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INTRANSITIVITY OF PREFERENCES

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It is shown that, under specified experimental conditions, consistent and predictable intransitivities can be demonstrated. The conditions under which intransitivities occur and their relationships to the structure of the alternatives and to processing strategies are investigated within the framework of a general theory of choice. Implications to the study of preference and the psychology of choice are discussed.

Whenever we choose which car to buy, which job to take, or which bet to play we exhibit preference among alternatives. The alternatives are usually multidimensional in that they vary along several attributes or dimensions that are relevant to the choice. In searching for the laws that govern such preferences, several decision principles have been proposed and investigated. The simplest and probably the most basic principle of choice is the transitivity condition.

A preference-or-indifference relation, denoted $\gtrsim$, is transitive if for all x, y, and z

$$x \gtrsim y \quad \text{and} \quad y \gtrsim z \quad \text{imply} \quad x \gtrsim z. \qquad [1]$$

Transitivity is of central importance to both psychology and economics. It is the cornerstone of normative and descriptive decision theories (Edwards, 1954, 1961; Luce & Suppes, 1965; Samuelson, 1953), and it underlies measurement models of sensation and value (Luce & Galanter, 1963; Suppes & Zinnes, 1963). The essential role of the transitivity assumption in measurement theories stems from the fact that it is a necessary condition for the existence of an ordinal (utility) scale, u, such that for all x and y,

$$u(x) \geq u(y) \quad \text{if and only if} \quad x \gtrsim y. \qquad [2]$$

Transitivity is also a sufficient condition for the existence of such a scale, provided the number of alternatives is finite, or countable.

Individuals, however, are not perfectly consistent in their choices. When faced with repeated choices between x and y, people often choose x in some instances and y in others. Furthermore, such inconsistencies are observed even in the absence of systematic changes in the decision maker's taste which might be due to learning or sequential effects. It seems, therefore, that the observed inconsistencies reflect inherent variability or momentary fluctuation in the evaluative process. This consideration suggests that preference should be defined in a probabilistic fashion. To do so, let $P(x,y)$ be the probability of choosing x in a choice between x and y, where $P(x,y) + P(y,x) = 1$. Preference can now be defined by

$$x \gtrsim y \quad \text{if and only if} \quad P(x,y) \geq \tfrac{1}{2}. \qquad [3]$$

The inconsistency of the choices is thus incorporated into the preference relation as x is said to be preferred to y only when it is chosen over y more than half the time. Restating the transitivity axiom in terms of this definition yields

$$P(x,y) \geq \tfrac{1}{2} \quad \text{and} \quad P(y,z) \geq \tfrac{1}{2} \quad \text{imply } P(x,z) \geq \tfrac{1}{2}. \qquad [4]$$

This condition, called weak stochastic transitivity, or WST, is the most general probabilistic version of transitivity. Violations of this property cannot be attributable to inconsistency alone.

Despite the almost universal acceptance of the transitivity axiom, in either algebraic

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or probabilistic form, one can think of several choice situations where it may be violated. Consider, for example, a situation in which three alternatives, x, y, and z, vary along two dimensions, I and II, and where their values on these dimensions are given by the following payoff matrix.

		Dimensions	
		I	II
	x	2ϵ	6ϵ
Alternatives	y	3ϵ	4ϵ
	z	4ϵ	2ϵ

The alternatives may be job applicants varying in intelligence (I) and experience (II), where the entries are the candidates' scores on the corresponding scales or dimensions. Suppose the subject (S) uses the following decision rule in choosing between each pair of alternatives: if the difference between the alternatives on Dimension I is (strictly) greater than ϵ, choose the alternative that has the higher value on Dimension I. If the difference between the alternatives on Dimension I is less than or equal to ϵ, choose the alternative that has the higher value on Dimension II. It is easy to see that this seemingly reasonable decision rule yields intransitive preferences when applied to the above matrix. Since the differences between x and y and between y and z on the first dimension are not greater than ϵ, the choice is made on the basis of the second dimension and hence x is chosen over y and y is chosen over z. But since the difference between x and z on the first dimension is greater than ϵ, z is chosen over x yielding an intransitive chain of preferences.

Formally, such a structure may be characterized as a lexicographic semiorder, abbreviated LS, where a semiorder (Luce, 1956) or a just noticeable difference structure is imposed on a lexicographic ordering. As an illustration, let us restate this rule in terms of the selection of applicants. An employer, regarding intelligence as far more important than experience, may choose the brighter of any pair of candidates. Cognizant that intelligence scores are not perfectly reliable, the employer may decide to regard one candidate as brighter than another one only if the difference between their IQ scores exceeds 3 points, for example. If the difference between the applicants is less than 3 points, the employer considers the applicants equally bright and chooses the more experienced candidate. Essentially the same example was discussed by Davidson, McKinsey, and Suppes (1955). Such a decision rule is particularly appealing whenever the relevant dimension is noisy as a consequence of imperfect discrimination or unreliability of available information. Where this decision rule is actually employed by indiduals, WST must be rejected.

Other theoretical considerations proposed by Savage (1951), May (1954), Quandt (1956), and Morrison (1962) suggest that WST may be violated under certain conditions. No conclusive violations of WST, however, have been demonstrated in studies of preferences although Morrison (1962) provided some evidence for predictable intransitivities in judgments of relative numerosity, and Shepard (1964) produced a striking circularity in judgments of relative pitch. Several preference experiments have tested WST, for example, Edwards (1953), May (1954), Papandreou, Sauerlander, Bownlee, Hurwicz, and Franklin (1957), Davis (1958), Davidson and Marschak (1959), Chipman (1960), and Griswold and Luce (1962). All these studies failed to detect any significant violation of WST.

The present paper attempts to explore the conditions under which transitivity holds or fails to hold. First, the LS described above is utilized to construct alternatives which yield stochastically intransitive data. The conditions under which WST is violated are studied within the framework of a general additive difference choice model and their implications for the psychology of choice are discussed.

EXPERIMENTS

General Considerations

The purpose of the following studies was to create experimental situations in which individuals would reveal consistent patterns of intransitive choices. The experiments are not addressed to the question of whether human preferences are, in general, transitive; but rather to the question of whether reliable intransitivities can be produced, and under what conditions. The construction of the alternatives was based on the LS described in the introduction. The application of the LS to a specific experimental situation, however, raises serious identification problems.

In the first place, the LS may be satisfied by some, but not all, individuals. One must identify, therefore, the *S*s that satisfy the model. This, however, is not an easy task since even if the LS is satisfied by all people, they may differ in the manner in which the alternatives are perceived or processed. Different individuals can characterize the same alternatives in terms of different sets of attributes. For example, one employer may evaluate job applicants in terms of their intelligence and experience whereas another employer may evaluate them in terms of their competence and sociability. Similarly, one *S* may conceptualize (two-outcome) gambles in terms of odds and stakes, while another may view them in terms of their expectation, variance, and skewness. Since the predictions of the model are based on the dimensional structure of the alternatives, this structure has to be specified separately for each *S*. In order to induce *S*s to use the same dimensional framework, alternatives that are defined and displayed in terms of a given dimensional representation have been employed.

Then, even if all individuals satisfy the LS relative to the same dimensions, they may still vary in their preference threshold as well as in the relative importance that they attribute to the dimensions. A difference between an IQ of 123 and an IQ of 127, for instance, may appear significant to some people and negligible to others.

