

Calculus: Applications and Integration

POLI 270 - Mathematical and Statistical Foundations

Sebastian M. Saiegh

Department of Political Science
University California, San Diego

October 7 2010



Calculus: Applications and Integration

- 1 Applications of the Derivative
 - Mean Value Theorems
 - Monotone Functions
- 2 Integration
 - Antidifferentiation: The Indefinite Integral
 - Definite Integrals



Introduction

Last week, we looked at the idea of instantaneous rate of change, and we learned how to find the derivative of a function.

Today, we are going to focus on some applications of the concept of the derivative.

In particular, we will find out how to use derivatives to locate the intervals in which a function is monotone and those points in the domain where a graph of a function presents some special characteristics.

These notions are critical for the study of optimization problems in political science.



Calculus: Applications and Integration

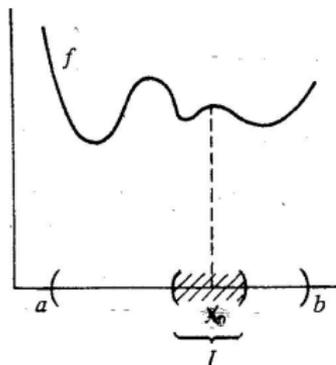
- 1 Applications of the Derivative
 - Mean Value Theorems
 - Monotone Functions
- 2 Integration
 - Antidifferentiation: The Indefinite Integral
 - Definite Integrals



Local Maxima and Minima

Let f be defined on an open interval (a, b) and let $x_0 \in (a, b)$.

We say that f has a *local maximum* at x_0 if $f(x) \leq f(x_0)$.

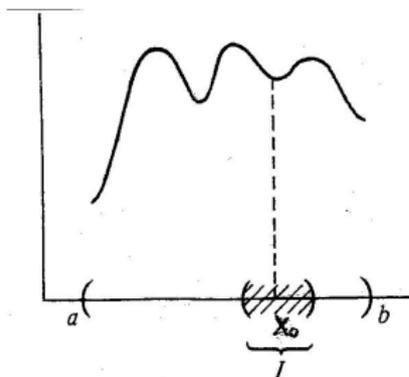


for all values of x in some open interval I which contains x_0 .

Local Maxima and Minima (cont.)

Local *minima* are defined similarly:

f has a *local minimum* at x_0 if $f(x) \geq f(x_0)$



for all values of x in some open interval I which contains x_0 .

Local Maxima and Minima (cont.)

In words, f has a local maximum at x_0 if its graph has a “little hill” above the point x_0 .

Similarly, f has a local minimum at x_0 if its graph has a “little valley” above the point x_0 .

If $f(x_0)$ is the maximum value of f on the whole interval (a, b) , then obviously f has a local maximum at x_0 . But the converse need not be true.

Absolute minima are defined similarly.



Maxima and Minima

If there is either a maximum or minimum at $x = x_0$, we sometimes combine these two possibilities by saying f has an *extremum* at x_0 .

Notice that once we know the place (value of x) where the largest or smallest value of f occurs, the value $y = f(x)$ is easy to calculate.

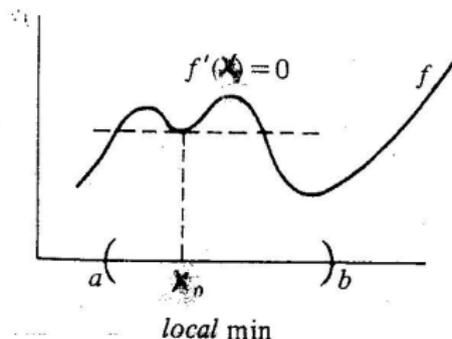
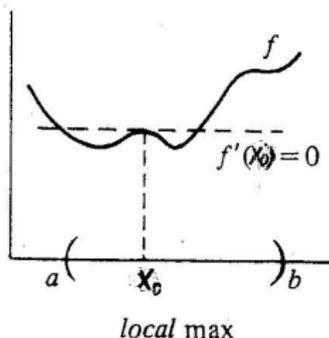
Now, we will use a few theorems and calculus methods to locate the appropriate x .



Locating Maxima and Minima

Theorem

Suppose that f is differentiable on (a, b) and that $x_0 \in (a, b)$. If f has a local maximum or minimum at x_0 , then $f'(x_0) = 0$.



Locating Maxima and Minima (cont.)

Proof. By definition

$$\frac{f(x) - f(x_0)}{x - x_0} \rightarrow f'(x_0) \text{ as } x \rightarrow x_0.$$

Suppose that $f'(x_0) > 0$.

From our discussion of limits we know that if

$$\lim_{x \rightarrow x_0} f(x) = L.$$

then if $L > 0$, $f(x) > 0$ for some $h > 0$ provided that $x_0 - h < x < x_0 + h$ and $x \neq x_0$.



Locating Maxima and Minima (cont.)

Therefore, it follows that for some open interval $I = (x_0 - h, x_0 + h)$

$$\frac{f(x) - f(x_0)}{x - x_0} > 0$$

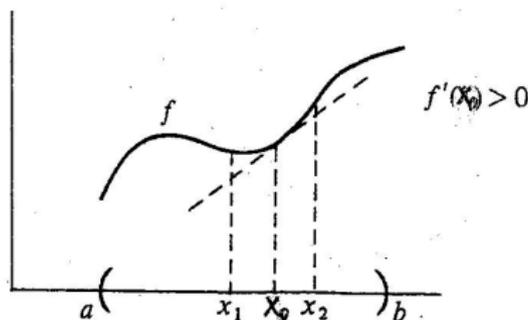
provided that $x \in I$ and $x \neq x_0$.

Let x_1 be *any* number in the interval $(x_0 - h, x_0)$. Then $x_1 - x_0 < 0$ and hence it follows from the last equation that $f(x_1) < f(x_0)$. Thus, f cannot have a local minimum at x_0 .

Let x_2 be *any* number in the interval $(x_0, x_0 + h)$. Then $x_2 - x_0 > 0$ and so it follows from the last equation that $f(x_2) > f(x_0)$. Thus, f cannot have a local maximum at x_0 .



Locating Maxima and Minima (cont.)



A similar argument deals with the case when $f'(x_0) < 0$.

