A Mathematical Appendix

Proposition 1. The first part of the claim is easy to see: since nobody is expected to contribute toward the action, no individual supporter has an incentive to contribute $x > 0$ because doing so would not cause the action to occur, and would thus be merely a cost. Since the action is not going to take place, no individual opponent has an incentive to spend $y > 0$ to block it. For the second claim, suppose $\pi = 1$ in equilibrium. If $Y > 0$, then any opponent who spends $y > 0$ could profit by deviating to $y = 0$ because the action will still be implemented, and he will consume more privately. Thus, no opponent can be spending, so $Y = 0$ in that equilibrium. This implies that $X = \theta$ or else any supporter could profit by spending $x' < x$ as long as $X \geq \theta$ holds. But if $X = \theta$, then any opponent could profit by spending some $y > 0$, no matter how small, and derail the action. $\square$

Proposition 2. Fix Q and consider the *ex ante* per-period equilibrium payoff for some player i:

$$u_i(\sigma) = \underbrace{\sum_{k=0}^{Q-2}(1)f(k)}_{\text{no action regardless of } i\text{'s vote}} + \underbrace{[p(1+a-x(Q)) + (1-p)(1)]\,f(Q-1)}_{\text{action occurs only if } i \text{ votes in favor}}$$
$$+ \underbrace{\sum_{k=Q}^{N-1}[p(1+a-x(k+1)) + (1-p)(1-a)]\,f(k),}_{\text{action occurs regardless of } i\text{'s vote}}$$

which simplifies to $u_i(\sigma) = 1 + \zeta_{\mathrm{w}}(Q)$, where $\zeta_{\mathrm{w}}(Q) < a$ is defined in the proposition. The equilibrium payoff from this strategy is $u_i(\sigma)/(1-\delta)$.

Consider first the implementation stage. Suppose first that $q \geq Q$ so the action should take place. Any supporter who deviates from $x(q)$ will cause the action to fail, making this unprofitable. Furthermore, there is no need to contribute more than the minimum necessary to implement it. Since this is an at-cost implementation, any opponent who invests against the action some y arbitrarily close to zero can derail it but then the game will revert to the unconditional SPE. Doing so would not be profitable if

$$1 - y + \frac{\delta(1)}{1-\delta} \leq 1 - a + \frac{\delta u_i(\sigma)}{1-\delta}$$

for $y \to 0$. We can rewrite this as $(1-\delta)a \leq \delta\zeta_{\mathrm{w}}(Q)$. The necessary condition for this inequality to work is $\zeta_{\mathrm{w}}(Q) > 0$. This condition is also sufficient to ensure that there exists δ high enough to satisfy the inequality. In that case any $\delta \geq \underline{\delta}_{\mathrm{w}}(Q)$, where the latter is defined in (1), will work. Note in particular that $a + \zeta_{\mathrm{w}}(Q) > 0$, and that $\zeta_{\mathrm{w}}(Q) > 0$ ensures that $\underline{\delta}_{\mathrm{w}}(Q) < 1$, so solutions exist.

Suppose now that $q < Q$ so the action is not supposed to take place. If $q < \theta$ then the action cannot be imposed because the supporters do not have enough resources to do so. Any attempt to do so would fail and would be unprofitable. If, on the other hand, $q \geq \theta$, then the (self-declared) supporters can implement the action if they wish to (because opponents are not spending anything against it) but doing so would result in the reversion to the unconditional SPE. This deviation will not be profitable if:

$$1 + a - x(q) + \frac{\delta(1)}{1-\delta} \leq 1 + \frac{\delta u_i(\sigma)}{1-\delta},$$

which we can rewrite as $(1-\delta)(a - x(q)) \leq \delta\zeta_{\mathrm{w}}(q)$. Recall now that the condition that prevents the deviation of an opponent is $(1-\delta)a \leq \delta\zeta_{\mathrm{w}}(q)$. Thus, if an opponent will not deviate, then supporters certainly would not do so in the implementation phase.

We now turn to the voting stage. Consider now a player who learns that he opposes the action. If he votes sincerely, then his expected payoff in this period will be:

$$u_o(\sigma) = \sum_{k=0}^{Q-2}(1)f(k) + (1)f(Q-1) + \sum_{k=Q}^{N-1}(1-a)f(k) = 1 - a(1 - F(Q-1)).$$

If he votes, falsely, in support of the action and then behaves as a supporter (so the action gets implemented), his payoff in this current period will be

$$\sum_{k=0}^{Q-2}(1)f(k) + (1 - a - x(Q))f(Q-1) + \sum_{k=Q}^{N-1}(1 - a - x(k+1))f(k) < u_o(\sigma).$$

