tarski 1966), "invariance under homomorphisms" (Feferman 1999), etc. The reason I use these terms is for general clarificatory purposes as well as to indicate that the relevant notion of invariance is intuitive rather than a term or art, and in particular, it is not committed to any specific mathematical theory, such as ZFC.

[25] In common terminology, **r** is a 1-1 and onto function (bijection) from a domain D of individuals to an equinumerous domain D' of individuals. We can also think of **r** as a 1-1 function on D, where D' is the (exact) range of **r**.

[26] Which holds of the properties *is-a-mammal* and *is-a-cell* but not of the properties *is-a-rock* or *is-a-number*.

operators (functions), property invariance (naturally adjusted to these types of properties) is very broad.

Thinking about properties in terms of invariance is fruitful in many ways. Among other things, the invariances of a given property (what 1-1 and onto replacements of individuals it is and it is not invariant under) are fruitful for zeroing in on what is essential in (and for) this property, what it takes into account or "notices" (so to speak) and what it does not. *Is-a-planet* does not notice if you replace Earth by Mars or Sun by Vega, but it does notice if you replace Earth by Vega. The differences between Earth and Mars are irrelevant for being a planet, but those between Earth and Vega are. Being a planet is not dependent on whether a given object sustains or does not sustain human life, has a richer atmosphere as that of Earth or a thinner atmosphere as that of Mars, is the size of Earth or the size of Mars, and so on. But it does depend on whether, like Earth, it orbits a star or, like Vega, it does not.

For invariance to distinguish what is and is not pertinent to a given property in all cases, it has to take into account not only actual individuals but also counterfactual individuals. For example, to show that having a kidney is not essential for having a heart we need to take into account counterfactual individuals, since the two properties are co-extensional, hence have the same invariances, among actual individuals. By extending Y to 1-1 and onto replacements of biologically-possible individuals, we obtain the right result. Similarly, to zero in on what matters for the property *is-subject-to-gravity*, we include counterfactual physical individuals (such as a second moon of Earth) in our invariance test. This is fruitful for showing that the scope of gravity includes all physically-possible individuals, not just actual physical individuals. Generally, the invariance test for properties encompasses actual-counterfactual individuals,[27] where the counterfactual individuals are of the relevant kind: biologically-possible individuals for biological

[27] An actual-counterfactual individual is an individual that is either actual or counterfactual. A domain of actual-counterfactual individuals may contain only actual individuals, only counterfactual individuals, or both actual and counterfactual individuals.

properties, physically-possible individuals for physical properties, and so on.[28]

We are now ready to discuss *maximal invariance,* or, as it is commonly called in the logical literature, "Invariance under Isomorphisms". All properties are invariant under at least one 1-1 and onto replacement of individuals, namely, the identity replacement. But some properties, as we have seen above – indeed, most properties – are invariant under other, non-trivial replacements as well. For example, *is-a-planet* is invariant under any replacement of planets by planets and of non-planets by non-planets, *is-a-human* is invariant under any replacement of humans by humans and of non-humans by non-humans, and so on. Most properties, however, are not invariant under *all* 1-1 and onto replacements. As we have seen above, *is-a-planet* is not invariant under a replacement of Earth by Vega, and it is easy to see that *is-a-human* is not invariant under a replacement of Tarski by Mt. Everest.

Let us say that a property is *maximally-invariant* if and only if it is invariant under all 1-1 and onto replacements of individuals simpliciter (i.e., without restriction to a specific category of individuals: "biological", "physical", etc.). Then the properties noted above are *not* maximally-invariant. And so are most of the properties we refer to in everyday life and the sciences.

Are there any maximally-invariant properties? – Yes. This discovery is commonly associated with the literature on logical constants by Mostowski (1957), Lindström (1966), Tarski

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[28] (i) Generally, this enables us to "extensionalize" intensionality or meaning. (For further discussion, see Sher 2021.)

(ii) What about, say, offsprings of a pair of mules? To the extent that there are no such individuals, they are not actual. Are they possible? They are certainly logically possible. Depending of how we conceive of physical possibility, they may also be physically possible. Whether we consider them biologically possible I leave for philosophers of biology to decide. If they are *not* biologically possible, they are not included among counterfactual biological individuals. Otherwise, they are.