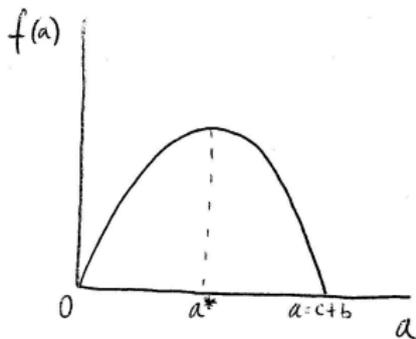
The only remaining possibility is $f'(x_0) = 0$.

Stationary Points

A point x_0 at which $f'(x_0) = 0$ is called a *stationary point* of f .

Example

Let $f(a) = ac + ab - a^2$. The value of f is zero when $a = 0$ and when $a = c + b$, and reaches a maximum in between.



Stationary Points (cont.)

The simplicity of the previous example is misleading, though.

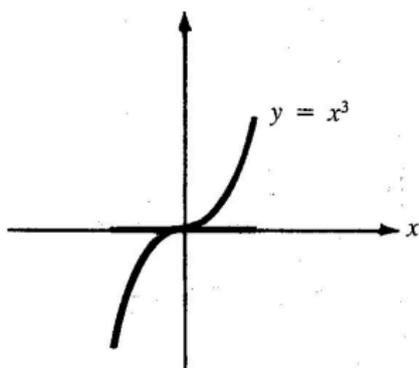
In general, not every point at which the derivative of a function is zero is a peak of its graph.

That is, not all stationary points give rise to a local maximum or a local minimum.



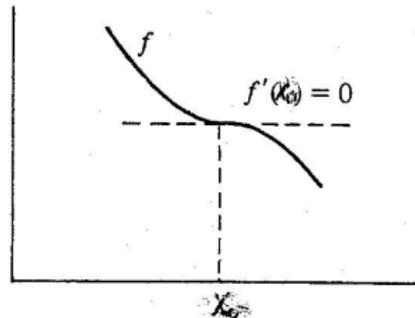
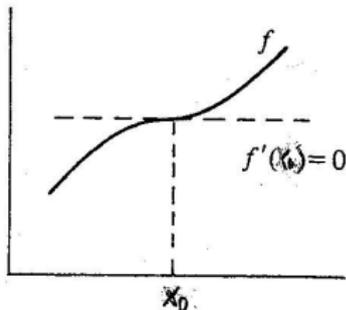
Stationary Points (cont.)

For example, the derivative of the function $y = x^3$ is zero when $x = 0$, but it has neither a maximum nor a minimum at this point.



Stationary Points (cont.)

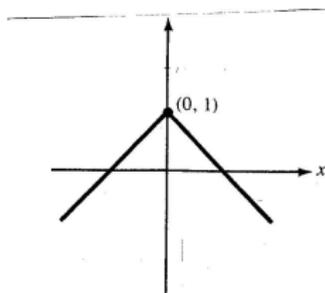
More generally, in those cases where $f'(x_0) = 0$ and the function has neither a maximum nor a minimum at that point, we say that f has a *point of inflection* at x_0 .



Stationary Points (cont.)

The situation is further complicated by the existence of functions with maxima or minima at points where the derivative is not even defined. For example, consider the following function,

$$f(x) = \begin{cases} 1 - x & \text{if } x \geq 0 \\ 1 + x & \text{if } x < 0. \end{cases}$$



Rolle's Theorem

We will now focus on a systematic procedure to locate and identify maxima and minima of differentiable functions.

Theorem

(Rolle's theorem) Suppose that f is continuous on $[a, b]$ and differentiable on (a, b) . If $f(a) = f(b)$, then, for some $x \in (a, b)$, $f'(x) = 0$.



Rolle's Theorem (cont.)

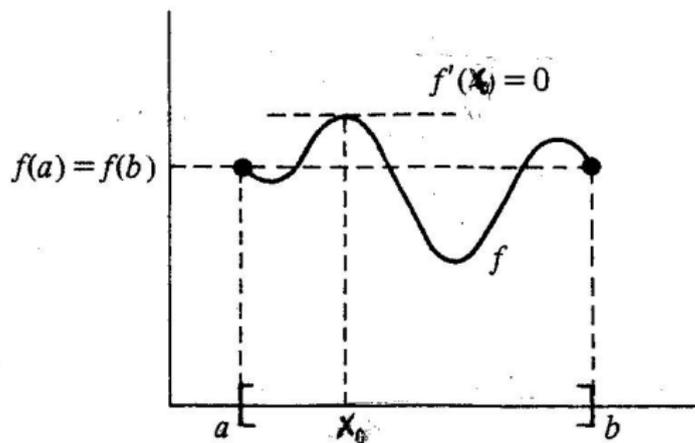
Proof. Since f is continuous on the compact interval $[a, b]$, it follows from the continuity property that f attains a maximum M at some point x_1 in the interval $[a, b]$ and attains a minimum m at some point x_2 in the interval $[a, b]$.

Suppose x_1 and x_2 are both endpoints of $[a, b]$. Because $f(a) = f(b)$ it then follows that $m = M$ and hence f is a constant on $[a, b]$. But then $f'(x) = 0$ for all $x \in (a, b)$.

Suppose that x_1 is not an endpoint of $[a, b]$. Then $x_1 \in (a, b)$ and f has a local maximum at x_1 . Thus, $f'(x_1) = 0$. Similarly, if x_2 is not an endpoint of $[a, b]$.



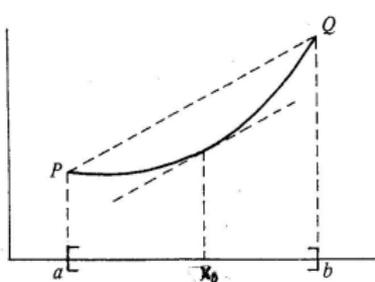
Rolle's Theorem (cont.)



Mean Value Theorem

Theorem

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b) . Then, for some $x_0 \in (a, b)$, $f'(x_0) = \frac{f(b)-f(a)}{b-a}$.



In the graph, the slope of the chord PQ is $\frac{f(b)-f(a)}{b-a}$. Thus, for some $x_0 \in (a, b)$, the tangent of f at x_0 is parallel to PQ .

Mean Value Theorem (cont.)

Theorem

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b) . If $f'(x) = 0$ for each $x \in (a, b)$, then f is constant on $[a, b]$.