These considerations suggest treating each *S* as a separate experiment and constructing the alternatives according to the dimensions and the spacing he uses. Alternatively, one may select, for a critical test, those *S*s who satisfy a specified criterion relative to a given representation. (It should be noted that the preselection of *S*s or alternatives, on the basis of an independent criterion, is irrelevant to the question of whether WST is consistently violated for any given *S*.) Both methods are employed in the following studies. The first study investigates choice between gambles while the second one is concerned with the selection of college applicants.

Experiment I

The present study investigates preferences between simple gambles. All gambles were of the form (x,p,o) where one receives a payoff of $\$x$ if a chance event p occurs, and nothing if p does not occur. The chance events were generated by spinning a spinner on a disc divided into a black and a white sector. The probability of winning corresponded to the relative size of the black sector. The gambles employed in the study are described in Table 1.

Each gamble was displayed on a card showing the payoff and a disc with the corresponding black and white sectors. An illustration of a gamble card is given in Figure 1. Note that, unlike the outcomes, the probabilities were not displayed in a numerical form. Consequently, no exact calculation of expected values was possible. The gambles were constructed so that the expected value increased with probability and decreased with payoff.

Since the present design renders the evaluation of payoff differences easier than that of probability differences, it was hypothesized that at least some *S*s would ignore small probability differences, and choose between adjacent gambles on the

TABLE 1

THE GAMBLES EMPLOYED IN EXPERIMENT I

Gamble	Probability of winning	Payoff (in \$)	Expected value (in \$)
a	7/24	5.00	1.46
b	8/24	4.75	1.58
c	9/24	4.50	1.69
d	10/24	4.25	1.77
e	11/24	4.00	1.83

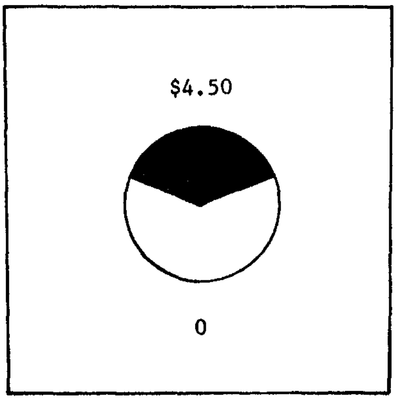

FIG. 1. An illustration of a gamble card.

basis of the payoffs. (Gambles are called adjacent if they are a step apart along the probability or the value scale.) Since expected value, however, is negatively correlated with payoff, it was further hypothesized that for gambles lying far apart in the chain, *S*s would choose according to expected value, or the probability of winning. Such a pattern of preference must violate transitivity somewhere along the chain (from *a* to *e*).

In order to identify *S*s who might exhibit this preference pattern, 18 Harvard undergraduates were invited to a preliminary session. The *S*s were run individually. On each trial the experimenter presented *S* with a pair of gamble cards and asked him which of the gambles he would rather play. No indifference judgment was allowed. The *S*s were told that a single trial would be selected at the end of the session and that they would be able to play the gamble they had chosen on that trial. They were also told that the outcome of this gamble would be their only payoff.

To minimize the memory of earlier choices in order to allow independent replications within one session a set of five "irrelevant" gambles was constructed. These gambles were of the same general form (x,p,o) but they differed from the critical gambles in probabilities and payoffs.

In the preliminary session, all *S*s were presented with all pairs of adjacent gambles (a,b; b,c; c,d; d,e) as well as with the single pair of extreme gambles (a,e). In addition, all 10 pair comparisons of the "irrelevant" gambles were presented. Each of the 15 pairs was replicated 3 times. The order of presentation was randomized within each of the three blocks.

The following criterion was used to identify the potentially intransitive *S*s. On the majority of the adjacent pairs (i.e., three out of the four) *S* had to prefer the alternative with the higher payoff, while on the extreme pair, he had to prefer the one with the higher expected value (i.e., choose *e* over *a*). A gamble was said to be preferred over another one if it was chosen on at least two out of the three replications of that pair. Eight out of the 18 *S*s satisfied the above criterion and were invited to participate in the main experiment.

The experiment consisted of five test sessions, one session every week. In each session, all 10 pair comparisons of the test gambles along with all 10 pair comparisons of the "irrelevant" gambles were presented. Each of the 20 pairs was replicated four times in each session. The position of the gambles (right-left) and the order of the pairs were randomized within each block of 20 pairs. The *S*s were run individually under the same procedure as in the preliminary session. Each of the test sessions lasted approximately $\frac{3}{4}$ of an hour. The choice probabilities of all eight *S*s between the five gambles are shown in Table 2. Violations of WST are marked by superscript x and violations of the LS are marked by superscript y.

The data indicate that although two *S*s (7 and 8) seemed to satisfy WST, it was violated by the rest of the *S*s. Furthermore, all violations were in the expected direction, and almost all of them were in the predicted locations. That is, people chose between adjacent gambles according to the payoff and between the more extreme gambles according to probability, or expected value. This result is extremely unlikely under the hypothesis that the intransitivities are due to random choices. Had this been the case, one should have expected the violations to be uniformly distributed with an equal number of violations in each of the two directions.

TABLE 2

PROPORTION OF TIMES THAT THE ROW GAMBLE WAS CHOSEN OVER THE COLUMN GAMBLE BY EACH OF THE EIGHT SUBJECTS

Subject	Gamble	Gamble a	b	c	d	e
1	a	—	.75	.70	.45[x]	.15[x]
	b		—	.85	.65	.40[x]
	c			—	.80	.60
	d				—	.85
	e					—
2	a	—	.40[y]	.65	.50	.25[x]
	b		—	.70	.40[xy]	.35[x]
	c			—	.75	.55
	d				—	.75
	e					—
3	a	—	.75	.70	.60	.25[x]
	b		—	.80	.65	.40[x]
	c			—	.95	.80
	d				—	1.00
	e					—
4	a	—	.50	.45	.20	.05
	b		—	.65[x]	.35	.10
	c			—	.70[x]	.40
	d				—	.85[x]
	e					—
5	a	—	.75	.65	.35[x]	.60[y]
	b		—	.80	.55	.30[x]
	c			—	.65	.65
	d				—	.70
	e					—
6	a	—	1.00	.90	.65	.20[x]
	b		—	.80	.75	.55
	c			—	.90	.65
	d				—	.75
	e					—
7	a	—	.45[y]	.65	.60	.60[y]
	b		—	.60	.40[xy]	.65
	c			—	.50	.75
	d				—	.70
	e					—
8	a	—	.60	.70	.75	.85[y]
	b		—	.65	.75	.85
	c			—	.60	.80
	d				—	.40[y]
	e					—

[x] Violations of WST.
[y] Violations of the LS.

To further test the statistical significance of the results, likelihood ratio tests of both WST and the LS were conducted for each S. This test compares a restrictive model (or hypothesis) denoted M_1 (such as WST or the LS) where the parameter space is constrained, with a nonrestrictive model, denoted M_0, which is based on an unconstrained parameter space. The test statistic is the ratio of the maximum value of the likelihood function of the sample under the restrictive model, denoted $L^*(M_1)$, to the maximum value of the likelihood function of the sample under the nonrestrictive model, denoted $L^*(M_0)$. For a large sample size, the quantity

$$Q(M_1,M_0) = -2\, ln \frac{L^*(M_1)}{L^*(M_0)}$$

has a chi-square distribution with a number of degrees of freedom that equals the number of constrained parameters. Using this distribution, one can test the null hypothesis that the data were generated by the restrictive model. For further details, see Mood (1950).