(iii) If either the category "actual" or "counterfactual" does not apply to a certain kind of individuals, we do not consider individuals of this category in our invariance tests. (For example, it is reasonable to question whether there are both actual and counterfactual mathematical individuals.)

(1966), and their followers.[29] In particular, *all* the properties denoted by the *standard logical constants* are maximally-invariant. Take, for example, the 1st-level 2-place property *is-identical-to*. For any domain D of actual-counterfactual individuals simpliciter, any bijection **r** on D, and any $a,b\in$D: $a=b$ (in D) if and only if **r**(a)=**r**(b) (in D' – the range of **r**).[30] Or take the 2nd-level 1-place property of 1-place 1st-level properties, IS-NON-EMPTY.[31] For any D, **r**, and 1st-level property P on D,[32] P IS-NON-EMPTY (in D) if and only if **r***(P) – the image of P under **r** – IS-NON-EMPTY (in D'). Or take the 2nd-level 2-place property denoted by negation in contexts of the form "~Φx": IS-IN-THE-COMPLEMENT-OF (in D). For any D, **r**, $a\in$D, and P (on D): a IS-IN-THE-COMPLEMENT-OF P (in D) if and only if **r**(a) IS-IN-THE-COMPLEMENT-OF **r***(P) (in D'). And the same holds for the properties denoted by all the standard logical constants.

But the properties denoted by the standard logical constants are not the only maximally-invariant properties. All higher level mathematical properties (e.g., as definable in ZFC) are also maximally-invariant (see Lindström 1966, Tarski 1966, and others, such as Sher 1991). In contrast, most 1st-level mathematical properties are not-maximally invariant.[33] These results have many philosophically interesting consequences concerning logic and mathematics. Most of these are outside the scope of the present paper, but the following consequence is of interest here: If we view mathematical individuals and non-maximally-invariant 1st-level mathematical properties as representing higher-level mathematical properties (see subsection on non-naive correspondence

[29] It should be noted that this discovery is already found in earlier writings (in a somewhat different context).

[30] Depending on what property is in question, "in D/D'" is either significant or not. In the case of *is-identical-to*, it is not significant, but in the case of, e.g., COMPLEMENT (see below), it is.

[31] "(∃x)Px" says: "P IS-NON-EMPTY".

[32] By "a property P on D" I mean a property considered vis-a-vis D. For example, if D includes all and only Israelis, *is-a-human* on D is the property of being an Israeli human.

[33] Here are two examples: (i) *is-identical-to-2*, (ii) *is-even*. Let D={2,4}, D'={3,5}, and let **r** be a bijection from D to D'. Then neither (i) nor (ii) is invariant under **r**.

above), all mathematical objects – individuals and properties – are either directly or indirectly maximally-invariant. I.e., mathematics as a discipline is maximally-invariant.

Now, the maximal invariance of mathematics explains its applicability in science. It does not explain why particular mathematical equations are used in particular physical laws or why particular symmetries are expected to be satisfied by particular types of physical laws. But it does contribute to the explanation of why mathematics is applicable to physics, why it is appropriate to use mathematical equations in physical laws and mathematical symmetries as guides for such laws, why using mathematics in physics does not interfere with, but on the contrary, may very well enhance, the necessity of physical laws, why mathematical principles and symmetries that work in one part of physics are transferrable to other parts, why mathematics contributes both to the precision (sharpness) and to the unity of physics, and more.

Spelling out the contribution of maximal invariance to these explanations in full is beyond the scope of the present paper, but briefly, here are two (clusters of) relevant points:

(a) *Broad Applicability and Strong Modal Force.* Maximally-invariant properties do not distinguish between individuals: they apply to actual individuals (properties of such individuals) if and only if they apply to counterfactual individuals (properties of such individuals), they apply to physical individuals (properties of physical individuals) if and only if they apply to non-physical individuals (properties of non-physical individuals), and so on. Since mathematical properties are maximally-invariant, they have an extremely broad applicability. In particular they apply to physics, and if they apply to one part of physics they may apply to others as well.[34]