Proof. Let $y \in (a, b)$. Then f satisfies the conditions of the mean value theorem on $[a, y]$. Hence for some $x_0 \in (a, y)$,

$$f'(x_0) = \frac{f(y) - f(a)}{y - a}.$$

But $f'(x_0) = 0$ and so $f(y) = f(a)$ for any $y \in [a, b]$.



Calculus: Applications and Integration

- 1 Applications of the Derivative
 - Mean Value Theorems
 - Monotone Functions
- 2 Integration
 - Antidifferentiation: The Indefinite Integral
 - Definite Integrals



Monotonicity

Let f be defined on a set S . We say that f *increases* on the set S if and only if, for each $x \in S$ and $y \in S$ with $x < y$, it is true that

$$f(x) \leq f(y).$$

If strict inequality always holds, we say that f is *strictly increasing* on the set S .

Similar definitions hold for decreasing and strictly decreasing.

A function that is either increasing or decreasing is called *monotone*.

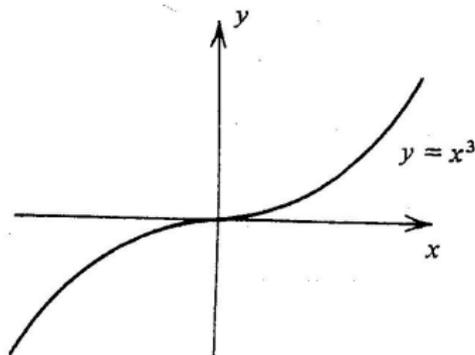
A function which is *both* increasing and decreasing must be a constant.



Strictly Increasing Functions

Example

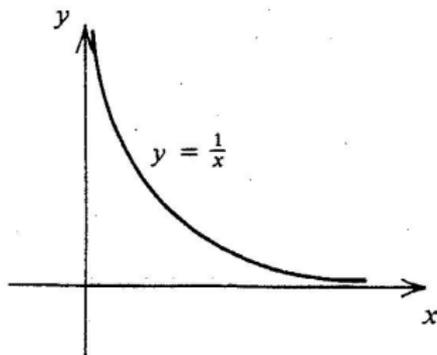
The function $f: \mathbf{R} \rightarrow \mathbf{R}$ defined by $f(x) = x^3$ is strictly increasing on \mathbf{R} .



Strictly Decreasing Functions

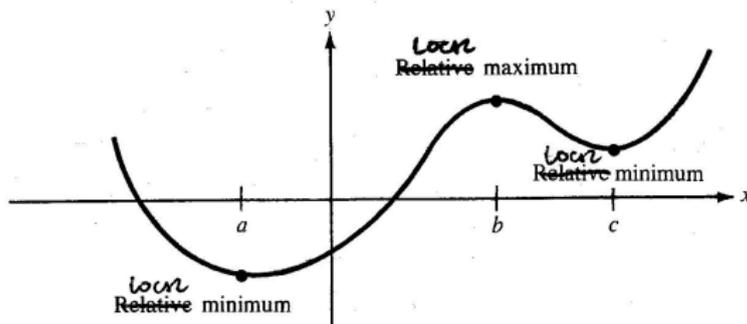
Example

The function $f: (0, \infty) \rightarrow \mathbf{R}$ defined by $f(x) = \frac{1}{x}$ is strictly decreasing on $(0, \infty)$.



Increasing and Decreasing Functions

We can observe that a function is increasing if its graph is rising as x increases and decreasing if its graph is falling as x increases.



This function, for example, is increasing for $a < x < b$ and for $x > c$. It is decreasing for $x < a$ and for $b < x < c$.

Local Maxima and Minima

If we know the intervals on which a function is increasing and decreasing, we can easily identify its local maxima and minima.

A local maximum occurs when the function stops increasing and starts decreasing.

In the previous graph, this happens when $x = b$.

A local minimum occurs when the function stops decreasing and starts increasing.

In this case, this happens when $x = a$ and $x = c$.



Differentiable monotone functions

Notice that we can find out where a differentiable function is increasing or decreasing by checking the sign of its derivative.

Theorem

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b) .

- (i) *If $f'(x) \geq 0$ for each $x \in (a, b)$, then f is increasing on $[a, b]$.
If $f'(x) > 0$ for each $x \in (a, b)$, then f is strictly increasing on $[a, b]$.*
- (ii) *If $f'(x) \leq 0$ for each $x \in (a, b)$, then f is decreasing on $[a, b]$.
If $f'(x) < 0$ for each $x \in (a, b)$, then f is strictly decreasing on $[a, b]$.*



Differentiable monotone functions (cont.)

Proof. Let c and d be any numbers in $[a, b]$ which satisfy $c < d$. Then f satisfies the conditions of the mean value theorem on $[c, d]$ and hence, for some $x_0 \in (c, d)$,

$$f'(x_0) = \frac{f(d) - f(c)}{d - c}.$$

If $f'(x) \geq 0$ for each $x \in (a, b)$, then $f'(x_0) \geq 0$ and hence $f(d) \geq f(c)$. Thus f increases on $[a, b]$.

If $f'(x) > 0$ for each $x \in (a, b)$, then $f'(x_0) > 0$ and hence $f(d) > f(c)$. Thus f is strictly increasing on $[a, b]$.

Similarly, for the other cases.



Differentiable monotone functions (cont.)

Example

Consider the function $f: \mathbf{R} \rightarrow \mathbf{R}$ defined by $f(x) = x(1 - x)$.

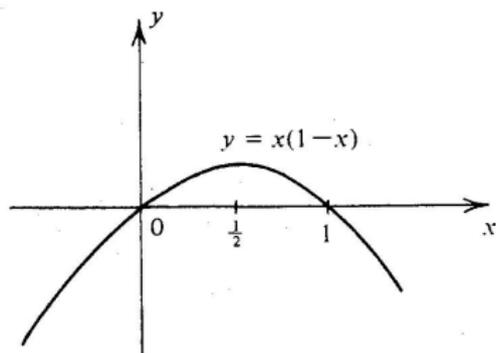
The derivative of this function is given by $f'(x) = 1 - 2x$.

Hence, $f'(x) \geq 0$ when $x \leq \frac{1}{2}$ and $f'(x) \leq 0$ when $x \geq \frac{1}{2}$.

It follows that f increases on $(-\infty, \frac{1}{2}]$ and decreases on $[\frac{1}{2}, \infty)$.



Differentiable monotone functions (cont.)



Notice what happens when $x = \frac{1}{2}$: the derivative of this function becomes $f'(x) = 0$.