In the present study, $L^*(M_0)$ is simply the product of the binomial probabilities, while $L^*(M_1)$ is obtained from it by substituting a value of one-half in the above product for those choice probabilities that were incompatible with the particular restrictive model. The tested version of the LS was that in the (four) pairs of adjacent gambles, preferences are according to payoff while in the most extreme pair of gambles, preferences are according to expected value. The obtained chi-square values with the associated degrees of freedom and significance levels are displayed in Table 3.

The table shows that WST is rejected at the .05 level for five Ss, while the LS is rejected for one S only. It is important to note that the test for rejecting WST is very conservative in that it depends only on the magnitude of the violations and ignores their (predicted) location and direction.

The last column of Table 3 reports the Q values corresponding to the ratio of the maximum likelihoods of WST and the LS. Since both models are of the restrictive type and the two chi-squares are not in-

TABLE 3

LIKELIHOOD RATIO TEST FOR ALL SUBJECTS UNDER WST AND THE LS

Subject	Q(WST,M₀)	df	p <	Q(LS,M₀)	df	p <	Q(WST,LS)
1	11.82	3	.01	.00	0	—	11.82
2	7.84	3	.05	.00	0	—	7.84
3	6.02	2	.05	.00	0	—	6.02
4	15.94	3	.01	.00	0	—	15.94
5	5.18	2	.10	.40	1	.75	4.78
6	7.36	1	.01	.00	0	—	7.36
7	.40	1	.75	1.80	3	.50	−1.20
8	.00	0	—	11.62	2	.01	−11.62

Note.— $Q(M_1,M_0) = -2\ ln \frac{L^*(M_1)}{L^*(M_0)}$.

dependent the distribution of this statistic is not known. Nevertheless, its values are substantially positive for six out of the eight *S*s, suggesting that the LS accounts for the data better than WST.

In a postexperimental interview, S_4 described his behavior as follows: "There is a small difference between Gambles *a* and *b* or *b* and *c* etc., so I would pick the one with the higher payoff. However, there is a big difference between Gambles *a* and *e* or *b* and *e* etc., so I would pick the one with the higher probability." This is, in fact, a good description of his actual choices. When asked whether this type of behavior might lead to intransitivities, he replied, "I do not think so, but I am not sure." The *S*s did not remember for sure whether any of the pairs were replicated during the experiment, although they were sure that most gambles appeared in more than one pair in any one of the sessions. When the transitivity assumption was explained to the *S*s, they reacted by saying that although they did not pay special attention to it, they were almost certain that their preferences were transitive.

The degree of intransitivity obtained in an experiment depends critically on the spacing of the alternatives and the selection of the display. To study the effects of changes in the payoff or the probability structure, three sets of gambles portrayed in Table 4 were compared.

Note that Set I is the one used in the main experiment. Set II was obtained from it by increasing the probability differences between adjacent gambles, and Set III by decreasing the payoff differences between them. All sets were constructed so that the expected value increased with the probability of winning and decreased with the payoff.

To compare the three sets, 36 Harvard undergraduates (who did not participate in the earlier sessions) were invited for a single session. Each *S* was presented with five pairs of gambles from each one of the three sets. These included the four pairs of adjacent gambles and the single pair of extreme gambles from each set. Each of the 15 pairs was replicated three times, in a randomized presentation order. The *S*s were run individually under the procedure employed in the earlier sessions. Further-

TABLE 4

GAMBLE SETS I, II, AND III

Set	Probability	Payoff	Expected value
I	7/24	5.00	1.46
	8/24	4.75	1.58
	9/24	4.50	1.69
	10/24	4.25	1.77
	11/24	4.00	1.83
II	8/24	5.00	1.67
	10/24	4.75	1.98
	12/24	4.50	2.25
	14/24	4.25	2.48
	16/24	4.00	2.67
III	7/24	3.70	1.08
	8/24	3.60	1.20
	9/24	3.50	1.31
	10/24	3.40	1.42
	11/24	3.30	1.51

more, the same criterion for circularity was investigated. That is, *S* had to choose between most (three out of four) adjacent gambles according to payoff and between the extreme gamble according to probability. The results showed that, out of the 36 *S*s, 13 satisfied the criterion in Set I, 6 satisfied the criterion in Set II and 8 satisfied the criterion in Set III. These findings indicate that the probability and the payoff differences used in Set I yield more intransitivities than those used in Sets II and III.

Experiment II

The second experimental task is the selection of college applicants. Thirty-six undergraduates were presented with pairs of hypothetical applicants and were asked to choose the one that they would rather accept. Each applicant was described by a profile portraying his percentile ranks on three evaluative dimensions, labeled I, E, and S. The *S*s were told that Dimension I reflects intellectual ability, Dimension E reflects emotional stability, and Dimension S reflects social facility. An illustrative profile is shown in Figure 2.

The *S*s were further told that the profiles were constructed by a selection committee on the basis of high school grades, intelligence and personality tests, letters of recommendation, and a personal interview. Using this information, all applicants were ranked with respect to the three dimensions and the three corresponding percentile ranks constitute the applicant's profile. The *S*s were then told that

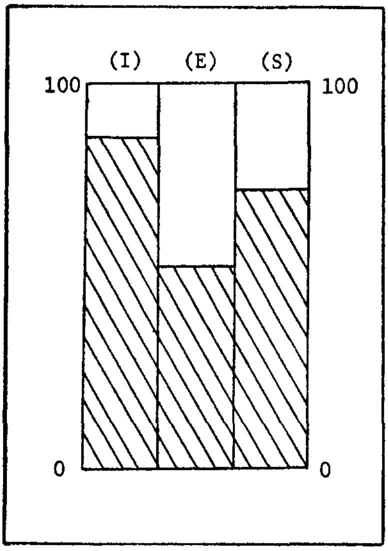

FIG. 2. An illustrative applicant's profile.

TABLE 5

THE 10 PROFILES USED IN THE PRELIMINARY SESSION OF EXPERIMENT II

Applicant	Dimensions		
	I	E	S
a	63	96	95
b	66	90	85
c	69	84	75
d	72	78	65
e	75	72	55
f	78	66	45
g	81	60	35
h	84	54	25
i	87	48	15
j	90	42	5

Note.—I = intellectual ability, E = emotional stability, S = social facility.

> The college selection committee is interested in learning student opinion concerning the type of applicants that should be admitted to the school. Therefore, you are asked to select which you would admit from each of several pairs of applicants. Naturally, intellectual ability would be the most important factor in your decision, but the other factors are of some value, too. Also, you should bear in mind that the scores are based on the committee's ranking and so they may not be perfectly reliable.

The study consisted of two parts: a preliminary session and a test session. The profiles used in the preliminary session are given in Table 5.

The profiles were constructed such that there was a perfect negative correlation between the scores on Dimension I and the scores on Dimensions E and S. The (absolute) difference between a pair of profiles on Dimension I is referred to as their I difference. A choice between profiles is said to be compatible with (or according to) Dimension I whenever the profile with the higher score on that dimension is selected, and it is said to be incompatible with Dimension I whenever the profile with the lower score on that dimension is selected.