Since this deviation will not be detected (and would not have been punished if it had), the game will continue as before. Thus, this deviation cannot be profitable. Suppose he votes for the action but then derails it. The optimal way of doing so would be to just consume privately — the other supporters, incorrectly expecting him to contribute $x(q)$ toward the "at cost" implementation would end up with $X < \theta$. Thus, his best possible payoff from a deviation for the current period will be 1. However, this deviation is observable and will be punished. This deviation will not be profitable if $1 + \delta/(1-\delta) \leq 1 - a(1 - F(Q - 1)) + \delta u_i(\sigma)/(1-\delta)$. This reduces to $(1-\delta)a(1-F(Q-1)) \leq \delta\zeta_{\mathrm{w}}(Q)$. However, since $(1 - F(Q-1))a < a$, this condition will be satisfied whenever the condition that prevents an opponent (who has voted sincerely) from derailing the implementation. (This makes sense: an insincere vote will increase the probability of having to derail the action, and thus the probability of the sanction relative to a sincere vote against it followed by derailing.)

Finally, consider a player who learns that he supports the action. If he votes sincerely, then his expected payoff will be:

$$u_s(\sigma) = \sum_{k=0}^{Q-2}(1)f(k) + (1 + a - x(Q))f(Q-1) + \sum_{k=Q}^{N-1}(1 + a - x(k+1))f(k).$$

If he deviates and votes insincerely and then does not derail the action (he has no incentive to vote insincerely and derail it), his payoff would be

$$u_s(\sigma') = \sum_{k=0}^{Q-2}(1)f(k) + (1)f(Q-1) + \sum_{k=Q}^{N-1}(1+a)f(k).$$

Since this deviation will go undetected, the game continues as before. Thus, the necessary and sufficient condition for this deviation to be unprofitable is $u_s(\sigma) - u_s(\sigma') \geq 0$, or

$$(a - x(Q))f(Q-1) \geq \sum_{k=Q}^{N-1} x(k+1)f(k),$$

which we can rewrite as (SC). This exhausts the possible deviations and completes the proof. $\square$

Lemma 1. We begin by showing that unconstrained maximization selects the complete information social optimum; that is $Q_{\mathrm{u}} = Q^*$. The payoff function will be increasing at Q if, and only if, $U(Q+1) - U(Q) = \zeta_{\mathrm{u}}(Q+1) - \zeta_{\mathrm{u}}(Q) > 0$, and decreasing if the difference is negative. We now obtain:

$$\begin{aligned}
&\zeta_{\mathrm{u}}(Q+1) - \zeta_{\mathrm{u}}(Q) \\
&\quad = p\left(a - \frac{\theta}{Q+1}\right) f(Q) - p\left(a - \frac{\theta}{Q}\right) f(Q-1) - \left[(2p-1)a - \frac{p\theta}{Q+1}\right] f(Q) \\
&\quad = (1-p)af(Q) - p\left(a - \frac{\theta}{Q}\right) f(Q-1).
\end{aligned}$$

Thus, $\zeta_{\mathrm{u}}(Q+1) - \zeta_{\mathrm{u}}(Q) > 0 \Leftrightarrow (1-p)af(Q) > p\left(a - \frac{\theta}{Q}\right) f(Q-1)$. The latter inequality is:

$$\begin{gathered}
(1-p)a\binom{N-1}{Q}p^{Q}(1-p)^{N-Q-1} > p\left(a - \frac{\theta}{Q}\right)\binom{N-1}{Q-1}p^{Q-1}(1-p)^{N-Q} \\
a\binom{N-1}{Q} > \left(a - \frac{\theta}{Q}\right)\binom{N-1}{Q-1} \\
\frac{a}{Q} > \frac{a - \theta/Q}{N-Q},
\end{gathered}$$

which yields

$$Q < \frac{N + \theta/a}{2} \equiv \widetilde{Q}.$$

Thus, the payoff is strictly increasing for all $Q < \widetilde{Q}$, and strictly decreasing for all $Q > \widetilde{Q}$, which implies that the unconstrained optimum is at $Q_{\mathrm{u}} = \lceil \widetilde{Q} \rceil = Q^*$. Clearly, if $\theta \leq Q^*$, then the first constraint will not be binding; otherwise, $Q_{\mathrm{u}} = \theta$ as long as the second constraint is not binding. We now turn to investigating the conditions under which it will.

We can rewrite (SC) as

$$\frac{a}{\theta} \geq \sum_{k=0}^{N-Q}\left[\frac{(N-Q)!(Q-1)!}{(Q+k)!(N-Q-k)!}\right]\left(\frac{p}{1-p}\right)^{k} \equiv T(p,Q). \tag{3}$$

Note that $a/\theta > 0$, but since

$$\frac{\partial T}{\partial p} = \sum_{k=0}^{N-Q}\left[\frac{(N-Q)!(Q-1)!}{(Q+k)!(N-Q-k)!}\right]\left[\frac{kp^{k-1}}{(1-p)^{k+1}}\right] > 0,$$

the inequality must be violated for p sufficiently high ($\lim_{p\to 1} T(p, Q) = \infty$ for any $Q < N$). On the other hand, $\lim_{p\to 0} T(p, Q) = 0$, and the inequality is satisfied for any Q.