As noted before, we call this a stationary or *critical point* of a function.



Critical Points

Critical Point. A point x in the feasible set of a function $f(x)$ is a critical point for f if $\frac{df}{dx} = 0$ there, or the derivative is undefined.

Every local extremum is a critical point.

However, as we saw earlier, not every critical point is necessarily a local extremum.



Critical Points (cont.)

If the derivative is positive to the left of a critical point and negative to the right of it, the function changes from increasing to decreasing and the critical point is a local maximum.

If the derivative is negative to the left of a critical point and positive to the right of it, the function changes from decreasing to increasing and the critical point is a local minimum.

If the sign of the derivative is the same on both sides of the critical point, the critical point is neither a local maximum nor a local minimum.



Higher-Order Derivatives

If a function $y = f(x)$ is differentiable, its derivative is another function $f'(x)$.

If $f'(x)$ is itself differentiable, we can apply the differentiation process to it.

The resulting function, called the *second derivative*, is denoted by any one of the following interchangeable symbols:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right), \frac{d^2f}{dx^2}, y'', f'', f^{(2)}$$



Higher-Order Derivatives (cont.)

Example

$$\text{Let } y = f(x) = x^3 - 4x^2 + 2x - 3.$$

Our rule for differentiating polynomials yields

$$\frac{dy}{dx} = 3x^2 - 8x + 2$$

We then differentiate the function $\frac{dy}{dx}$ to obtain the second derivative

$$\frac{d^2y}{dx^2} = 6x - 8.$$



Higher-Order Derivatives (cont.)

There is no reason why this process cannot be continued, provided that we obtain differentiable functions at each step.

Differentiating the second derivative, we obtain a new function called the third derivative; differentiating the third derivative yields the fourth derivative, and so on.

We are particularly interested in the second derivative, though, because there is a very useful geometric interpretation of it.



Concavity

We can see the usefulness of the second derivative by looking at the notion of concavity.

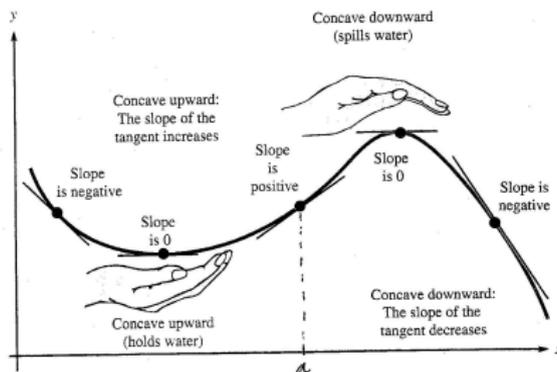
A curve is said to be *concave upward* if the slope of its tangent increases as it moves along the curve from left to right.

A curve is said to be *concave downward* if the slope of its tangent decreases as it moves along the curve from left to right.



Concavity (cont.)

This picture illustrates the idea of concavity. The curve is concave upward to the left of $x = a$ and concave downward to the right.



A useful “trick” : a graph can be described as “holding water” where it is concave upward, and as “spilling water” where it is concave downward.



Concavity (cont.)

There is a simple characterization of concavity in terms of the sign of the second derivative.

It is based on the fact (established earlier in this class) that a quantity increases when its derivative is positive and decreases when its derivative is negative.

The second derivative comes into the picture when this fact is applied to the first derivative (or slope of the tangent).



Concavity (cont.)

Here is the argument: suppose the second derivative f'' is positive on an interval.

This implies that the first derivative f' must be increasing on the interval. But f' is the slope of the tangent.

Hence, the slope of the tangent is increasing, and so the graph of f is concave upward on the interval.

On the other hand, if f'' is negative on an interval, then f' is decreasing. This implies that the slope of the tangent is decreasing, and so the graph of f is concave downward on the interval.



Test for Concavity

Test for Concavity.

If $f''(x) > 0$ on the interval $a < x < b$, then f is concave upward on this interval.

If $f''(x) < 0$ on the interval $a < x < b$, then f is concave downward on this interval.



Test for Concavity (cont.)

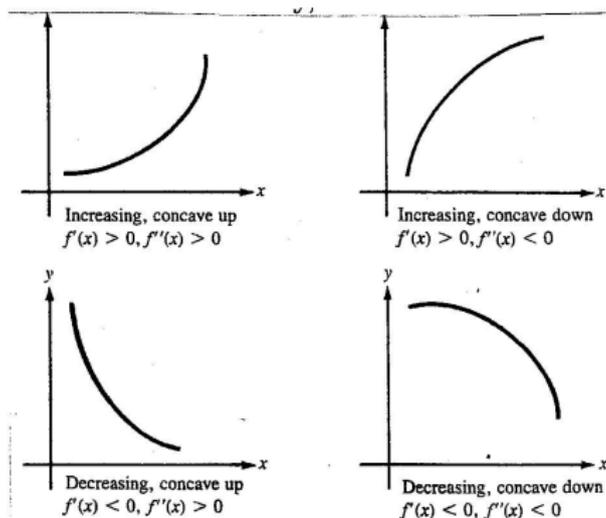
We have to be careful, though. You should not confuse the concavity of a curve with its increase or decrease.

A curve that is concave upward on an interval may be either increasing or decreasing on that interval.

Similarly, a curve that is concave downward may be increasing or decreasing.



Test for Concavity (cont.)



Test for Concavity (cont.)

Here is a table listing some useful functional forms for $x > 0$:

$f(x)$	$f'(x)$	$f''(x)$	Slope	Curvature
$\log(x)$	$\frac{1}{x}$	$-\frac{1}{x^2}$	Increasing	C-down
\sqrt{x}	$\frac{1}{2\sqrt{x}}$	$-\frac{1}{4x^{\frac{3}{2}}}$	Increasing	C-down
x^2	$2x$	2	Increasing	C-up
$\frac{1}{x}$	$-\frac{1}{x^2}$	$\frac{2}{x^3}$	Decreasing	C-up
$7 - x^2$	$-2x$	-2	Decreasing	C-down
$7x - x^2$	$7 - 2x$	-2	Increasing/Decreasing	C-down



Using the second derivative to find extrema

We can also use the second derivative to identify whether there is a point on the graph of a function at which the concavity of the function changes.

We usually call this point an *inflection point*.

If the second derivative of a function is defined at an inflection point, its value there must be zero.