Since Dimension I is the most important to the present task, and since the graphical display hinders the evaluation of small I difference it was hypothesized that the LS would be satisfied by some of the *S*s. For small I differences these *S*s would choose according to Dimensions E and S, but for large I differences they would choose according to Dimension I. The purpose of the preliminary session was to identify *S*s who behaved in that fashion and to collect preference data that could be employed in constructing new sets of profiles to be used in the test session.

The *S*s were run individually. On each trial the experimenter presented *S* with a pair of profiles and asked him to make a choice. Indifference judgment was not allowed. The *S*s were presented with all 45 pair comparisons of the 10 profiles in the same randomized order.

The criterion for participation in the test session was that at least six out of nine choices between the adjacent profiles (a,b; b,c; c,d; d,e; e,f; f,g; g,h; h,i; i,j) were according to Dimensions E and S and at least seven out of the 10 choices between the extreme profiles (a,j; a,i; a,h; a,g; b,j; b,i; b,h; c,j; c,i; d,j) were according to Dimension I. Fifteen out of the 36 *S*s satisfied this criterion and were invited to the test session.[2]

Using the data obtained in the preliminary session, the following procedure was employed to construct a special set of five profiles for each *S*. Let $n(\delta)$ denote the number of choices (made by a given *S* in the preliminary session) between profiles whose I difference was at most δ and that were incompatible with Dimension I. Similarly, let $m(\delta)$ denote the number of choices between profiles whose I difference was at least δ and that were compatible with Dimension I. Note that $\delta = 3, 6, 9, \cdots, 27$ and that, by the selection criterion employed, $n(3) \geq 6$ and $m(18) \geq 7$ for all the selected *S*s. The values of $n(\delta)$ and $m(\delta)$ were computed for each *S* and the value of δ' for which $n(\delta) + m(\delta)$ is maximized was obtained.

To illustrate the procedure, the choices made by S_8 in the preliminary session, along with the values of $n(\delta)$, $m(\delta)$, and $n(\delta) + m(\delta)$ are shown in Table 6. A value of 1 in an entry indicates that the profile in that row was selected over the profile in that column. A value of 0 indicates the opposite.

Note that the diagonals of Table 6 represent pairs of profiles that have equal I differences, and that these differences increase with the distance from the main (lowest) diagonal. Thus, pairs of adjacent profiles are on the lowest diagonal while pairs of extreme profiles are on the higher diagonals. Inspection of Table 6 reveals

[2] In some pilot studies in which *S*s were run in a group and only two dimensional profiles were used the proportion of *S*s satisfying the above criterion was considerably lower.

TABLE 6

CHOICES MADE BY S_8 IN THE PRELIMINARY SESSION AND THE RESULTING VALUES OF $n(\delta)$, $m(\delta)$, AND $n(\delta) + m(\delta)$

Profile	a	b	c	d	e	f	g	h	i	j	δ	$n(\delta)$	$m(\delta)$	$n(\delta) + m(\delta)$
a		1	1	0	0	0	1	0	0	0	27	23	1	24
b			1	1	1	0	0	0	1	0	24	23	3	26
c				1	0	1	1	0	0	0	21	23	5	28
d					1	1	0	1	1	0	18	22	8	30
e						0	1	1	0	1	15	21	11	32
f							1	1	1	0	12	19	15	34
g								1	0	0	9	17	18	35
h									0	1	6	13	20	33
i										1	3	7	22	29
j														

Note.—The preference of a row profile over a column profile is denoted by a 1, and the reverse preference is denoted by 0.

that most choices on the three lower diagonals were incompatible with Dimension I, while most choices on the six upper diagonals were compatible with Dimension I. The value of δ which maximizes $n(\delta) + m(\delta)$ is 9, which is taken as an estimate of the preference threshold ϵ. (It should be noted that ϵ was originally defined as a subjective rather than objective difference. Consequently, it need not be independent of the location of the scores, and different estimates of δ' may be obtained for different parts of the scale. In the present study, however, only a single estimate of ϵ was obtained for each *S*.)

On the basis of these estimates, *S*s were divided into four groups, and a special set of profiles was constructed for each group. The new sets were constructed so that the intermediate I differences equaled the estimated threshold, ϵ, for each *S*. More specifically, the I differences in the four pairs of adjacent profiles $(a,b;\ b,c;\ c,d;\ d,e)$ were smaller than ϵ, the I differences in the three pairs of extreme profiles $(a,e;\ a,d;\ b,e)$ were larger than ϵ, and the I differences in the three pairs of intermediate profiles $(a,c;\ b,d;\ c,e)$ equaled ϵ. The four sets of profiles, constructed for the test session, are shown in Table 7. Note that in each of the sets of Table 7 there is a perfect negative correlation between Dimension I and Dimensions E and S, and that the profiles are equally spaced. The four sets differ from each other in the location and the spacing of the profiles. The differences between adjacent profiles on Dimensions I, E, and S respectively are 3, 6, and 10 in Set I; 6, 10, and 15 in Set II; 9, 12, and 20 in Set III; 12, 16, and 23 in Set IV. Under the hypothesized model this construction was designed to yield preference patterns where choices between the four adjacent profiles are incompatible with Dimension I, whereas choices between the three extreme profiles are compatible with Dimension I.

The test session took place approximately 2 weeks after the preliminary session. The *S*s were reminded of the instructions and the nature of the task. They were run individually, and each one was presented with all 10 pair comparisons of the five new

TABLE 7

FOUR SETS OF PROFILES CONSTRUCTED FOR EXPERIMENT II

Set	Profiles	Dimensions		
		I	E	S
I	a	69	84	75
	b	72	78	65
	c	75	72	55
	d	78	66	45
	e	81	60	35
II	a	66	90	85
	b	72	80	70
	c	78	70	55
	d	84	60	40
	e	90	50	25
III	a	54	90	95
	b	63	78	75
	c	72	66	55
	d	81	54	35
	e	90	42	15
IV	a	42	96	96
	b	54	80	73
	c	66	64	50
	d	78	48	27
	e	90	32	4

Note.—I = intellectual ability, E = emotional stability, S = social facility.

profiles along with all 10 pair comparisons of five "irrelevant" profiles introduced to minimize recall of the earlier decisions. Each of the 20 pairs was replicated three times during the session. The order of presentation was identical for all *S*s and it was randomized within each block of 20 pairs. The choice frequencies of all 10 critical pairs of profiles are shown in Table 8 for each *S*. Since only three replications of each pair comparison were obtained, the likelihood ratio test could not be properly applied to these data. Instead, the maximum likelihood estimates of the choice probabilities, under both WST and the LS, were obtained for each *S*. The observed proportion of triples violating WST, denoted π, was then compared with the expected proportions, based on the maximum likelihood estimates under WST and the LS, denoted WST(π) and LS(π) respectively. Table 8 shows that the observed values of π exceed the maximum likelihood estimates of π under WST for all but one *S* ($p < .01$ by a