Take now $Q_{\mathrm{u}} = \max\{Q^*, \theta\}$ so that the first constraint is satisfied. For p sufficiently low condition (SC) will be met (with $n(p) = 0$), but as we increase p, it must eventually fail. Since $T(p, Q)$ is continuous in p, there must exist some $\hat{p}$ where (3) is satisfied with equality, so that the condition will fail for any $p > \hat{p}$. We now show that it is necessary to increase Q to restore the condition. First, note that $T(p, Q)$ is strictly decreasing in Q. Since Q changes in discrete jumps, we can rewrite $T(p, Q+1) - T(p, Q) = D(p, Q)$ as:

$$D(p, Q) = \sum_{k=0}^{N-Q} \left[\frac{(N-Q-1)!(Q-1)!}{(Q+k+1)!(N-Q-k)!} \right] \left(\frac{p}{1-p} \right)^k [Q - (k+1)N] < 0,$$

where the inequality follows from the fact that the first two terms in the summation are positive but the third is negative for any $k \geq 0$.

We now show that it is possible to satisfy (3) at $p > \hat{p}$ by choosing some $Q > Q_{\mathrm{u}}$. For this, it is sufficient to establish that there exists $\varepsilon > 0$ such that $T(\hat{p} + \varepsilon, Q_{\mathrm{u}} + 1) < T(\hat{p}, Q_{\mathrm{u}})$. Since $T(p, Q) = T(p, Q+1) - D(p, Q)$, we can write this as:

$$T(\hat{p} + \varepsilon, Q_{\mathrm{u}} + 1) - T(\hat{p}, Q_{\mathrm{u}}) = T(\hat{p} + \varepsilon, Q_{\mathrm{u}} + 1) - T(\hat{p}, Q_{\mathrm{u}} + 1) + D(\hat{p}, Q_{\mathrm{u}}).$$

But since $\lim_{\varepsilon\to 0} [T(\hat{p} + \varepsilon, Q_{\mathrm{u}} + 1) - T(\hat{p}, Q_{\mathrm{u}} + 1)] = 0$ but $D(\hat{p}, Q_{\mathrm{u}}) < 0$, the fact that this difference is continuous in ε implies that that there exists $\hat{\varepsilon} > 0$ such that $T(\hat{p}+\varepsilon, Q_{\mathrm{u}}+1) - T(\hat{p}, Q_{\mathrm{u}} + 1) + D(\hat{p}, Q_{\mathrm{u}}) < 0$ for all $\varepsilon < \hat{\varepsilon}$. In other words, (3) must be satisfied at $T(\hat{p}+\varepsilon, Q_{\mathrm{u}}+1)$. Thus, the optimal quota for these values of p will be $Q_{\mathrm{u}}+1$, or $n(p) = 1$. Continuing in this way, we find that as p increases, $n(p)$ must increase by one unit in a step-wise manner as well until the quota reaches unanimity, in which case the condition will be satisfied regardless of the value of p because then $T(p, N) = 1/N < a/\theta$. □

Proposition 3. Fix Q and consider the voting phase assuming that players will contribute if the quota is met. With everyone contributing when they have to there is no incentive not to vote sincerely. If a supporter votes against the action, it will fail if he happens to be pivotal, and he will contribute if it gets implemented even without his vote. Clearly such a deviation cannot be profitable. If an opponent votes for the action, he will only cause it to be implemented if he happens to be pivotal, an unprofitable deviation. Thus, it is only necessary to ensure that the contribution is properly enforced.

Consider now the phase in which players have voted and there are $q \geq Q$ in support so the action should take place under the equilibrium strategies. Since $x = \theta/N$, any player who fails to contribute will derail the action. The consequences of not contributing x are the same regardless of how one has voted, so we can analyze the deviation in this phase of the stage game without reference to the vote of the player. It is easy to see that if an opponent can be induced to contribute, then a supporter will surely do so: the continuation game is the same for both and the current payoff from the equilibrium strategy is lower for the opponent. Thus, it is sufficient to provide an incentive to the opponent. If he does not contribute, the action will fail to take place, and the game will revert to the non-cooperative equilibrium. If the player follows the equilibrium strategy σ and contributes x, the action will take place now and in every future period in which the quota is met. To calculate the latter, we need the *ex ante* expected payoff to an arbitrary player (i.e., the expected payoff before he learns his preferences). Since the action takes place for any $q \geq Q$, the per-period expected payoff is:

$$u_i(\sigma) = \underbrace{\sum_{k=0}^{Q-2} (1) f(k)}_{\text{no action regardless of } i\text{'s vote}} + \underbrace{[p(1+a-x)+(1-p)(1)] f(Q-1)}_{\text{action occurs only if } i \text{ votes in favor}}$$
$$+ \underbrace{\sum_{k=Q}^{N-1} [p(1+a)+(1-p)(1-a)-x] f(k)}_{\text{action occurs regardless of } i\text{'s vote}},$$

which simplifies to:

$$u_i(\sigma) = 1 + p(a-x)f(Q-1) + \sum_{k=Q}^{N-1} [(2p-1)a-x] f(k).$$

Thus, the condition for an opponent to follow the equilibrium strategy and invest for the action today is:

$$1 - a - x + \frac{\delta u_i(\sigma)}{1-\delta} \geq 1 + \frac{\delta(1)}{1-\delta},$$

which we can rewrite as $\delta u_i(\sigma) \geq \delta + (1-\delta)(a+x)$, or $\delta \zeta_{\mathrm{u}}(Q) \geq (1-\delta)(a+x)$. Since $a + x + \zeta_{\mathrm{u}}(Q) > 0$, this yields $\delta \geq \underline{\delta}_{\mathrm{u}}(Q)$, with $\underline{\delta}_{\mathrm{u}}(Q)$ defined in (2).To ensure that $\underline{\delta}_{\mathrm{u}}(Q) < 1$, we require that $\zeta_{\mathrm{u}}(Q) > 0$, as stated.

Finally, we need to consider $q < Q$ when the action will not take place. Clearly, no opponent would contribute anything if the supporters follow the equilibrium strategy, so we only need to make sure that the supporters do so. If $q < \theta$, then the action is beyond the combined capabilities of the group. This deviation would result in wasted spending and no action, so it cannot be profitable. The only possibly tempting deviation is for them to implement the action, which they can do when $q \geq \theta$ (since the opponents are spending $Y = 0$). In this case, the action can take place now (with opponents consuming privately) but the game will revert to the private consumption SPE from the following period. The condition for supporters to follow their equilibrium strategy and not impose the action today is:

$$1 + \frac{\delta u_i(\sigma)}{1-\delta} \geq 1 + a - x(q) + \frac{\delta(1)}{1-\delta},$$

which simplifies to $\delta \zeta_{\mathrm{u}}(Q) \geq (1-\delta)(a - x(q))$. Since this inequality must hold for all realizations of $q < Q \leq N$ and because the RHS is increasing in q (since $x(q) = \theta/q$ is decreasing), it is necessary that it be satisfied at $q = N$. Thus, we end up with $\delta \zeta_{\mathrm{u}}(Q) \geq (1-\delta)(a-x)$. Recalling that the condition that prevents deviation by opponents is $\delta \zeta_{\mathrm{u}}(Q) \geq (1-\delta)(a+x)$, we conclude that whenever the latter is satisfied, the supporters will have no incentive to impose the action either. □

Lemma 2. Since $U(Q) = 1 + \zeta_{\mathrm{u}}(Q)$, the payoff function will be increasing at Q if, and only if, $U(Q+1) - U(Q) = \zeta_{\mathrm{u}}(Q+1) - \zeta_{\mathrm{u}}(Q) > 0$, and decreasing if the difference is negative. We now obtain:

$$
\begin{aligned}
&\zeta_{\mathrm{u}}(Q+1) - \zeta_{\mathrm{u}}(Q) \\
&\quad = p(a-x) f(Q) - p(a-x) f(Q-1) + [(2p-1)a - x]\,(F(Q) - F(Q-1)) \\
&\quad = p(a-x) f(Q) - p(a-x) f(Q-1) - [(2p-1)a - x]\, f(Q) \\
&\quad = (1-p)(a+x) f(Q) - p(a-x) f(Q-1) \\
&\quad = (1-p)(a+x)\binom{N-1}{Q} p^{Q}(1-p)^{N-Q-1} - p(a-x)\binom{N-1}{Q-1} p^{Q-1}(1-p)^{N-Q} \\
&\quad = p^{Q}(1-p)^{N-Q}\left[(a+x)\binom{N-1}{Q} - (a-x)\binom{N-1}{Q-1}\right] \\
&\quad = p^{Q}(1-p)^{N-Q}\left[(a+x)\frac{(N-1)!}{Q!(N-Q-1)!} - (a-x)\frac{(N-1)!}{(Q-1)!(N-Q)!}\right] \\
&\quad = \left[\frac{p^{Q}(1-p)^{N-Q}(N-1)!}{(Q-1)!(N-Q-1)!}\right]\left[\frac{a+x}{Q} - \frac{a-x}{N-Q}\right].
\end{aligned}
$$