Inflection points can also occur at points in the domain of the function where the second derivative is undefined.



Using the second derivative to find extrema (cont.)

Points on the graph of a function $f(x)$ where $f'(x) = 0$ or $f'(x)$ does not exist are sometimes called *first-order critical points*.

A number c in the domain of f where $f''(c) = 0$ or $f''(x)$ does not exist is called a *second-order critical value*, and the corresponding point $(c, f(c))$ on the graph of f is called a *second-order critical point*.

Second-order critical points are to inflection points as first-order critical points are to local extrema.

In particular, every inflection point is a second-order critical point.



The Second Derivative Test

The second derivative can also be used to classify the first-order critical points of a function as local maxima or local minima.

The procedure is usually known as the *second derivative test*.

The Second Derivative Test.

Suppose $f'(a) = 0$.

If $f''(a) > 0$, then f has a local minimum at $x = a$.

If $f''(a) < 0$, then f has a local maximum at $x = a$.

However, if $f''(a) = 0$, the test is inconclusive and f may have a local maximum, a local minimum, or no local extremum at all at $x = a$.



The Second Derivative Test (cont.)

Example

Let $f(x) = 2x^3 + 3x^2 - 12x - 7$. We will use the second derivative test to find the local extrema.

First, we calculate the first derivative of this function,

$$f'(x) = 6x^2 + 6x - 12$$

We can now re-write this expression as $6(x + 2)(x - 1)$ to see more easily how $f'(x) = 0$ when $x = -2$ and when $x = 1$.

And that their corresponding points $(-2, 13)$ and $(1, -14)$ are the first-order critical points of f .



The Second Derivative Test (cont.)

Next, to test these points we compute the second derivative

$$f''(x) = 12x + 6$$

and we evaluate it at $x = -2$ and $x = 1$.

Since $f''(-2) = -18 < 0$, it follows that the critical point $(-2, 13)$ is a local maximum.

And since $f''(1) = 18 > 0$, it follows that the critical point $(1, -14)$ is a local minimum.



Absolute Maxima and Minima

We already know how to find local extrema, right?

But suppose we want to find now the absolute maximum (or minimum) of some function f .

We already know that places where the derivative is equal to zero are important, so we can use the notion of a critical point to locate the absolute extremum.



Absolute Maxima and Minima (cont.)

Locating Absolute Extrema on a Bounded Interval. Suppose f is differentiable on a closed, bounded interval $a \leq x \leq b$.

Step 1 Locate all critical points x_1, \dots, x_k within the interval.

Step 2 Calculate the values $y = f(x)$ at these critical points and at the endpoints $x = a$ and $x = b$.

Step 3 Compare the values $f(a), f(x_1), \dots, f(x_k), f(b)$. The largest of these is the absolute maximum for f on the interval $a \leq x \leq b$; the smallest is the absolute minimum.



Absolute Maxima and Minima (cont.)

Notice that when the interval on which we wish to maximize or minimize a continuous function is not of the form $a \leq x \leq b$, this procedure no longer applies.

This is because there is no longer any guarantee that the function actually has an absolute maximum or minimum on the interval in question.

On the other hand, if an absolute extremum does exist and the function is continuous on the interval, the absolute extremum will still occur at a local extremum or endpoint contained in the interval.



Absolute Maxima and Minima (cont.)

Therefore, to find the absolute extrema of a continuous function on an interval that is not of the form $a \leq x \leq b$, we still evaluate the function at all the critical points and endpoints that are contained in the interval.

It turns out that in many cases, the interval at hand will contain only one first-order critical value of the function.

When this happens, we can use the second derivative test to identify its absolute extremum, even though this test is really a test for only local extrema.

The reason is that, in this special case, every local maximum or minimum is necessarily also an absolute maximum or minimum.



Second Derivative Test for Absolute Extrema

The Second Derivative Test for Absolute Extrema. Suppose that f is continuous on an interval on which $x = a$ is the only first-order critical value and that $f'(a) = 0$.

If $f''(a) > 0$, then $f(a)$ is the absolute minimum of f on the interval.

If $f''(a) < 0$, then $f(a)$ is the absolute maximum of f on the interval.

The second derivative test for absolute extrema can be applied to any interval, whether closed or not.

The only requirement is that the function be continuous and have only one critical value on the interval.



Calculus: Applications and Integration

- 1 Applications of the Derivative
 - Mean Value Theorems
 - Monotone Functions
- 2 Integration
 - Antidifferentiation: The Indefinite Integral
 - Definite Integrals



Introduction

Differentiation is one of the two central concepts in calculus.

The other one is *integration*.

The first operation, calculating the derivative of a function, is used to find the exact slope of the tangent to a function's curve at any specified point along the curve.

The second operation, integration, can be used to find the area under the graph of a continuous function.



Introduction (cont.)

Both problems, finding the tangent (to) and the area (under) a curve, can be solved following different and independent paths, but at the end of the day, they are inter-related as the calculation of areas can be reduced to the calculation of the *antiderivative* of a function.

In practical terms, integration comes in handy when the derivative of a function is known and the goal is to find the function itself.

For example, say we know the rate of inflation, we may wish to use this information to estimate future prices.



Antiderivative

In a sense, integration is the inverse of differentiation, so a good starting point is to say a few words about reversing the process of differentiation.

An *indefinite integral* or antiderivative of a function $f(x)$ is any function $F(x)$ whose derivative is the original function

$$\frac{dF}{dx} = f(x).$$

For example, what is an indefinite integral for $f(x) = x^2$? We want to find a function $F(x)$ whose derivative is $f(x) = x^2$.



Antiderivative (cont.)

This is a new and unfamiliar problem.

However, we do know a lot about derivatives, so it is not hard to make an informed guess.

Recall the power rule: the derivative of any power x^r is a constant times a power of one lower degree.

If we want to find a function $F(x)$ such that $\frac{dF}{dx} = x^2$, this suggests we consider something like x^3 . Now

$$\frac{d}{dx}(x^3) = 3x^2$$

so $F(x) = x^3$ is not quite the correct choice.



Antiderivative (cont.)

However, if we choose $F(x) = \frac{1}{3}x^3$ then

$$\frac{dF}{dx} = \frac{d}{dx} \left(\frac{1}{3}x^3 \right) = \frac{1}{3} \frac{d}{dx} (x^3) = \frac{1}{3} (3x^2) = x^2.$$

Thus, $F(x) = \frac{1}{3}x^3$ is an indefinite integral of $f(x) = x^2$.