TABLE 8

FREQUENCIES OF SELECTING THE FIRST ELEMENT OF EACH PAIR OVER THE SECOND, TOTALED OVER THE THREE REPLICATIONS

Subject	Set	Pair										π	$WST(\pi)$	$LS(\pi)$
		a, b	b, c	c, d	d, e	a, c	b, d	c, e	a, d	b, e	a, e			
1	I	3	3	2	1	2	1	2	1	0	0	.4	.213	.316
2	I	2	1	2	3	2	1	2	0	0	0	.2	.196	.292
3	I	3	2	3	2	1	2	3	0	0	0	.4	.262	.303
4	II	3	3	2	3	1	0	0	0	0	0	.3	.125	.241
5	II	0	0	0	0	0	0	0	0	0	0	.0	.000	.300
6	II	3	2	3	1	2	1	2	2	0	1	.3	.171	.285
7	II	3	3	3	1	2	1	1	0	0	0	.2	.197	.295
8	II	3	3	3	1	2	1	1	0	0	0	.3	.125	.242
9	II	3	2	3	3	1	2	3	2	1	0	.4	.237	.281
10	III	3	3	2	3	2	2	2	1	0	0	.5	.324	.391
11	III	3	3	3	3	2	3	3	1	2	1	.4	.238	.366
12	III	2	1	2	3	2	1	2	0	0	0	.2	.196	.292
13	III	3	2	2	3	2	1	0	1	1	0	.3	.228	.275
14	IV	2	2	3	3	1	0	2	1	1	0	.3	.228	.275
15	IV	3	2	3	3	2	2	2	2	0	0	.4	.238	.372
Total												.307	.199	.302

Note.—The values of π denote the observed proportions of intransitive triples, whereas the values of $LS(\pi)$ and $WST(\pi)$ denote the expected proportions under the two models, respectively.

sign test). Furthermore, the LS predicted the observed proportions better than WST for 11 out of 15 *S*s. Finally, the overall proportion of intransitive triples (.307) is significantly higher ($p < .01$) than the value expected under WST (.199), but it is not significantly different from the value expected under the LS (.302), according to a chi-square test. Hence, WST is rejected because both the overall proportion of intransitive triples and the π values of a significant majority of *S*s exceed their expected value under WST.

The *S*s were interviewed at the end of the test session. None of the *S*s realized that his preferences were intransitive. Moreover, a few *S*s denied this possibility emphatically and asked to see the experimenter's record. When faced with his own intransitivities one *S* said "I must have made a mistake somewhere." When the LS was explained to that *S*, however, he commented, "It is a reasonable way to make choices. In fact, I have probably made some decisions that way." The relation between the model and its logical consequences was obviously not apparent to our *S*.

THEORY

The empirical studies showed that, under appropriate experimental conditions, the behavior of some people is intransitive. Moreover, the intransitivities are systematic, consistent, and predictable. What type of choice theory is needed to explain intransitive preferences between multidimensional alternatives?

The lexicographic semiorder that was employed in the construction of the alternatives for the experiments is one such model. It is not, however, the only model that can account for the results. Furthermore, despite its intuitive appeal, it is based on a noncompensatory principle that is likely to be too restrictive in many contexts. In this section, two choice theories are introduced and their relationships to the transitivity principle are studied.

Let $A = A_1 \times \cdots \times A_n$ be a set of multidimensional alternatives with elements of the form $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n)$, where x_i $(i = 1, \cdots, n)$ is the value of Alternative x on Dimension i. Note that the components of x may be nominal scale values rather than real numbers. A theory of choice between such alternatives is essentially a decision rule which determines when x is preferred to y, or when $p(x,y) > \frac{1}{2}$. A more elaborate theory may also provide an explicit formula for $P(x,y)$.

In examining the process of choice between multidimensional alternatives, two different methods of evaluation have been considered (Morrison, 1962). The first is based on independent evaluations. According to this method, one evaluates the two alternatives, x and y, separately, and assigns scale values, $u(x)$ and $u(y)$, to each of them. Alternative x is, then, preferred to Alternative y if and only if $u(x) > u(y)$. The scale value assigned to an alternative is a measure of its utility, or subjective value, which is assumed to depend on the subjective values of its components. More specifically, there are scales $u_1, \cdots, u_n$ defined on $A_1, \cdots, A_n$ respectively such that $u_i(x_i)$ is the subjective value of the ith component of Alternative x. It is further assumed that the overall utility of an alternative is expressable as a specified function of the scale values of its components. Among the various possible functional relations, the additive combination rule has been most thoroughly investigated. According to the additive (conjoint measurement) model, the subjective value of an alternative is simply the sum of the subjective value of its components.

Stated formally, a preference structure satisfies the *additive model* if there exist real-valued functions $u, u_1, \cdots, u_n$ such that

$$x \gtrsim y \quad \text{if and only if}$$

$$u(x) = \sum_{i=1}^{n} u_i(x_i) \geq \sum_{i=1}^{n} u_i(y_i) = u(y). \qquad [5]$$

Axiomatic analyses of this model, which are based on ordinal assumptions, have been provided by Debreu (1960), Luce and Tukey (1964), Krantz (1964), and Luce (1966) under solvability conditions. Necessary and sufficient conditions for additivity have been discussed by Adams and Fagot (1959), Scott (1964), and Tversky (1967b). For some of the empirical applications of the model, see Shepard (1964) and Tversky (1967a). Note that the commonly applied multiple-regression model is a special case of the additive model where all the subjective scales are linear.

The second method of evaluation is based on comparisons of component-wise differences between the alternatives. According to this method one considers quantities of the form $\delta_i = u_i(x_i) - u_i(y_i)$ which correspond to the difference between the subjective values of x and y on the ith dimension. To each such quantity, one applies a difference function, ϕ_i, which determines the contribution of the particular subjective difference to the overall evaluation of the alternatives. The quantity $\phi_i(\delta_i)$ can be viewed, therefore, as the "advantage" or the "disadvantage" (depending on whether δ_i is positive or negative) of x over y with respect to Dimension i. With this interpretation in mind, it is natural to require that $\phi_i(-\delta) = -\phi_i(\delta)$. The obtained values of $\phi_i(\delta_i)$ are, then, summed over all dimensions, and x is preferred over y whenever the resulting sum is positive.

Stated formally, a preference structure satisfies the *additive difference model* if there exist real-valued functions $u_1, \cdots, u_n$ and increasing continuous functions $\phi_1, \cdots, \phi_n$ defined on some real intervals such that

$$x \gtrsim y \quad \text{if and only if}$$

$$\sum_{i=1}^{n} \phi_i[u_i(x_i) - u_i(y_i)] \geq 0$$

where $\qquad [6]$

$$\phi_i(-\delta) = -\phi_i(\delta) \quad \text{for all } i.$$

An axiomatic analysis of the additive difference model will be presented elsewhere. Essentially the same model was proposed by Morrison (1962). A set of ordinal axioms yielding a (symmetric) additive difference model of similarity (rather than preference) judgments has been given by Beals, Krantz, and Tversky (1968).

A comparison of the additive model (Equation 5) with the additive difference model (Equation 6) from a psychological viewpoint reveals that they suggest different ways of processing and evaluating the alternatives. A schematic illustration of the difference is given below.

$$x = (x_1, \cdots, x_i, \cdots, x_n) \rightarrow \sum_{i=1}^{n} u_i(x_i)$$

$$y = (y_1, \cdots, y_i, \cdots, y_n) \rightarrow \sum_{i=1}^{n} u_i(y_i)$$

$$\downarrow$$

$$\sum_{i=1}^{n} \phi_i[u_i(x_i) - u_i(y_i)]$$

In the simple additive model, the alternatives are first processed "horizontally," by adding the scale values of the components, and the resulting sums are then compared to determine the choice. In the additive difference model, on the other hand, the alternatives are first processed "vertically," by making intradimensional evaluations, and the results of these vertical comparisons are then added to determine the choice. Although the two models suggest different processing strategies, the additive model is formally a special case of the additive difference model where all the difference functions are linear. To verify this fact, suppose $\phi_i(\delta_i) = t_i\delta_i$ for some positive t_i and for all i. Consequently,

$$\sum_{i=1}^{n} \phi_i[u_i(x_i) - u_i(y_i)] = \sum_{i=1}^{n} t_i u_i(x_i) - \sum_{i=1}^{n} t_i u_i(y_i).$$

Thus, if we let $v_i(x_i) = t_i u_i(x_i)$ for all i, then Equation 6 can be written as $x \gtrsim y$ if and only if

$$\sum_{i=1}^{n} v_i(x_i) \geq \sum_{i=1}^{n} v_i(y_i)$$

which is the additive model of Equation 5.