Since the first bracketed term is always positive, it follows that

$$
\zeta_{\mathrm{u}}(Q+1) - \zeta_{\mathrm{u}}(Q) > 0 \Leftrightarrow \frac{a+x}{Q} - \frac{a-x}{N-Q} > 0.
$$

Solving the second inequality yields $(a+x)N > 2aQ$, which, after substituting $x = \theta/N$ ends in:

$$
Q < \frac{N + \theta/a}{2} \equiv \widetilde{Q}. \tag{4}
$$

Thus, if $Q < \widetilde{Q}$, then $U(Q+1) > U(Q)$, and the payoff function is increasing; but if $Q > \widetilde{Q}$, then $U(Q+1) < U(Q)$, so it is decreasing. Since for any $Q < \widetilde{Q}$ we would pick $Q+1$ for a higher payoff, it follows that the best possible payoff is at $Q_{\mathrm{u}} = \lceil \widetilde{Q} \rceil = Q^{*}$. $\square$

Proposition 4. We need to show that $U_{\mathrm{w}}(Q) = U_{\mathrm{u}}(Q) \Leftrightarrow \zeta_{\mathrm{w}}(Q) = \zeta_{\mathrm{u}}(Q)$. We can rewrite this equation as:

$$
\begin{aligned}
p\left(a - \frac{\theta}{Q}\right) f(Q-1) &+ \sum_{k=Q}^{N-1}\left[(2p-1)a - \frac{p\theta}{k+1}\right] f(k) \\
&= p\left(a - \frac{\theta}{N}\right) f(Q-1) + \sum_{k=Q}^{N-1}\left[(2p-1)a - \frac{\theta}{N}\right] f(k),
\end{aligned}
$$

which simplifies to:

$$
\sum_{k=Q}^{N-1}\left(\frac{1}{N} - \frac{p}{k+1}\right) f(k) = p\left(\frac{1}{Q} - \frac{1}{N}\right) f(Q-1). \tag{5}
$$

We need to prove (5) for an arbitrary Q, which we now do by induction. First, we show that it holds for $Q = N$. Since the summation term is zero (the lower bound exceeds the upper bound), it is sufficient to show that the right-hand side is zero too:

$$p\left(\frac{1}{N} - \frac{1}{N}\right) f(N-1) = 0.$$

For the inductive step, assume that (5) holds for some $Q > 1$. We now prove that the claim holds for $Q - 1$ as well. Rewriting the claim at $Q - 1$ yields:

$$\begin{aligned} p\left(\frac{1}{Q-1} - \frac{1}{N}\right) f(Q-2) &= \sum_{k=Q-1}^{N-1} \left(\frac{1}{N} - \frac{p}{k+1}\right) f(k) \\ &= \left(\frac{1}{N} - \frac{p}{Q}\right) f(Q-1) + \sum_{k=Q}^{N-1} \left(\frac{1}{N} - \frac{p}{k+1}\right) f(k), \end{aligned}$$

and since the claim is assumed to hold at Q, we substitute the second term using (5):

$$\begin{aligned} &= \left(\frac{1}{N} - \frac{p}{Q}\right) f(Q-1) + p\left(\frac{1}{Q} - \frac{1}{N}\right) f(Q-1) \\ &= \left(\frac{1-p}{N}\right) f(Q-1). \end{aligned}$$

Using the definition of the probability mass function, we can rewrite this as:

$$\left(\frac{1}{Q-1} - \frac{1}{N}\right)\binom{N-1}{Q-2} p^{Q-1}(1-p)^{N-Q+1} = \left(\frac{1}{N}\right)\binom{N-1}{Q-1} p^{Q-1}(1-p)^{N-Q+1}$$

which, after canceling the probability terms on both sides, yields

$$\left[\frac{N-Q+1}{N(Q-1)}\right]\left[\frac{(N-1)!}{(Q-2)!(N-Q+1)!}\right] = \left(\frac{1}{N}\right)\left[\frac{(N-1)!}{(Q-1)!(N-Q)!}\right]$$

and since $(N - Q + 1)! = (N - Q + 1)(N - Q)!$, and $(Q - 1)(Q - 2)! = (Q - 1)!$, cancellations on both sides yield

$$\frac{1}{(Q-1)!(N-Q)!} = \frac{1}{(Q-1)!(N-Q)!},$$

so the claim holds at $Q - 1$. By induction, it must hold for all $Q = 1, 2, \ldots, N$. $\square$

Proposition 5. Consider first the continuation game after the vote. Whenever the agent invests toward the action, it will succeed because $x_0(Q)$ ensures that any groups of opponents at $q \geq Q$ does not have enough resources left to derail it (even though supporters consume privately). If $q < Q$, the agent reimburses the players. Since everyone is consumes privately, no supporter can benefit by deviating and attempting to implement the action. Thus, neither opponents nor supporters have an incentive to deviate after the vote.