Antiderivative (cont.)

Example

Let $F(x) = \frac{1}{3}x^3 + 5x + 2$. Can $F(x)$ be the antiderivative of $f(x) = x^2 + 5$?

Well, $F(x)$ is an antiderivative of $f(x)$ if and only if $F'(x) = f(x)$.

As we differentiate F we find that

$$\frac{dF}{dx} = x^2 + 5 = f(x).$$



Antiderivative (cont.)

A function $f(x)$ may have more than one indefinite integral.

For example, one antiderivative of the function $f(x) = 3x^2$ is $F(x) = x^3$, since

$$\frac{d}{dx}(x^3) = 3x^2 = f(x).$$

But so is $G(x) = x^3 + 12$, since the derivative of the constant 12 is zero and

$$\frac{d}{dx}(x^3 + 12) = 3x^2 = f(x).$$



Antiderivative (cont.)

In general, if F is one antiderivative of f , any function obtained by adding a constant to F is also an antiderivative of f .

This leads us to ask: How can we find *all* indefinite integrals of a function $f(x)$?

Given one indefinite integral $F(x)$, there are others of the form

$$F(x) + \text{constant}.$$

In fact, these are all there are.



Uniqueness of the Indefinite Integral

In other words, if $f(x)$ has an indefinite integral at all, this indefinite integral is “unique up to an added constant.”

For example, $f(x) = x^2$ has $F(x) = \frac{1}{3}x^3$ as an indefinite integral; all possible indefinite integrals are of the form

$$\frac{1}{3}x^3 + c \quad (c \text{ any constant}).$$

Uniqueness of the Indefinite Integral. Suppose that $f(x)$ has an indefinite integral $F(x)$ on an interval. Then all other indefinite integrals of $f(x)$ have the form $F(x) + c$, where c is an arbitrary constant.



Uniqueness of the Indefinite Integral (cont.)

Proof. Suppose $G(x)$ is any other indefinite integral for $f(x)$. Then

$$\frac{dF}{dx} = f(x) \quad \text{and} \quad \frac{dG}{dx} = f(x).$$

Their difference $H(x) = G(x) - F(x)$ has derivative zero because

$$\frac{dH}{dx} = \frac{d}{dx}(G(x) - F(x)) = \frac{dG}{dx} - \frac{dF}{dx} = f(x) - f(x) = 0.$$

That is, the rate of change of $H(x)$ is identically zero for all x . Intuitively, we can see that this forces $H(x)$ to be a constant function, $H(x) = c$ for all x (c some constant).

Because $G(x) - F(x) = c$, we obtain $G(x) = F(x) + c$.



Integral Notation

Indefinite integrals of $f(x)$ are indicated by the special symbol

$$\int f(x) dx$$

The symbol \int is called an *integral sign* and indicates that you are to find the most general form of the antiderivative of the function following it.

The integral sign is merely an elongated “s”, and stands for “the sum of”. In the expression

$$\int f(x) dx,$$

the function $f(x)$ that is to be integrated is called the *integrand*.



Integral Notation (cont.)

The symbol dx that appears after the integrand may seem mysterious. Its role is to indicate that x is the variable with respect to which the integration is to be performed.

The symbol d merely means “a little bit of.” Thus, dx means a little bit of x , and $\int dx$ means the sum of all the little bits of x .

Hence, it is customary to write

$$\int f(x)dx = F(x) + c$$

to express the fact that every antiderivative of $f(x)$ is of the form $F(x) + c$.



Integral Notation (cont.)

Example

Let $f(x) = 3x^2$.

We can express the fact that every antiderivative of f is of the form $x^3 + c$ by writing $\int 3x^2 dx = x^3 + c$.

In the above expression

$$\int f(x) dx = F(x) + c$$

the function f is the integrand, and the (unspecified) constant c that is added to $F(x)$ to give the most general form of the antiderivative is known as the *constant of integration*.



The Indefinite Integral

Here is a restatement of the definition of the indefinite integral (or antiderivative) in integral notation.

The Indefinite Integral.

$$\int f(x)dx = F(x) + c$$

if and only if

$$F'(x) = f(x)$$

for every x in the domain of f .



Integration Rules

As stated before, integration is the reverse of differentiation.

Therefore, many rules for integration can be obtained by stating the corresponding rules for differentiation in reverse.

Example

Let $f(x) = x^7$. An indefinite integral of f is given by $F(x) = \frac{1}{8}x^8$:

$$\frac{dF}{dx} = \frac{d}{dx} \left(\frac{1}{8}x^8 \right) = x^7.$$



Integration Rules (cont.)

More generally, by reversing the differentiation formula

$$\frac{d}{dx} \left(\frac{1}{r+1} x^{r+1} \right) = \frac{r+1}{r+1} x^r = x^r$$

we see that the function $f(x) = x^r$ has an indefinite integral of the form

$$F(x) = \frac{x^{r+1}}{r+1}$$

except when $r = -1$, in which case $\frac{1}{r+1}$ is undefined.



Power Rule for Integrals

The Power Rule for Integrals. For $r \neq -1$,

$$\int x^r dx = \frac{x^{r+1}}{r+1} + c$$

That is, to integrate x^r (for $r \neq -1$), increase the power of x by 1 and divide by the new power.



Power Rule for Integrals (cont.)

Example

$$\text{Let } f(x) = x^{\frac{3}{5}}$$

To find the indefinite integral we should simply increase the power of x by 1 and then divide by the new power to get

$$\int x^{\frac{3}{5}} dx = \frac{5}{8} x^{\frac{8}{5}} + c$$

Example

$$\text{Let } f(x) = \frac{1}{x^3}$$

The function f has an indefinite integral

$$F(x) = \frac{x^{-3+1}}{-3+1} = \frac{x^{-2}}{-2} = \frac{-1}{2x^2}.$$



Power Rule for Integrals (cont.)

Example

Let $f(x) = \sqrt{x}$

The function f has an indefinite integral

$$F(x) = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3}x^{\frac{3}{2}}.$$

This is the first of several integration rules we shall derive.



Power Rule for Integrals (cont.)

To avoid errors, some special indefinite integrals are worth noting.