Hence, if the difference functions are linear the two models (but not necessarily the processing strategies) coincide. The vertical processing strategy is, thus, compatible with the additive model if and only if the difference functions are linear.

The proposed processing strategies, as well as the models associated with them, are certainly affected by the way in which the information is displayed. More specifically, the additive model is more likely to be used when the alternatives are displayed sequentially (i.e., one at a time), while the additive difference model is more likely to be used when the dimensions are displayed sequentially. Two different types of political campaigns serve as a case in point. In one type of campaign, each candidate appears separately and presents his views on all the relevant issues. In the second type the various issues are raised separately and each candidate presents his view on that particular issue. It is argued that the "horizontal" evaluation method, or the simple additive model, is more likely to be used in the former situation, while the "vertical" evaluation method, or the additive difference model, is more likely to be used in the latter situation.

Although different evaluation methods may be used in different situations, there are several general considerations which favor the additive difference model. In the first place, it is considerably more general, and can accommodate a wider variety of preference structures. The LS, for example, is a limiting case of this model where one (or more) of the difference functions approaches a step function where $\phi(\delta) = 0$ whenever $\delta \leq \epsilon$. Second, intradimensional comparison may simplify the evaluation task. If one alternative is slightly better than another one on all relevant dimensions, it will be immediately apparent in a componentwise comparison and the choice will indeed be easy. If the alternatives, however, are evaluated independently this dominance relation between the alternatives may be obscured, which would certainly complicate the choice process. But even if no such dominance relation exists, it may still be easier to use approximation methods when the evaluation is based on component-wise comparisons. One common approximation procedure is based on "canceling out" differences that are equal, or nearly equal, thus reducing the number of dimensions that have to be considered. In deciding which of two houses to buy, for example, one may feel that the differences in style and location cancel each other out and the choice problem reduces to one of deciding whether it is worth spending $\$x$ more for a larger house. It is considerably more difficult to employ this procedure when the two alternatives are evaluated independently.

Finally, intradimensional evaluations are simpler and more natural than interdimensional ones simply because the compared quantities are expressed in terms of the same units. It is a great deal simpler to evaluate the difference in intelligence between two candidates than to evaluate the combined effect of intelligence and emotional stability. In choosing between two n-dimensional alternatives, one makes $2n$ interdimensional evaluations when the alternatives are evaluated independently according to the additive model, but only n interdimensional evaluations along with n intradimensional evaluations according to the additive difference model.

Now that the two models have been defined and compared, their relationships to the transitivity principle are investigated. It can be readily seen that the simple additive model satisfies the transitivity principle, for the assumptions that x is preferred to y, and y is preferred to z imply that $u(x) > u(y)$ and $u(y) > u(z)$. Hence, $u(x) > u(z)$, which implies that x must be preferred to z. Note that the argument does not depend on the additivity assumption. Transitivity must, therefore, be satisfied by any model where a scale value is assigned to each alternative and the preferences are compatible with Equations 2 or 3.

Under what conditions does the additive difference model satisfy the transitivity principle? The answer to this question is given by the following result, which depends on the dimensionality of the alternatives.[3]

Theorem: If the additive difference model (Equation 6) is satisfied then the following assertions hold whenever the difference functions are defined.

1. For $n \geq 3$, transitivity holds if and only if all difference functions are linear. That is, $\phi_i(\delta) = t_i\delta$ for some positive t_i and for all i.
2. For $n = 2$, transitivity holds if and only if $\phi_1(\delta) = \phi_2(t\delta)$ for some positive t.
3. For $n = 1$, transitivity is always satisfied.

The proof is given in the appendix. The theorem shows that the transitivity assumption imposes extremely strong constraints on the form of the difference functions. In the two-dimensional case, the difference functions applied to the two dimensions must be identical except for a change of unit of their domain. If the alternatives have three or more dimensions, then transitivity is both necessary and sufficient for the linearity of all the difference functions. Recall that under the linearity assumption, the additive difference model reduces to the simple additive model, which has already been shown to satisfy transitivity. The above theorem asserts, however, that this is the only case in which the transitivity assumption is compatible with the additive difference model. Put differently, if the additive difference model is satisfied and if even one difference function is nonlinear, as is likely to be the case in some situations, then transitivity must be violated somewhere in the system. The experimental identification of these intransitivities in the absence of knowledge of the form of the difference functions might be very difficult indeed. The LS employed in the design of the experimental research is based on one extreme form of nonlinearity where one of the difference functions is, or can be approximated by, a step function. The above theorem suggests a new explanation of the intransitivity phenomenon, in terms of the form of the difference functions, which may render it more plausible than it seemed before.

Most of the choice mechanisms that have been purported to yield intransitivities (including the LS) are based on the notion of shifting attention, or switching dimensions, from one choice to another. Consequently, they assume that some relevant information describing the alternatives is ignored or dis-

[3] In referring to the dimensionality of the alternatives, denoted n, only nontrivial dimensions having more than one value are considered. The fact that transitivity holds whenever $n = 2$ and $\phi_1 = \phi_2$ has been recognized by Morrison (1962, p. 19).

carded on particular choices. In contrast to this notion, intransitivities can occur in the additive difference model in a fully compensatory system where all the information is utilized in the evaluation process.

Both the additive model and the additive difference model can be extended in a natural way. To do so, let F be an increasing function and suppose that all choice probabilities are neither 0 and 1. The (extended) additive model is said to be satisfied whenever Equation 5 holds and

$$P(x,y) = F\left[\sum_{i=1}^{n} u_i(x_i) - \sum_{i=1}^{n} u_i(y_i)\right]. \quad [7]$$

Similarly, the (extended) additive difference model is said to be satisfied whenever Equation 6 holds and

$$P(x,y) = F\left(\sum_{i=1}^{n} \phi_i[u_i(x_i) - u_i(y_i)]\right). \quad [8]$$

Both models are closely related to the Fechnerian or the strong utility model (see Luce & Suppes, 1965). This model asserts that there exists a function u and a distribution function F such that

$$P(x,y) = F[u(x) - u(y)]. \quad [9]$$

Note that Equation 7 is a special case of Equation 9, where the utilities are additive, while Equation 8 is an additive generalization of Equation 9 to the multidimensional case. The two most developed probabilistic models of Thurstone (1927, Case V) and Luce (1959) can be obtained from the Fechnerian model by letting F be the normal or the logistic distribution function respectively.