We now examine the voting stage given that the continuation game after the vote will be played according to the equilibrium strategies. Consider a player who learns that he is an opponent. If he votes sincerely, the action will be implemented if there are $q \geq Q$ supporters among the remaining $N-1$ players. If, on the other hand, he votes insincerely in support of the action, the agent would implement it when there are $q \geq Q-1$ supporters among the remaining players. Since the player would not be able to block the action whenever implementation is attempted, this deviation simply increases the likelihood of implementation and decreases the likelihood that he will get back some of his payment to the agent, making him strictly worse off.

Consider now a player who learns that he is a supporter. If he votes sincerely, the action will be implemented if there are $q \geq Q-1$ supporters among the remaining players, and his payoff would be:

$$U_s = \sum_{k=0}^{Q-2}(1-w)f(k) + \sum_{k=Q-1}^{N-1}(1-x_0(Q)+a)f(k) = 1-w+(a-\hat{x}(Q))\sum_{k=Q-1}^{N-1} f(k). \tag{6}$$

If he deviates and votes against the action, then the agent will attempt implementation when there are $q \geq Q$ supporters among the remaining players. Since he will not even try to implement the action with fewer votes, there is no point in the supporter spending anything toward it. Since the action will succeed in all other cases, his payoff will simply be:

$$\hat{U}_s = \sum_{k=0}^{Q-1}(1-w)f(k) + \sum_{k=Q}^{N-1}(1-x_0(Q)+a)f(k) = 1-w+(a-\hat{x}(Q))\sum_{k=Q}^{N-1} f(k) < U_s,$$

making this deviation unprofitable. Thus, any supporter has strict incentives to vote sincerely as well. □

Lemma 3. Delegating with Q means that every player contributes $x_0(Q)$, votes sincerely after observing his preference, and consumes privately. The agent commits the resources toward the action if there are $q \geq Q$ supporting votes and reimburses the players (net his fee) otherwise. The expected payoff to an opponent from a sincere vote is:

$$U_o = \sum_{k=0}^{Q-1}(1-w)f(k) + \sum_{k=Q}^{N-1}(1-x_0-a)f(k) = 1-w-(a+\hat{x}(Q))\sum_{k=Q}^{N-1} f(k), \tag{7}$$

where we used (NBC) to obtain

$$x_0(Q) - w = \frac{(1-w)(N-Q)+\theta}{2N-Q} \equiv \hat{x}(Q).$$

That is, $\hat{x}(Q) = x_0(Q) - w$ is the portion of the contribution that can be used for implementation. For any agreed-upon Q, the *ex ante* expected payoff to player i is:

$$
\begin{aligned}
U_{\mathrm{a}} = {} & p\left[1 - w + (a - \hat{x}(Q)) \sum_{k=Q-1}^{N-1} f(k)\right] \\
& + (1-p)\left[1 - w - (a + \hat{x}(Q)) \sum_{k=Q}^{N-1} f(k)\right] \\
= {} & 1 - w + p(a - \hat{x}(Q)) f(Q-1) + [(2p-1)a - \hat{x}(Q)] \sum_{k=Q}^{N-1} f(k), \qquad (8)
\end{aligned}
$$

where we used (6) for the payoff in case he turns out to be a supporter (with probability p), (7) for the payoff in case he turns out to be an opponent (with probability $1-p$). To see how U_{a} changes with Q, note that:

$$
\begin{aligned}
U_{\mathrm{a}}(Q+1) - U_{\mathrm{a}}(Q) = {} & (1-p)\,[a + \hat{x}(Q+1)]\, f(Q) \\
& - p\,[a - \hat{x}(Q)]\, f(Q-1) + [\hat{x}(Q) - \hat{x}(Q+1)] \sum_{k=Q}^{N-1} f(k)
\end{aligned}
$$

or, with $\gamma(Q) = \hat{x}(Q) - \hat{x}(Q+1)$, and $\beta(Q) = a(N-2Q) + N\hat{x}(Q+1) + Q\gamma(Q)$,

$$
\begin{aligned}
= {} & \beta(Q)\left[\frac{(N-1)!}{Q!(N-Q)!}\right] p^{Q}(1-p)^{N-Q} \\
& + \gamma(Q) \sum_{k=Q}^{N-1} \left[\frac{(N-1)!}{k!(N-1-k)!}\right] p^{k}(1-p)^{N-1-k},
\end{aligned}
$$