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[34] By "applicability" in this section I mean, roughly: (i) Properties in general can be had by objects of some kinds but not of other kinds. They are applicable to the former but not to the latter. For example: the property *is-even (divisible-by-2)* is not applicable to physical individuals, but the property *is-a-water-molecule* is. Some properties – in particular, maximally-invariant properties – are applicable to all objects (of the appropriate levels: individuals, 1st-level properties, etc.). This does not mean that every object has these properties, but that every object can be properly said to have or lack this property. This is the case with, e.g., identity and non-identity. (ii) Similarly, statements, principles, and laws in general are applicable to objects of some kinds but not others. For example, physical laws are applicable to physical objects but not

It follows that the laws (principles) governing/describing mathematical (maximally-invariant) properties have an especially strong modal force: if a given property does not distinguish between any actual-counterfactual individuals, the principles (laws) governing/describing it cannot distinguish between them either. Otherwise, they would not be laws of this property.[35] Mathematical principles, therefore, have an extremely large actual-counterfactual scope, hence an extremely strong modal force. Accordingly, the inclusion of mathematical principles in physical laws does not reduce, and may even increase, the modal force of these laws.[36]

We can, in fact, divide principles governing/describing maximally-invariant properties to two kinds, conditional and unconditional. The logical-mathematical principle "Every individual is identical to itself" is unconditional; the mathematical principle "Every binary operation is

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to mathematical objects. But laws can also be applicable to objects of all kinds, as do the logical-mathematical laws of identity and cardinality.

[35] Note: laws can be thought of in a variety of ways. Here I focus on laws that can be thought of as governing or describing properties. For example, the laws of identity can be thought of as governing/describing the property (relation) of identity, and Newton's law of universal gravity can be thought of as governing/describing the property of gravity (the relation of gravity between two physical objects).

[36] What I mean by saying that the inclusion of mathematical principles in physical laws may increase the modal force of these laws can be explained by the following example-schema: Suppose we compare two formulations of a given physical law or principle, L, L_1 and L_2. L_1 is a qualitative formulation; L_2 is a quantitative formulation. Then, due to the greater sharpness of the quantitative formulation (see (b) below), it is not unlikely that L_1 will miss a certain range of actual-counterfactual cases to which L applies while L_2 will not. In this case, L_1 will have a weaker modal force than L_2.

To take a concrete example, Newton's law of gravity,

$$F = G\frac{m_1 m_2}{r^2},$$

employs the mathematical operations (properties) of multiplication, division, and exponentiation, relying on the laws/principles governing them. This does not weaken its modal force. Indeed, employing mathematical machinery may enable Newton's law to have a stronger modal force than a merely "qualitative" counterpart (which is limited to non-mathematical machinery).

associative if it is a group operation" is conditional. Both have extremely strong modal force, and both are applicable to physics.

(b) *Pure Patterns: Precision, Sharpness, Unity.* To present this point it will be useful to note that the concept of *maximal invariance* is equivalent to that of *invariance under all isomorphisms:* for any property P, P is maximally-invariant if and only if P is invariant under all isomorphisms (of its argument-structures). To see what invariance under all isomorphisms means, consider any property P and any (non-empty) domain D of individuals. Let β be any argument of P in D.[37] Then <D,β> is an argument-structure for P. Let **r** (a replacement function) be any 1-1 function from D onto any equinumerous domain D'. Let β' be the image of β under **r**. Then the structures <D,β> and <D',β'> are isomorphic. To say that P is invariant under isomorphisms is to say that for any isomorphic structures <D,β> and <D',β'> as above, P holds of β in D if and only if P holds of β' in D'.

Invariance under isomorphisms is distinctive of *mathematics.* Mathematical theories (equations, principles) do not distinguish between any isomorphic structures[38]. Mathematical structures are closed under isomorphisms. This means that mathematics distinguishes only *pure patterns of objects*. What mathematics notices is only *patterns* of individuals having properties and standing in relations, not the individuals themselves.[39] This is important for understanding the

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[37] If P is a 1-place 1st-level property, β is an individual in D; if P is an n-place, n>1, 1st-level property, β is an n-tuple of individuals in D; if P is a 1-place 2nd-level property of 1-place 1st-level properties, β is a subset of D, i.e., the extension of 1-place 1st-level property P'; etc.

[38] Or something analogous to isomorphic structures, such as congruent polygons in geometry.