- The constant function $f(x) = x^0$ has indefinite integral

$$F(x) = \frac{x^1}{1} = x.$$

- The function $f(x) = x^{-1}$ has indefinite integral

$$\int \frac{1}{x} dx = \ln |x| + c$$

When x is negative, it turns out that the function $\ln |x|$ is an antiderivative of $\frac{1}{x}$, because if x is negative, then $|x| = -x$ and

$$\frac{d}{dx}(\ln |x|) = \frac{d}{dx}[\ln(-x)] = \left(\frac{1}{-x}\right)(-1) = \frac{1}{x}.$$



Power Rule for Integrals (cont.)

- The function e^x has indefinite integral

$$\int e^x dx = e^x + c$$

Integration of the exponential function e^x is trivial since e^x is its own derivative.



Constant Multiple Rule for Integrals

The Constant Multiple Rule for Integrals.

For any constant c ,

$$\int cf(x)dx = c \int f(x)dx$$

That is, the integral of a constant times a function is equal to the constant times the integral of the function.



Constant Multiple Rule for Integrals (cont.)

Again, some special indefinite integrals are worth noting.

- The constant function whose value is k , $f(x) = k$ has $F(x) = kx$ for an indefinite integral. This gives the integration formula

$$\int k dx = kx + c \quad (k \text{ any constant})$$

- Taking $k = 1$ or $k = 0$ we get two commonly encountered special cases:

$$\int 1 dx = x + c$$

$$\int 0 dx = c$$



Sum Rule for Integrals

The Sum Rule for Integrals.

$$\int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx$$

That is, the integral of a sum is the sum of the individual integrals.



Sum Rule for Integrals (cont.)

Example

Let $f(x) = 3e^x + \frac{2}{x} - \frac{1}{2}x^2$.

The integral of f is given by

$$\begin{aligned}\int \left(3e^x + \frac{2}{x} - \frac{1}{2}x^2 \right) dx &= 3 \int e^x dx + 2 \int \frac{1}{x} dx - \frac{1}{2} \int x^2 dx \\ &= 3e^x + 2 \ln |x| - \frac{1}{6}x^3 + c\end{aligned}$$



Integration of Products and Quotients

You may have noticed that we did not look at any general rules for the integration of products and quotients.

This is because there are no general rules.

Occasionally, we may be able to rewrite a product or a quotient in a form in which you can integrate it using the techniques we have learned today.



Integration of Products and Quotients (cont.)

Example

$$\text{Let } f(x) = \frac{3x^5 + 2x - 5}{x^3}.$$

We can rewrite f as

$$\frac{3x^5 + 2x - 5}{x^3} = 3x^2 + \frac{2}{x^2} - \frac{5}{x^3} = 3x^2 + 2x^{-2} - 5x^{-3}$$

and then integrate term by term:

$$\begin{aligned} \int \frac{3x^5 + 2x - 5}{x^3} dx &= \int (3x^2 + 2x^{-2} - 5x^{-3}) dx \\ &= x^3 - 2x^{-1} + \frac{5}{2}x^{-2} + c \\ &= x^3 - \frac{2}{x} + \frac{5}{2x^2} + c \end{aligned}$$



Practical Application

Lets look at a concrete problem in which the rate of change of a quantity is known and the goal is to find an expression for the quantity itself.

Since the rate of change is the derivative of the quantity, we can find the expression for the quantity itself by antidifferentiation.

Example

A manufacturer has found that marginal cost is $3q^2 - 60q + 400$ dollars per unit when q units have been produced. The total cost of producing the first 2 units is \$900. What is the total cost of producing the first 5 units?



Practical Application

First, note that the marginal cost is the derivative of the total cost function $C(q)$.

Thus, $C'(q) = 3q^2 - 60q + 400$ and so $C(q)$ must be the antiderivative

$$C(q) = \int C'(q) dq = \int (3q^2 - 60q + 400) dq = q^3 - 30q^2 + 400q + k$$

for some constant k .

The value of k is determined by the fact that $C(2) = 900$:

$$900 = (2)^3 - 30(2)^2 + 400(2) + k \quad \text{or} \quad k = 212$$



Practical Application (cont.)

Therefore,

$$C(q) = q^3 - 30q^2 + 400q + 212$$

and the cost of producing the first 5 units is

$$C(5) = (5)^3 - 30(5)^2 + 400(5) + 212$$

which amounts to \$1,587.



Calculus: Applications and Integration

- 1 Applications of the Derivative
 - Mean Value Theorems
 - Monotone Functions
- 2 Integration
 - Antidifferentiation: The Indefinite Integral
 - Definite Integrals



Definite Integral: Definition

We have already learned the techniques of antidifferentiation.

Now, we are going to define the *definite integral* of a function in terms of its indefinite integral and use it to calculate areas.

We start by defining the definite integral of a function $f(x)$ over the interval $a \leq x \leq b$. It is indicated by the symbol

$$\int_a^b f(x) dx$$

The number b is called the “upper limit,” and a is the “lower limit” of the definite integral.

In contrast with the indefinite integral, which is a function, the definite integral is a number.



Calculating the Definite Integral

Rule for Calculating the Definite Integral. Let $\int_a^b f(x)dx$

Step 1 Find an indefinite integral $F(x)$ of $f(x)$ on the interval $a \leq x \leq b$.

Step 2 Compute the value of the indefinite integral $F(x)$ at the upper limit $x = b$; this is $F(b)$.

Step 3 Compute the value of $F(x)$ at the lower limit $x = a$; this is $F(a)$.

Step 4 Subtract the number in Step 3 from the number in Step 2; this is the definite integral

$$\int_a^b f(x)dx = F(b) - F(a)$$



Calculating the Definite Integral (cont.)

In carrying out this procedure it does not matter which indefinite integral $F(x)$ we use; the number obtained in Step 4 will always be the same.

To see why, observe that if $G(x)$ is some other indefinite integral of $f(x)$, then $G(x) = F(x) + c$ for some constant c .

Therefore

$$\begin{aligned}G(b) - G(a) &= (F(b) + c) - (F(a) + c) \\ &= F(b) + c - F(a) - c \\ &= F(b) - F(a)\end{aligned}$$

and the added constant c cancels out of the final result.



Calculating the Definite Integral (cont.)

Example

Let $\int_2^6 x dx$

In order to evaluate this definite integral, we follow the steps delineated above.

First, note that $F(x) = \frac{1}{2}x^2$ is an indefinite integral of $f(x) = x$.