It can be easily shown that Equation 9 and, hence, Equation 7 satisfy not only WST, but also a stronger probabilistic version of transitivity called strong stochastic transitivity, or SST. According to this condition, if $P(x,y) \geq \frac{1}{2}$ and $P(y,z) \geq \frac{1}{2}$ then both $P(x,z) \geq P(x,y)$ and $P(x,z) \geq P(y,z)$.

Clearly, SST implies WST but not conversely. However, if Equation 8 is valid with $n \geq 3$, and if WST is satisfied, then according to the above theorem, Equation 8 reduces to Equation 7 which satisfies SST as well. Thus, we obtain the somewhat surprising result that, under the extended additive difference model, with $n \geq 3$, WST and SST are equivalent.

Discussion

In the introduction, a choice model (the LS) yielding intransitive preferences was described. This model was employed in the design of two studies which showed that, under specified experimental conditions, consistent intransitivities can be obtained. The theoretical conditions under which intransitivities occur were studied within the framework of a general additive difference model. The results suggest that in the absence of a model that guides the construction of the alternatives, one is unlikely to detect consistent violations of WST. The absence of an appropriate model combined with the lack of sufficiently powerful statistical tests may account for the failure to reject WST in previous investigations.

Most previous tests of WST have been based on comparison between the observed proportion of intransitive triples, π, and the expected proportion under WST. As Morrison (1963) pointed out, however, this approach leads to difficulties arising from the fact that in a complete pair comparison design only a limited proportion of triples can, in principle, be intransitive. Specifically, the expected value of π for an S who is diabolically (or maximally) intransitive is $\frac{k+1}{4(k-2)}$ where k is the number of alternatives. As k increases, this expression approaches one-fourth, which is the expected value of π under the hypothesis of random choice (i.e., $P(x,y) = \frac{1}{2}$ for all x,y). Morrison argued, therefore, that unless the intransitive triples can be identified in advance, it is practically impossible (with a large number of alternatives) to distinguish between the diabolically intransitive S and the random S on the basis of the observed value of π. These considerations suggest that a more powerful test of WST can be obtained by using many replications of a few well-chosen alternatives rather than by using a few replications of many alternatives. The latter approach, however, has been employed in most studies of preference.

What are the implications of the present results for the analysis of choice behavior?

Casual observations, as well as the comments made by *S*s, suggest that the LS (or some other nonlinear version of the additive difference model) is employed in some real-world decisions, and that the resulting intransitivities can also be observed outside the laboratory. Consider, for example, a person who is about to purchase a compact car of a given make. His initial tendency is to buy the simplest model for $2089. Nevertheless, when the saleman presents the optional accessories, he first decides to add power steering, which brings the price to $2167, feeling that the price difference is relatively negligible. Then, following the same reasoning, he is willing to add $47 for a good car radio, and then an additional $64 for power brakes. By repeating this process several times, our consumer ends up with a $2593 car, equipped with all the available accessories. At this point, however, he may prefer the simplest car over the fancy one, realizing that he is not willing to spend $504 for all the added features, although each one of them alone seemed worth purchasing.

When interviewed after the experiment, the vast majority of *S*s said that people are and should be transitive. Some *S*s found it very difficult to believe that they had exhibited consistent intransitivities. If intransitivities of the type predicted by the additive difference model, however, are manifest in choice behavior why were *S*s so confident that their choices are transitive?

In the first place, transitivity is viewed, by college undergraduates at least, as a logical principle whose violation represents an error of judgment or reasoning. Consequently, people are not likely to admit the existence of consistent intransitivities. Second, in the absence of replications, one can always attribute intransitivities to a change in taste that took place between choices. The circular preferences of the car buyer, for example, may be explained by the hypothesis that, during the choice process, the consumer changed his mind with regard to the value of the added accessories. If this hypothesis is misapplied, the presence of genuine intransitivities is obscured. Finally, most decisions are made in a sequential fashion. Thus, having chosen y over x and then z over y, one is typically committed to z and may not even compare it with x, which has already been eliminated. Furthermore, in many choice situations the eliminated alternative is no longer available so there is no way of finding out whether our preferences are transitive or not. These considerations suggest that in actual decisions, as well as in laboratory experiments, people are likely to overlook their own intransitivities.

Transitivity, however, is one of the basic and the most compelling principles of rational behavior. For if one violates transitivity, it is a well-known conclusion that he is acting, in effect, as a "money-pump." Suppose an individual prefers y to x, z to y, and x to z. It is reasonable to assume that he is willing to pay a sum of money to replace x by y. Similarly, he should be willing to pay some amount of money to replace y by z and still a third amount to replace z by x. Thus, he ends up with the alternative he started with but with less money. In the context of the selection of applicants, intransitivity implies that, if a single candidate is to be selected in a series of pair comparisons, then the chosen candidate is a function of the order in which the pairs are presented. Regardless of whether this is the case or not, it is certainly an undesirable property of a decision rule.

As has already been mentioned, the normative character of the transitivity assumption was recognized by *S*s. In fact, some evidence (MacCrimmon, 1965) indicates that when people are faced with their own intransitivities they tend to modify their choices according to the transitivity principle. Be this as it may, the fact remains that, under the appropriate experimental conditions, some people are intransitive and these intransitivities cannot be attributed to momentary fluctuations or random variability.

Is this behavior necessarily irrational? We tend to doubt it. It seems impossible to reach any definite conclusion concerning human rationality in the absence of a detailed analysis of the sensitivity of the criterion and the cost involved in evaluating the alternatives. When the difficulty (or the cost) of the evaluations and the consistency (or the error) of the judgments are taken into account, a model based on component-wise evaluation, for example, may prove superior to a model based on independent evaluation despite the fact that the former is not necessarily transitive while the latter is. When faced with complex multidimensional alternatives, such as job offers, gambles, or candidates, it is extremely difficult to utilize properly all the available information. Instead, it is contended that people employ various approximation methods that enable them to process the relevant information in making a decision. The particular approximation scheme depends on the nature of the alternatives as well as on the ways in which they are presented or displayed. The lexicographic semiorder is one such an approximation. In general, these simplification procedures might be extremely useful in that they can approximate one's "true preference" very well. Like any approximation, they are based on the assumption that the approximated quantity is independent of the approximation method. That is, in using such methods in making decisions we implicitly assume that the world is not designed to take advantage of our approximation methods. The present experiments, however, were designed with exactly that goal in mind. They attempted to produce intransitivity by capitalizing on a particular approximation method. This approximation may be very good in general, despite the fact that it yields intransitive choices in some specially constructed situations. The main interest in the present results lies not so much in the fact that transitivity can be violated but rather in what these violations reveal about the choice mechanism and the approximation method that govern preference between multidimensional alternatives.

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APPENDIX

Theorem

If the additive difference model (Equation 6) is satisfied, then the following assertions hold whenever the difference functions are defined.

1. For $n \geq 3$, transitivity holds if and only if all difference functions are linear. That is, $\phi_i(\delta) = t_i\delta$ for some real t_i and for all i.
2. For $n = 2$, transitivity holds if and only if $\phi_1(\delta) = \phi_2(t\delta)$ for some real t.
3. For $n = 1$, transitivity is always satisfied.