[39] This is a structuralist outlook on mathematical individuals. In the literature, it is often expressed as the "places in structures" view of mathematical individuals (see, e.g., Shapiro 1997). Another way to think of it is in terms of the conception described above of mathematical individuals as posits representing "formal-structural" properties in the sense of maximal invariance / invariance under isomorphisms. For further discussion of the relation between invariance under isomorphisms and structurality, see Sher (2016) and references there.

applicability of mathematics to physics. Pure patterns do not distinguish between types of individuals (actual and non-actual, physical and non-physical, photons and stars), therefore they can, in principle, appear in physics. I.e., physical objects can, in principle, exhibit mathematical patterns. And, in fact, they do. For example, physical objects stand in such relations as *has-a-greater-mass-than* to others, and these relations have mathematical properties, hence the objects standing in these relations exhibit mathematical patterns. *Has-a-greater-mass-than* is an ordering relation; therefore, mathematical principles governing ordering relations apply to this and other physical ordering relations. This, in turn, explains why looking for mathematical patterns can, in principle, contribute to physical discovery (correct guessing in physics). The point is that these patterns are more likely to be applicable than non-mathematical patterns. That is why guesses that are informed by mathematics are more likely to be successful than those that are not. There is no limit to the number, variety, and intricacy of pure mathematical patterns. Therefore, mathematics provides especially rich and useful tools for physics.

Furthermore, the appeal to mathematical patterns add sharpness, precision, and unity to physics. Since mathematical patterns are closed under all isomorphisms, there are myriad diverse structures – structures with different kinds of individuals – in which the same pattern repeats again and again.[40] This enables us to identify sharply and precisely what the pattern is: what belongs to the pattern and what does not. Moreover, the reappearance of the same patterns, and the same types of patterns, in different parts of physics increases its unity.

Mathematical operators and principles are used everywhere in physics and on all levels: from multiplication, division, and exponentiation in Newton's law of gravity, to an array of mathematical operators in Einstein's field equation. Symmetries, too, are used throughout physics, taking advantage of the above features of mathematical principles. David Gross's account of symmetry brings all this to clear view:

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[40] For example, the group pattern (and various specific variants of this pattern, such as Lie group), appears again and again in various branches of physics.

> Symmetry principles ... summarize the regularities of the laws that are independent of the specific dynamics. Thus invariance principles provide a structure and coherence to the laws of nature just as the laws of nature provide a structure and coherence to the set of events. Indeed, it is hard to imagine that much progress could have been made in deducing the laws of nature without the existence of certain symmetries. The ability to repeat experiments at different places and at different times is based on the invariance of the laws of nature under space-time translations. Without regularities embodied in the laws of physics we would be unable to make sense of physical events; without regularities in the laws of nature we would be unable to discover the laws themselves. Today we realize that symmetry principles are even more powerful—they dictate the form of the laws of nature. [Gross 1996: 14256]

Looking back at the mind-world correspondence, we see that mathematics is a major locus of this correspondence. Mathematical, maximally-invariant, patterns appear in nature, and since we, humans, have cognitive access both to some natural phenomena and to maximal invariance and some of the mathematical patterns it gives rise to, we are able to identify, and look for, mathematical patterns in natural phenomena.

Peirce and Steiner claimed that humans' ability to make correct guesses about the world goes beyond chance. In this paper I have discussed the conditions that make correct guessing possible for humans as well as those that make mathematics an especially suitable catalyst of correct guessing. Correct guessing requires a cognitive setup in which the mind can make fruitful hypotheses about the world. My discussion of the possibility of such a setup begins with Kant, proceeds to Quine, and continues with my own proposals. Kant provides a systematic account of the mind-world connection, but his account is too rigid to explain the flexibility, initiative, and freedom-to-maneuver that are central to most acts of fruitful guessing. Quine frees us from this rigidity but is unable to account for intellect's role in such guessing. His empiricist pragmatism confines intellect to a largely pragmatic, service role (reflected in his indispensabilist account of mathematical knowledge). My own account makes further steps forward in explaining the flexibility, directedness, and active initiative available to us for making fruitful guesses on all levels. I further connect mathematics' guiding role in scientific guessing to its extremely high degree of invariance. This explains the universal applicability of mathematical patterns to the world, including those facets of the world that are the target of scientific guessing.

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