Next, in Step 2 we have:

The value of F at the upper limit $b = 6$ is

$$F(6) = \frac{1}{2}(6)^2 = 18$$



Calculating the Definite Integral (cont.)

In Step 3 we have:

The value of F at the lower limit $a = 2$ is

$$F(2) = \frac{1}{2}(2)^2 = 2$$

Finally, in step 4 we obtain the definite integral by taking the difference of these values,

$$\int_2^6 x dx = F(6) - F(2) = 18 - 2 = 16$$



Calculating the Definite Integral (cont.)

With practice, the steps in this procedure can be combined by introducing a useful symbol used universally to indicate the difference of endpoint values of a function $F(x)$.

If we define $F(x)\Big|_a^b = F(b) - F(a)$, the procedure for calculating the definite integral takes on a simple form.

$$\int_a^b f(x) dx = F(x)\Big|_a^b$$

where $F(x)$ is any indefinite integral of f .



Calculating the Definite Integral (cont.)

Example

Let $\int (3 - x^2) dx$ and $\int_{-1}^2 (3 - x^2) dx$.

First we have that

$$\int (3 - x^2) dx = 3x - \frac{1}{3}x^3 + c$$

To evaluate the definite integral, we may choose the simplest value for the constant in the indefinite integral - namely, zero.



Calculating the Definite Integral (cont.)

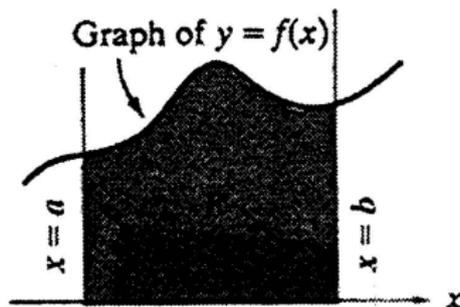
Then using the indefinite integral $F(x) = 3x - \frac{1}{3}x^3$, we calculate the definite integral:

$$\begin{aligned}\int_{-1}^2 (3 - x^2) dx &= \left(3x - \frac{1}{3}x^3\right) \Big|_{-1}^2 \\ &= \left[3(2) - \frac{1}{3}(2)^3\right] - \left[3(-1) - \frac{1}{3}(-1)^3\right] \\ &= \left(6 - \frac{8}{3}\right) - \left(-3 - \frac{1}{3}(-1)\right) \\ &= \frac{10}{3} - \left(-\frac{8}{3}\right) = 6\end{aligned}$$



Areas

Finally, we now turn to the problem of area. We shall consider general regions in the plane such as the one shown in the following figure.



Such a region R is obtained by taking the part of the plane bounded by a continuous curve $y = f(x)$, the x axis, and the two vertical lines $x = a$ and $x = b$.



Fundamental Theorem of Calculus

We will assume that $a < b$, as in the figure, and restrict the attention to the case where $f(x) \geq 0$; so the curve $y = f(x)$ stays above the horizontal axis.

Then the area, $\text{Area}(R)$, can be evaluated by applying a truly remarkable result connecting areas and integrals.

Fundamental Theorem of Calculus. For any plane region R of the form shown in the previous figure, bounded by the graph of a continuous function $y = f(x)$, the area of R is equal to the definite integral of $f(x)$ for $a \leq x \leq b$:

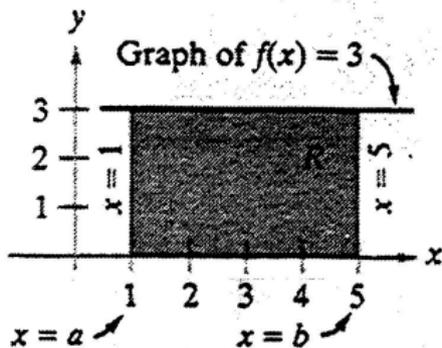
$$\text{Area}(R) = \int_a^b f(x) dx$$



Fundamental Theorem of Calculus (cont.)

Example

Let $f(x) = 3$, for $1 \leq x \leq 5$. The region R bounded by the graph of f is a rectangle.



Fundamental Theorem of Calculus (cont.)

We can now use the Fundamental Theorem of Calculus to get the area for R . Its upper boundary is given by the constant function $f(x) = 3$, and the vertical lines are given by $a = 1$ and $b = 5$.

Using the Fundamental Theorem of Calculus and the indefinite integral $F(x) = 3x$ we obtain

$$\begin{aligned} \text{Area}(R) &= \int_a^b f(x) dx = \int_1^5 3 dx \\ &= 3x \Big|_1^5 = 15 - 3 \\ &= 12 \end{aligned}$$

Note that this calculation agrees with the usual formula:
 $\text{Area} = \text{height} \times \text{width}$, because the height is 3, and the width is $5-1=4$.

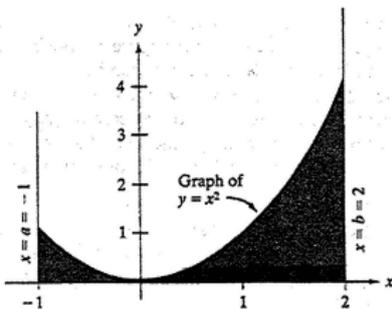


Fundamental Theorem of Calculus (cont.)

Lets examine now a region whose area cannot be calculated by the usual methods of plane geometry (finding its area would be difficult without the Fundamental Theory of Calculus).

Example

Let $f(x) = x^2$, for $-1 \leq x \leq 2$. The area R of the region lying above the interval in the x axis and below the graph of f looks like this.



Fundamental Theorem of Calculus (cont.)

The upper boundary of R is given by $y = x^2$, and the vertical boundary lines are given by $a = -1$ and $b = 2$.

We have already found indefinite integrals for $f(x) = x^2$ at the beginning of today's class.

We may use the particular indefinite integral $F(x) = \frac{1}{3}x^3$.



Fundamental Theorem of Calculus (cont.)

Applying the Fundamental Theorem we obtain,

$$\begin{aligned} \text{Area}(R) &= \int_{-1}^2 x^2 dx \\ &= \left. \frac{1}{3}x^3 \right|_{-1}^2 \\ &= \left(\frac{1}{3}(2)^3 \right) - \left(\frac{1}{3}(-1)^3 \right) \\ &= \left(\frac{8}{3} \right) - \left(-\frac{1}{3} \right) \\ &= \frac{9}{3} = 3 \end{aligned}$$