where we note that

$$
\gamma(Q) = \frac{(1-w)N - \theta}{(2N-Q)(2N-Q-1)} > 0.
$$

Thus, $U_{\mathrm{a}}(Q+1) - U_{\mathrm{a}}(Q) \gtreqless 0$ if, and only if,

$$
\begin{aligned}
& \beta(Q)\left[\frac{(N-1)!}{Q!(N-Q)!}\right] p^{Q}(1-p)^{N-Q} \\
& \qquad + \gamma(Q) \sum_{k=Q}^{N-1} \left[\frac{(N-1)!}{k!(N-1-k)!}\right] p^{k}(1-p)^{N-1-k} \gtreqless 0,
\end{aligned}
$$

or, after dividing both sides by $(N-1)!\, p^{Q}(1-p)^{N-Q}$, if, and only if,

$$
\frac{\beta(Q)}{Q!(N-Q)!} + \left[\frac{\gamma(Q)}{1-p}\right] \sum_{k=Q}^{N-1} \left[\frac{1}{k!(N-1-k)!}\right] \left(\frac{p}{1-p}\right)^{k-Q} \gtreqless 0.
$$

We re-index the summation term and multiply both sides by $Q!(N-Q)!$ to obtain:

$$\beta(Q) + \left[\frac{\gamma(Q)}{1-p}\right] \sum_{i=0}^{N-1-Q} \left[\frac{Q!(N-Q)!}{(Q+i)!(N-1-Q-i)!}\right] \left(\frac{p}{1-p}\right)^i \gtreqqless 0.$$

Using the definition of $\beta(Q)$, and dividing both sides by Q, we can rewrite this as:

$$\begin{aligned} &\frac{aN + (N-Q)\hat{x}(Q+1)}{Q} + \hat{x}(Q) \\ &\qquad + \left[\frac{\gamma(Q)}{Q(1-p)}\right] \sum_{i=0}^{N-1-Q} \left[\frac{Q!(N-Q)!}{(Q+i)!(N-1-Q-i)!}\right] \left(\frac{p}{1-p}\right)^i \gtreqqless 2a. \end{aligned} \tag{9}$$

Observe now that all three terms on the left-hand side of this inequality are positive. Furthermore, at $Q = 1$ the left-hand side is strictly larger because it reduces to aN plus three non-negative terms and $N \geq 2$. Thus, at $Q = 1$, the difference is strictly positive, so the payoff function is increasing. We now prove that the function is concave. For this, we only need to show that the left-hand side of (9) (which is essentially the first derivative of U_{a}) is decreasing in Q. First, note that

$$\hat{x}(Q) = \frac{(1-w)(N-Q)+\theta}{2N-Q} \Rightarrow \frac{\mathrm{d}\hat{x}(Q)}{\mathrm{d}Q} = \frac{\theta - (1-w)N}{(2N-Q)^2} < 0,$$

where the inequality follows from $w < \overline{w}$. This means that the first two terms on the left-hand side of (9) are decreasing in Q. If $Q > N-1$, then the third term is zero, and the claim holds. Consider then $Q \leq N-1$. We now wish to show that the third term decreases as well. Letting

$$D(Q) = \left(\frac{1}{1-p}\right)\left[\frac{\gamma(Q)}{Q}\right] \sum_{i=0}^{N-1-Q} \left[\frac{Q!(N-Q)!}{(Q+i)!(N-1-Q-i)!}\right] \left(\frac{p}{1-p}\right)^i,$$

we note that $D(Q+1) - D(Q) < 0$ if, and only if,

$$\begin{aligned} &\sum_{i=0}^{N-1-(Q+1)} \left[\frac{\gamma(Q+1)(Q+1)!(N-(Q+1))!}{(Q+1)(Q+1+i)!(N-1-(Q+1)-i)!}\right. \\ &\qquad\qquad \left. - \frac{\gamma(Q)Q!(N-Q)!}{Q(Q+i)!(N-1-Q-i)!}\right] \left(\frac{p}{1-p}\right)^i \\ &\qquad\qquad\qquad - \left[\frac{\gamma(Q)Q!(N-Q)!}{Q(N-1)!}\right] \left(\frac{p}{1-p}\right)^{N-1-Q} < 0. \end{aligned}$$

Since the second term is positive but is being subtracted, the inequality must hold whenever the summation is negative. Simplifying the summation, this requirement becomes:

$$\begin{aligned} &\sum_{i=0}^{N-1-(Q+1)} \left(\frac{p}{1-p}\right)^i \left[\frac{Q!(N-(Q+1))!}{(Q+i)!(N-1-(Q+1)-i)!}\right] \\ &\qquad\qquad \times \left[\frac{\gamma(Q+1)}{Q+1+i} - \frac{\gamma(Q)(N-Q)}{Q(N-1-Q-i)}\right] < 0, \end{aligned}$$

and since the first two multiplicative terms in this summation are positive, the inequality will certainly hold if the third term is negative. But since the first term in that expression is decreasing in i while the second one is increasing, it is sufficient to show that the inequality holds at $i = 0$, or that