Proof

By WST, $P(x,y) = \frac{1}{2}$ and $P(y,z) = \frac{1}{2}$ imply $P(x,z) = \frac{1}{2}$. Hence, according to the additive difference model there exist functions $u_1, \cdots, u_n$ and increasing continuous functions $\phi_1, \cdots, \phi_n$ defined on some real intervals of the form $(-\delta_i,\delta_i)$ such that

$$\sum_{i=1}^{n} \phi_i[u_i(x_i) - u_i(y_i)] = 0$$

and

$$\sum_{i=1}^{n} \phi_i[u_i(y_i) - u_i(z_i)] = 0$$

imply

$$\sum_{i=1}^{n} \phi_i[u_i(x_i) - u_i(z_i)] = 0,$$

where $\phi_i(-\delta) = -\phi_i(\delta)$. Letting

$$\alpha_i = u_i(x_i) - u_i(y_i) \quad \text{and} \quad \beta_i = u_i(y_i) - u_i(z_i)$$

yields

$$(*) \quad \sum_{i=1}^{n} \phi_i(\alpha_i) = 0 \quad \text{and} \quad \sum_{i=1}^{n} \phi_i(\beta_i) = 0$$

imply

$$\sum_{i=1}^{n} \phi_i(\alpha_i + \beta_i) = 0.$$

First, suppose $n = 1$, hence (*) reduces to:

$$\phi(\alpha) = 0, \quad \phi(\beta) = 0 \quad \text{imply} \quad \phi(\alpha + \beta) = 0.$$

But since ϕ is increasing, and $\phi(0) = -\phi(0) = 0, \alpha = \beta = 0 = \alpha + \beta$, and hence the above equation is always satisfied.

Next, suppose $n = 2$, hence, by (*), $\phi_1(\alpha_1) + \phi_2(\alpha_2) = 0$ and $\phi_1(\beta_1) + \phi_2(\beta_2) = 0$ imply $\phi_1(\alpha_1 + \beta_1) + \phi_2(\alpha_2 + \beta_2) = 0$. Since $\phi_i(-\delta) = -\phi_i(\delta)$, the above relation can be rewritten as $\phi_1(\alpha_1) = \phi_2(-\alpha_2)$ and $\phi_1(\beta_1) = \phi_2(-\beta_2)$ imply $\phi_1(\alpha_1 + \beta_1) = \phi_2(-\alpha_2 - \beta_2)$. Consequently, by letting $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$, and repeating the argument n times, we obtain $\phi_1(\alpha) = \phi_2(\beta)$ implies $\phi_1(n\alpha) = \phi_2(n\beta)$ for any positive integer n for which both $\phi_1(n\alpha)$ and $\phi_2(n\beta)$ are defined.

Since all difference functions are continuously increasing and since they all vanish at zero, one can select positive a, b such that $\phi_1(a)$, $\phi_2(b)$ are defined and such that $\phi_1(a) = \phi_2(b)$. Hence, for any positive integers m, n for which $\phi_1\left(\frac{m}{n}a\right)$ and $\phi_2\left(\frac{m}{n}b\right)$ are defined, $\phi_1\left(\frac{a}{n}\right)$ and $\phi_2\left(\frac{b}{n}\right)$ are also defined. Furthermore, $\phi_1\left(\frac{a}{n}\right) = \phi_2\left(\frac{b}{n}\right)$, for otherwise a strict inequality must hold. Suppose $\phi_1\left(\frac{a}{n}\right) < \phi_2\left(\frac{b}{n}\right)$, hence there exists c such that $\phi_1\left(\frac{a}{n}\right)$

$= \phi_2(c) < \phi_2\left(\frac{b}{n}\right)$. Consequently, $c < \frac{b}{n}$, or $nc < b$, and $\phi_2(nc)$ is defined. Hence, $\phi_2(nc) = \phi_1(a) = \phi_2(b)$ and $nc = b$, a contradiction. By the symmetry of the situation, a similar contradiction is obtained if $\phi_2\left(\frac{a}{n}\right) > \phi_2\left(\frac{b}{n}\right)$. Therefore, $\phi_1(a) = \phi_2(b)$ implies $\phi_1\left(\frac{a}{n}\right) = \phi_2\left(\frac{b}{n}\right)$ for all n.

Next, let $t = \frac{b}{a}$ and suppose that both $\phi_1(c)$ and $\phi_2(tc)$ are defined. Thus, for any $\delta \geq 0$, there exist m, n such that $c - \delta \leq \frac{m}{n}a \leq c$, and hence $\frac{b}{a}(c - \delta) \leq \frac{b}{a}\frac{m}{n}a \leq \frac{b}{a}c$. Consequently, $\phi_1(c - \delta) \leq \phi_1\left(\frac{m}{n}a\right) \leq \phi_1(c)$ and $\phi_2\left[\frac{b}{a}(c - \delta)\right] \leq \phi_2\left(\frac{m}{n}b\right) \leq \phi_2\left(\frac{b}{a}c\right)$. As δ approaches 0, however, $\phi_1(c - \delta) = \phi_1(c)$ and $\phi_2\left[\frac{b}{a}(c - \delta)\right] = \phi_2\left(\frac{b}{a}c\right)$, and since $\phi_1\left(\frac{m}{n}a\right) = \phi_2\left(\frac{m}{n}b\right)$, by hypothesis, $\phi_1(c) = \phi_2(tc)$ as required. Conversely, if $\phi_1(c) = \phi_2(tc)$ it follows readily that Equation (*) is satisfied which completes the proof of this case.

Finally, suppose $n \geq 3$. Since we can let all but three differences be zero, we consider the case where $n = 3$. Hence,

$$\phi_1(\alpha_1) + \phi_2(\alpha_2) + \phi_3(\alpha_3) = 0$$

and

$$\phi_1(\beta)_1 + \phi_2(\beta_2) + \phi_3(\beta_3) = 0$$

imply $\phi_1(\alpha_1 + \beta_1) + \phi_2(\alpha_2 + \beta_2) + \phi_3(\alpha_3 + \beta_3) = 0$. By the earlier result, however, $\phi_i(\delta) = \phi_j(t_j\delta)$ for $i, j = 1, 2, 3$. Hence, the above implication is expressible as

$$\phi(\alpha) + \phi(\beta) = \phi(\delta) \quad \text{and} \quad \phi(\alpha') + \phi(\beta') = \phi(\delta')$$

imply

$$\phi(\alpha + \alpha') + \phi(\beta + \beta') = \phi(\delta + \delta').$$

Define ψ such that $\phi(\alpha) + \phi(\beta) = \phi[\psi(\alpha, \beta)]$ for all α, β. Hence,

$$\begin{aligned}\phi(\alpha+\alpha') + \phi(\beta+\beta') &= \phi[\psi(\alpha+\alpha', \beta+\beta')] \\ &= \phi(\delta+\delta') \\ &= \phi[\psi(\alpha, \beta) + \psi(\alpha', \beta')].\end{aligned}$$

Hence, $\psi(\alpha, \beta) + \psi(\alpha', \beta') = \psi(\alpha + \alpha', \beta + \beta')$ and ψ is linear in α, β. Therefore, $\phi(\alpha) + \phi(\beta) = \phi(p\alpha + q\beta)$ for some real p, q. If we let $\beta = 0$, we get $\phi(\alpha) = \phi(p\alpha)$ hence $p = 1$. Similarly, if we let $\alpha = 0$, we get $\phi(\beta) = \phi(q\beta)$ hence $q = 1$. Consequently, $\phi(\alpha) + \phi(\beta) = \phi(\alpha + \beta)$ and ϕ if linear as required. The converse for any $n \geq 3$ is immediate which completes the proof of this theorem.

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