$$\frac{\gamma(Q+1)}{Q+1} - \frac{\gamma(Q)(N-Q)}{Q(N-1-Q)} < 0,$$

because if this is true, then the term above will be negative for any $i > 0$ as well. Rearranging terms gives us:

$$Q(N-1-Q)\left[\gamma(Q+1) - \gamma(Q)\right] < \gamma(Q)N.$$

Using the definition of $\gamma(Q)$, dividing both sides by $(1-w)N - \theta$, and multiplying them by $2N - Q - 1$ gives us:

$$Q(N-1-Q)\left(\frac{1}{2N-Q-2} - \frac{1}{2N-Q}\right) < \frac{N}{2N-Q},$$

which, after simplifying and multiplying both sides by $2N - Q$, yields:

$$2Q(N-1-Q) < N(2N-Q-2)$$

or, after adding and subtracting QN on the right-hand side and re-arranging terms,

$$2(N-Q) < 2(N-Q)^2 + QN,$$

which simply reduces to

$$0 < 2(N-Q)(N-Q-1) + QN,$$

which holds because we have been considering the case with $Q \leq N - 1$. Thus, all three terms on the left-hand side of (9) are decreasing in Q. We conclude that the payoff function is concave, which implies that it has a unique maximizer, which we denote $Q^*_{\mathrm{a}}(w, p)$. (It is the smallest integer for which the left-hand side of (9) is less than the right-hand side.) It is immediate that the optimal quota must be $Q_{\mathrm{a}}(w, p) = \min(Q^*_{\mathrm{a}}(w, p), \overline{Q}_{\mathrm{a}})$.

We finally show that $Q_{\mathrm{a}}(w, p)$ is non-decreasing in p. Since only the interior solution depends on p, we only need to prove the claim for $Q^*_{\mathrm{a}}(w, p)$. From the FOC given by (9), it is sufficient to show that the summation term (the only one involving p) is increasing in p. Taking the derivative of that term with respect to p produces

$$\left[\frac{\gamma(Q)}{Q}\right] \sum_{i=0}^{N-1-Q} \left[\frac{Q!(N-Q)!}{(Q+i)!(N-1-Q-i)!}\right]\left[\frac{p^i}{(1-p)^{2+i}}\right]\left(1 + \frac{i}{p}\right) > 0,$$

so the claim holds. To see why this is so, fix some p and consider the optimum $Q^*_{\mathrm{a}}(w, p)$, which is the smallest integer for which the left-hand side of (9) is less than the right-hand side (that is, increasing the quota would make the payoff worse). If increasing p causes the left-hand side to increase, it will eventually exceed the right-hand side for some $\hat{p} > p$. But then $Q^*_{\mathrm{a}}(w, \hat{p})$ will no longer be the smallest integer that makes the left-hand side less than the right-hand side (i.e., it will no longer be optimal). Since the left-hand side is decreasing in Q, the requirement for optimality can be restored by increasing the quota to $Q_{\mathrm{a}}^{*}(w, \hat{p}) = Q_{\mathrm{a}}^{*}(w, p)+1$, which will make the left-hand side less than the right-hand side again. Continuing in this manner, we see that increasing p will cause the quota to increase in step-wise fashion until it reaches the ceiling $\overline{Q}_{\mathrm{a}}$. □

Lemma 4. Note now that $\lim_{p\to 1} U_{\mathrm{a}} = 1-w+a-\hat{x}(Q_{\mathrm{a}})$. This is strictly preferable to private consumption whenever this is greater than 1, or, after rearranging terms, whenever $aN + (a-1)(N - Q_{\mathrm{a}}) > wN + \theta$. Since $N \geq Q_{\mathrm{a}}$ and $a - 1 > 0$, the second term on the left-hand side is non-negative at the optimum quota. It then follows that it is sufficient to establish that $aN > wN + \theta$ holds. Since the right-hand side is increasing in w, we only need to establish the claim at $\overline{w}$, where it reduces to $aN > \overline{w}N + \theta = N \Leftrightarrow a > 1$, which holds. □

Proposition 6. Since the strategies are unconditional, deviation does not affect future play, and the discount factor is irrelevant. The only possibly profitable deviation is therefore limited to the stage-game. However, since delegation with Q_{a} is preferable to private consumption and because the strategies from Proposition 5 specify an equilibrium in the stage-game, no such deviation exists. □

Abiding by the Vote: Between-Groups Conflict in International Collective Action

Christina J. Schneider
Department of Political Science, University of California – San Diego
Branislav L. Slantchev
Department of Political Science, University of California – San Diego

Contact:	Branislav L. Slantchev Department of Political Science University of California, San Diego 9500 Gilman Drive La Jolla, CA 92093-0521
Phone:	858-822-2684
Fax:	858-534-7130
Email:	slantchev@ucsd.edu

Short Bio